Kamil — Equivariant cohomology & rings of functions

Carrell–Liebermann '77, Akyildiz–Carrell '87 } $\Rightarrow$ recover non-equiv. cohomology of a smooth proj. variety from isolated zeroes of a vector field

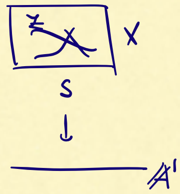
$B_2 := B(SL_2) = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \det = 1 \}$; $B_2 \curvearrowright X$ smooth proj. variety $/\mathbb{C}$

$S = \underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\underset{\shortparallel}{e}} + \underset{\{\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}\}}{\underset{\shortparallel}{\mathfrak{t}}}$ – affine line in $\mathfrak{b}_2$

$\exists$ vector field on $\mathfrak{b}_2 \times X$, on $\{v\} \times X$ comes from infinitesimal B_2-action

a section of $\pi^* TX$

Restrict it to $S \times X$; $\mathcal{Z}$ = zero scheme of this v. field

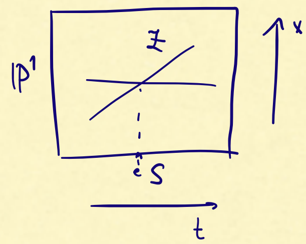


Thm (Brion–Carrell)
Assume the action $B_2 \curvearrowright X$ is regular, i.e. $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has a single zero.
Then $\mathcal{Z}$ is affine, every irr. comp. isomorphic to $\mathbb{A}^1$,
and $\exists$ natural graded iso $H^*_{\mathbb{C}^*}(X, \mathbb{C}) \to \mathbb{C}[\mathcal{Z}]$ of algebras.

Rmk For grading on $\mathbb{C}[\mathcal{Z}]$, $\exists$ $\mathbb{C}^* \curvearrowright S \times X$ preserving $\mathcal{Z}$: $t \cdot \left(\begin{pmatrix} v & 1 \\ 0 & -v \end{pmatrix}, x \right) = \left(\begin{pmatrix} t^{-2}v & 1 \\ 0 & -t^{-2}v \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot x \right)$

Example $B_2 \curvearrowright \mathbb{P}(\text{irrep of } SL_2)$ — otherwise the action is not regular

E.g. for vector rep



$\mathbb{C}[\mathcal{Z}] = \mathbb{C}[t, x] / x(x-t)$
$\rightsquigarrow$ at $t = 0$, $\mathbb{C}[x]/x^2 \cong \mathbb{P}^1$

More general groups

G reductive or parabolic

Define $S \subset \mathfrak{g}$ – Kostant section; parametrizes regular conj. orbits

If G semisimple, $S = e + C_{\mathfrak{g}}(f)$, (e, f, h) – principal $\mathfrak{sl}_2$-triple

G solvable, $S = e + \mathfrak{t}$, $\mathfrak{t}$ max torus

$\exists$ $\mathfrak{g}$-family of v. fields on X as before;

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{Z}^{\text{total}} \arrow[d] \\ S \times X \arrow[r] & \mathfrak{g} \times X \end{tikzcd}$$

$\mathcal{Z}^{\text{total}}$ ← zero scheme of v. field

Thm (Hausel–R.)
Assume that the action is regular. Then $\mathcal{Z}$ affine, and $H^*_G(X, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}[\mathcal{Z}^{\text{tot}}]^G \xrightarrow{\sim} \mathbb{C}[\mathcal{Z}]$, isos of graded algs over $H^*_G(pt)$.

If $\mathcal{E} \to X$ G-equiv. vector bundle, $c_k(\mathcal{E}) \longmapsto [(v, x) \in \mathfrak{g} \times X \longmapsto \mathrm{Tr}_{\Lambda^k v}(\Lambda^k \mathcal{E}_x)]$

Rmk X spherical $\not\Rightarrow$ regular!

Till — Stable envelopes

Motivation: → certain coh. classes
→ can be used to construct braidings.

Setup:
- (X, ω) smooth, cpx symplectic, quasi-projective
- $T = T_1 \times T_2 \curvearrowright X$ torus action, s.t. ω is T_1-invariant, $\mathrm{rk}\, T_2 = 1$, $T_2 \curvearrowright \omega$ with weight 1.
- $X \to X_0$ proper, T-equivariant to affine variety
- X^{T_1} is finite.

Example $X = T^*Gr(k,n) \longrightarrow X_0 = \{ A \in \mathfrak{gl}_n : A^2 = 0 \}$

Attracting cells Fix $\sigma : \mathbb{C}^* \to T_1$ s.t. $X^\sigma = X^{T_1}$

Def $p \in X^{T_1} \rightsquigarrow \mathrm{Attr}_\sigma(p) = \{ x \in X : \lim_{t \to 0} \sigma(t).x = p \}$

$\preceq$ partial order on X^{T_1}, $\quad p \preceq q \iff p \in \overline{\mathrm{Attr}_\sigma(q)}$

Fact $\mathrm{Attr}^f_\sigma(p) := \coprod_{q \preceq p} \mathrm{Attr}_\sigma(q)$ is a closed subvariety.

Thm (Maulik–Okounkov) $\exists!\ \mathrm{Stab}_\sigma : X^{T_1} \longrightarrow H_T^{\dim(X)}(X)$ s.t. ← even b/c X is symplectic!

1) normalization: $\forall p \in X^{T_1}$: $i_p^*(\mathrm{Stab}_\sigma p) = e\left(T_p(X)^-_\sigma\right)$
2) support: $\forall p \in X^{T_1}$: $\mathrm{Stab}_\sigma p$ is supported on $\mathrm{Attr}^f_\sigma(p)$
3) smallness: $\forall p, q \in X^{T_1}, p \neq q$: $i_q^* \mathrm{Stab}_\sigma(p)$ is divisible by $\hbar$ ← equiv. param. of $H^*_{T_2}$

Cor $(\mathrm{Stab}_\sigma(p))_{p \in X^{T_1}}$ is a basis of $H_T^*(X)_{loc}$

Ex $X = T^*\mathbb{P}^1$, $\sigma : t \mapsto (t, t^2)$

$\mathrm{Stab}_\sigma([0:1]) = [\mathrm{Fiber}_{[0:1]}]$

$\mathrm{Stab}_\sigma([1:0]) = [\mathbb{P}^1] + [\mathrm{Fiber}_{[1:0]}]$

R-matrices

Notation $i < j \rightsquigarrow T_{ij} := \{ (t_1, \ldots, t_r) \in T_1 : t_i = t_j \}$ $\quad T_1 / T_{ij}$ is a torus of rank 1.

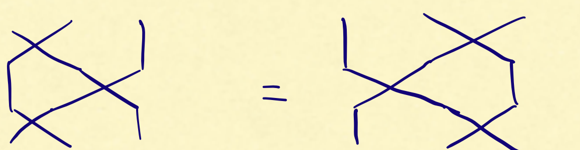
$\sigma_{ij} : \mathbb{C}^* \to T_1/T_{ij} \quad t \longmapsto (1, \ldots, \underset{\text{pos } i}{t}, -, 1)$

$\tau_{ij} : \mathbb{C}^* \to T_1/T_{ij} \quad t \longmapsto (1, \ldots, \underset{\text{pos } j}{t}, -, 1)$

Setup
1) If w is a T_1-weight of T_pX then $w = t_i - t_j$ for some $i \neq j$
2) $X^{T_1} = X_1 \times \ldots \times X_r$
3) $X^{T_{ij}} = X_{ij} \times \prod_{k \neq i,j} X_k$, $\quad X_{ij} \subset X$ closed subvariety.

Def $R_{ij} := H_T(X_i \times X_j) \xrightarrow{\mathrm{Stab}_{\sigma_{ij}}} H_T(X_{ij}) \xrightarrow{\mathrm{Stab}^{-1}_{\tau_{ij}}} H_T(X_i \times X_j) \xrightarrow{\mathrm{Flip}} H_T(X_j \times X_i)$

Thm (Maulik–Okounkov) (in localized cohomology)



$H(X_i) \otimes H(X_j) \otimes H(X_k) \qquad\qquad H(X_i) \otimes H(X_j) \otimes H(X_k)$

Miguel — Hitchin map for minuscule Lagrangians

C smooth proj. curve $/\mathbb{C}$, $g \geq 2$, K_C canonical bundle.

Def A Higgs bundle is (E, Φ), where
- E v.bun of rk n, deg d on C
- $\varphi : E \to E \otimes K_C$

Ex $E_0 = \mathcal{O} \oplus K^{-1} \oplus K^{-2} \oplus \dots \oplus K^{-n+1}$

$$\varphi_0 = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & 1 & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

$\exists\ \mathcal{M}(n,d)$ – moduli space of (polystable) Higgs bundles. It is qproj, symplectic, has $\mathbb{C}^*$-action, sympl form has wt 1.

Def $h : \mathcal{M}(n,d) \longrightarrow \mathcal{A} = \bigoplus_{i=1}^{n} H^0(C, K^i)$, $h(E,\varphi) = (a_1, \dots, a_n)$, where $\sum_{i=0}^{n} a_{n-i} \lambda^i = \det(\lambda - \varphi)$

It is proper, complete integrable system

Ex $h(E_0, \varphi_0) = 0$

h is a "global version" of $\chi : \mathfrak{gl}_n \to \mathfrak{gl}_n /\!/ GL_n$

Let $S = e + C_{\mathfrak{gl}_n}(f)$ be the Kostant section. Let $C_{\mathfrak{gl}_n}(f) = \langle f_1, \dots, f_n \rangle$

$$e = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}$$

We can define $\underset{\text{(of } h)}{\text{Hitchin section}}$ $\mathcal{A} \to \mathcal{M}(n,d)$, $a \longmapsto (E_0, \varphi_a)$; $\varphi_a = \varphi_0 + a_1 f_1 + \dots + a_n f_n$

The image is $W_0^+ = \{(E_0, \varphi_a) : a \in \mathcal{A}\} \subset \mathcal{M}(n,d)$

Fix $c \in C$, $V \subset E_0|_c$ with $\varphi_a|_c(V) \subset V$

Def
$$\begin{array}{ccccccccc} 0 & \to & E_V & \longrightarrow & E_0 & \longrightarrow & E_0|_c / V & \longrightarrow & 0 \\ & & \downarrow \varphi_{V,a} & & \downarrow \varphi_a & & \downarrow \bar{\varphi}_a & & \\ 0 & \to & E_V \otimes K & \longrightarrow & E_0 \otimes K & \longrightarrow & E_0|_c / V \otimes K & \longrightarrow & 0 \end{array}$$

$\rightsquigarrow (E_V, \varphi_{V,a})$ Higgs field

$W_k^+ := \{(E_V, \varphi_{V,a}) : a \in \mathcal{A},\ \varphi_a(V) \subset V,\ \dim V = k\}$

$S_k = \{(A,V) \in S \times Gr(k,n) : A(V) \subset V\}$

$$\begin{array}{ccccc} W_k^+ & \longrightarrow & S_k & \longrightarrow & \mathrm{Spec}(H^\bullet_G(Gr(k,n))) \\ \downarrow h & & \downarrow & & \downarrow \\ \mathcal{A}_n & \xrightarrow{ev_c} & S & \longrightarrow & \mathrm{Spec}(H^\bullet_G(pt)) \end{array}$$

Jakub Equivariant cohomology, K-theory & fixed point schemes

$G = GL_n$, $T \subset G$ max. torus $\qquad G \curvearrowright X$

$\mathrm{Fix}_G(X) = \{(g,x) \in G \times X : gx = x\} \quad \curvearrowleft G$

$\mathrm{Zero}_{\mathfrak{g}}(X) = \{(v,x) \in \mathfrak{g} \times X :$ vector field v vanishes at $x\}$

Claim: $K_0^G(X) \simeq \mathbb{C}[\mathrm{Fix}_G X /\!/ G]$, $\quad H_G^*(X) \simeq \mathbb{C}[\mathrm{Zero}_{\mathfrak{g}} X /\!/ G]$

Ex 1 $X = pt$

$G = T$ torus: $\quad \mathrm{Fix}_T(pt) = T \rightsquigarrow \Gamma(T, \mathcal{O}) = \mathbb{C}[t_1^\pm, \dots, t_n^\pm] \qquad K_0^T(pt, \mathbb{C}) = \mathbb{C}[t_1^\pm, \dots, t_n^\pm]$

$G = GL_n$: $\quad \Gamma(G, \mathcal{O})^G = \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g)\} \qquad K_0^G(pt, \mathbb{C}) = \mathbb{C}[t_1^\pm, \dots, t_n^\pm]^W \cong \mathbb{C}[c_1, \dots, c_{n-1}, c_n^\pm]$

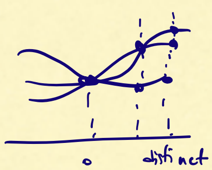
$(g \mapsto \mathrm{tr}_g V) \longleftarrow [V]$

Similar for H_G^* & $\mathrm{Zero}_{\mathfrak{g}}$.

Ex 2 $G = GL_n$, $X = \mathbb{P}^{n-1}$

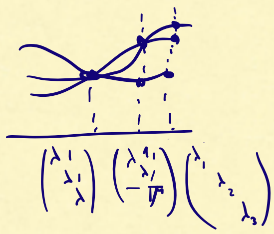
$H_T^*(\mathbb{P}^{n-1}, \mathbb{C}) = \mathbb{C}[t_1, \dots, t_n][\xi] / (\xi - t_1) \dots (\xi - t_n) \qquad H_G^*(\mathbb{P}^{n-1}, \mathbb{C}) = \mathbb{C}[c_1, \dots, c_n][\xi] / \xi^n - c_1 \xi^{n-1} + c_2 \xi^{n-2} - \dots$

$\mathrm{Spec}\, H_G^*(\mathbb{P}^{n-1})$
$\downarrow$
$\mathrm{Spec}\, H_G^*(pt)$



$\mathbb{A}^n = S$

$\mathrm{Zero}_S(\mathbb{P}^{n-1})$



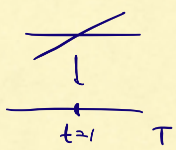
Upshot: $G \curvearrowright G/P$

$\mathrm{Zero}_{\mathfrak{g}}(G/P) \to \mathfrak{g}$ is the partial Grothendieck-Springer alteration

$\Rightarrow H_G^*(G/P) \simeq \mathbb{C}^G[\mathrm{Zero}_{\mathfrak{g}}(G/P)]$

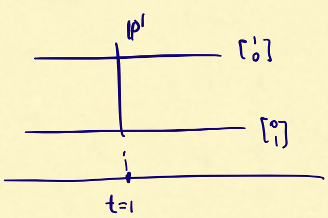
Ex 3 $T = \mathbb{G}_m = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \curvearrowright \mathbb{P}^1$

$K_0^T(\mathbb{P}^1) = \mathbb{C}[t^\pm][\xi] / (\xi - t)(\xi - 1)$



$0 \to \mathbb{C}[t^\pm][\xi] / (\xi - t)(\xi - 1) \to \mathbb{C}[t^\pm] \oplus \mathbb{C}[t^\pm] \to \mathbb{C} \to 0$ exact sequence.

$\mathrm{Fix}_T \mathbb{P}^1$:



$\Rightarrow \mathbb{C}[\mathrm{Fix}_T \mathbb{P}^1] =$ first term of the s.e.s. above.

Tanguy Intersection cohomology rings of nilpotent orbit closures

$n \geq 1$, $N \geq 2$ $\qquad \lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N_{\geq 0}$ – composition of n

$\overline{\mathbb{O}}_\lambda$: closure of nilp. orbit in $\mathfrak{gl}_n$, Jordan blocks $\lambda_1, \dots, \lambda_N$

Fl_λ : partial flags $\quad 0 \subset F_1 \subset F_2 \subset \dots \subset F_N = \mathbb{C}^n \quad ; \quad \lambda_i = F_i / F_{i-1}$

$$\begin{array}{ccc} X = T^* Fl_\lambda & & (x, F_\bullet) \\ \downarrow & & \downarrow \\ X_0 = \overline{\mathbb{O}}_\lambda & & x \end{array} \qquad \text{symplectic resolution}$$

Set $T = A \times \mathbb{C}^\times_\hbar = (\mathbb{C}^\times)^n \times \mathbb{C}^\times_\hbar \curvearrowright X$ $\qquad R := H^\bullet_T(pt) = \mathbb{C}[a_1, \dots, a_n, \hbar] \qquad K = \mathrm{Frac}(R)$

Consider $V_n = \bigoplus_\lambda H^\bullet_T(T^* Fl_\lambda) \curvearrowleft \tilde{Y}_\hbar(\mathfrak{gl}_N) \supset \mathfrak{gl}_N$

After localization: $K \otimes_{\mathbb{C}} (\mathbb{C}^N)^{\otimes n} \xrightarrow{\mathrm{Stab}} K \otimes_R V_n$ $\quad$ map of $\tilde{Y}_\hbar(\mathfrak{gl}_N)$-modules.
$\circlearrowleft$ $\mathfrak{gl}_N$

Goal: construct a graded ring structure on $IH^\bullet_T(\overline{\mathbb{O}}_\lambda) \subseteq H^\bullet_T(T^* Fl_\lambda)$ using quantum cup product on $H^\bullet_T(T^* Fl_\lambda)$
(after McBreen–Proudfoot)

Strategy: Yangian action $\rightsquigarrow IH^\bullet_T(\mathbb{O}_\lambda)$ [DHSM '23]
Formulas for $QH^\bullet_T(T^* Fl_\lambda)$ [Gorbounov-Rimanyi-Tarasov-Varchenko '13]

Running example $n = N = 2$

$$\begin{array}{cccc} V_2 = & H^\bullet_T(pt) & \oplus H^\bullet_T(\mathbb{P}^1) & \oplus H^\bullet_T(pt) \\ & (2,0) & (1,1) & (0,2) \\ \mathrm{Stab} \uparrow & & & \\ R \otimes (\mathbb{C}^2)^{\otimes 2} & e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_2 \\ & & e_2 \otimes e_1 & \end{array}$$

Intersection cohomology [DHSM]

$\{\lambda_1 \leq \dots \leq \lambda_N,\ IH^\bullet_T(\overline{\mathbb{O}}_\lambda)\} \subseteq H^\bullet_T(T^* Fl_\lambda) \subseteq V_n$ $\quad$ are the spaces of lowest weight vectors in V_n.

$$(\mathbb{C}^N)^{\otimes n} = \bigoplus_{\mu_1 \geq \dots \geq \mu_n} L(\mu)^{\oplus f_\mu} \qquad f_\mu = \#\text{ of std Young tableaux of shape } \mu.$$

Ex $n = N = 2$ $\qquad (\mathbb{C}^2)^{\otimes 2} = \mathrm{Sym}^2 \mathbb{C}^2 \oplus \Lambda^2 \mathbb{C}^2$
$\qquad e_2 \otimes e_2 \qquad e_1 \otimes e_2 - e_2 \otimes e_1$ $\quad$ lowest weight vector

$IH^\bullet_T(\overline{\mathbb{O}}_{(0,2)}) \otimes_R K$ spanned by $\mathrm{Stab}(e_2 \otimes e_2)$
$IH^\bullet_T(\overline{\mathbb{O}}_{(1,1)}) \otimes_R K$ — " — $\mathrm{Stab}(e_1 \otimes e_2 - e_2 \otimes e_1)$

Ex $\lambda = (2,1)$ $\rightsquigarrow (\mathbb{C}^2)^{\otimes 3} = \mathrm{Sym}^3 \mathbb{C}^2 \oplus (L(2,1))^{\oplus 2}$

$IH^\bullet_T(\overline{\mathbb{O}}_{(1,2)})$ spanned by Stabs of
$e_2 \otimes e_2 \otimes e_1 - e_1 \otimes e_2 \otimes e_2$ $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$
$e_2 \otimes e_1 \otimes e_2 - e_1 \otimes e_2 \otimes e_2$ $\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$

Quantum cohomology: deformed cup product on $QH^\bullet_T(T^*Fl_\lambda) = H^\bullet_T(Fl_\lambda) \otimes \mathbb{C}(q_1, \dots, q_N)$

Maulik-Okounkov: quantum mult. by divisors on $H^\bullet_T(T^*Fl_\lambda) \otimes \mathbb{C}(q)$ is given by action of Bethe subalgs. of Yangian.

GRTV: $QH^\bullet_T(T^*Fl_\lambda) = \mathcal{H}^q_\lambda := R(q_1 \dots q_N)[\gamma_{ij} \ {\scriptstyle 1 \le i \le N \atop 1 \le j \le \lambda_i}] \Big/ \langle W^q(u) = \prod_{i<j}(q_i - q_j) \times \hat{\prod_{k=1}}(u - a_k) \rangle$

where $W^q(u) := \det\left(q_i^{N-j} \prod_{k=1}^{\lambda_i} (u - \gamma_{ik} + \hbar(i-j)) \right)_{1 \le i,j \le N}$

e.g. $\mathcal{H}^q_\lambda = R(q_1, q_2)[\gamma_1 = \gamma_{11}, \gamma_2 = \gamma_{21}] \Big/ \begin{array}{l} \gamma_1 + \gamma_2 = a_1 + a_2 \\ \gamma_1\gamma_2 + \frac{q_2}{q_1 - q_2}\hbar(\gamma_1 - \gamma_2 + \hbar) = a_1 a_2 \end{array}$

McBreen-Proudfoot conjecture:

- specialize $q_1 = \dots = q_N = 1$
- quotient out $\mathrm{Ann}(\hbar)$

Conj

$$\begin{tikzcd} IH^\bullet_T(\overline{\mathcal{D}}_\chi) \arrow[r, hook] \arrow[rr, bend right, "\simeq \text{ of graded } R\text{-modules}"'] & QH^\bullet_T(T^*Fl_\lambda) \arrow[r, two heads] & QH^\bullet_T(\text{specialized}) \end{tikzcd}$$

Ex

$N = n = 2$, $\lambda = (1,1)$ $\quad R[\gamma_1, \gamma_2] \Big/ \begin{array}{l} \gamma_1 + \gamma_2 = a_1 + a_2 \\ \hbar(\gamma_2 - \gamma_1 - \hbar) = 0 \end{array} \rightsquigarrow R[\gamma_1, \gamma_2] \Big/ \begin{array}{l} \gamma_1 + \gamma_2 = a_1 + a_2 \\ \gamma_2 - \gamma_1 - \hbar = 0 \end{array} \simeq R.$

$e_1 \otimes e_2 - e_2 \otimes e_1 \rightsquigarrow \gamma_1 - a_2 - (\gamma_1 - a_1 + \hbar) = a_1 - a_2 - \hbar.$

Tamas Big algebras

Classic version – motivation from mirror symmetry

Quantum version – Verma, Harish-Chandra modules

$\mathfrak{g}$ semisimple Lie algebra (e.g. $G = SL_n, PGL_n$)

$\mu \in \Lambda^+(G) =: \Lambda^+ \rightsquigarrow \rho^\mu : G \to GL(V^\mu)$ $\quad \rho_\mu = \mathrm{Lie}(\rho^\mu) \rightsquigarrow U(\mathfrak{g}) \to \mathrm{End}(V^\mu)$

V^μ – rep. of highest weight μ. $\quad$ filtered $\quad U(\mathfrak{g}) = \bigcup_{p=0}^{\infty} U_p(\mathfrak{g})$

PBW: $\mathrm{gr}\, U(\mathfrak{g}) \simeq S(\mathfrak{g})$

$\pi : S(\mathfrak{g}) \to U(\mathfrak{g})$ – symmetrization map (of $\mathfrak{g}$-modules!)

$R = R(\mathfrak{g}) = (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^G$ ← diagonal action – universal Kostant algebra $/ Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$ $\qquad Z(\mathfrak{g}) := U(\mathfrak{g})^G$

$C = C(\mathfrak{g}) = (S(\mathfrak{g}) \otimes U(\mathfrak{g}))^G$ – universal Kirillov algebra $/ S(\mathfrak{g})^G \otimes Z(\mathfrak{g})$

$R^\mu := C^\mu(R) = (U(\mathfrak{g}) \otimes \mathrm{End}\, V^\mu)^G$ – Kostant algebra $/ Z(\mathfrak{g})$

$C^\mu := C^\mu(C) = (S^\cdot(\mathfrak{g}) \otimes \mathrm{End}\, V^\mu)^G$ – Kirillov algebra $/ S^\cdot(\mathfrak{g})^G$

Claim (Higson) Irred. R^μ-mod $\xleftrightarrow{1:1}$ $(\mathfrak{g}, K)$-modules V where $\mathrm{Hom}_K(V^\mu, V) \neq 0$.

$$\begin{array}{ccc} Z & \xhookrightarrow{\Delta} & R \\ \cap & & \cap \\ U & \longrightarrow & U \otimes U \end{array}$$
← coproduct

$Z = \bigoplus Z^p$ ← here we use the symmetrization iso

$\Omega^p := \pi(q^p)$ for some $q^p \in Z^p$; $\quad \Delta(\Omega^p) = \bigoplus_{k=0}^{p} \underbrace{D^k(\Omega)^p}_{U^{p-k} \otimes U^k} \quad \rightsquigarrow \quad D^k(\Omega)^p \in R$

Thm (Hausel–Zveryk '22; for $\mathfrak{sl}_n$)

Let $c_p \in S^p(\mathfrak{g})^G$ – p-th elementary symm. polynomial $\quad \simeq \mathbb{C}[\mathfrak{t}]^{S_n}$

Then $\{D^k(\Omega^p)\}_{p=2,\dots,n}^{k=0,\dots,p}$ commute.

In fact, the subalg. gen. by $D^k(\Omega^p)$ is isom. to Gaudin algebra (Feigin–Frenkel–Reshetikhin); it is a homomorphic image of Feigin–Frenkel center.

Denote this algebra by $\mathcal{G} \subset R$; this is a polynomial algebra.

$\mathcal{G}^\mu \subset C^\mu(\mathcal{G}) \subset R^\mu \quad /Z$ – quantum big algebra

$\mathcal{B}^\mu = \bar{\mathcal{G}}^\mu = C^\mu(\bar{\mathcal{G}}) \subset C^\mu$ – big algebra

$\mathcal{G}^\mu_\hbar = IH^*_{G^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu) \circlearrowleft Z^\mu_\hbar = Z(R^\mu) \simeq H^*_{G^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu)$

$\chi : Z \to \mathbb{C}$ character $\rightsquigarrow R^\mu_\chi = R^\mu / \ker\chi = \mathrm{End}(M_\chi \otimes V^\mu)^G$ $\qquad M_\chi$ is Verma module with infinitesimal char χ.

Conj $Z(R^\mu_\chi) = Z(R^\mu)_\chi$.

Catharina: Verma modules

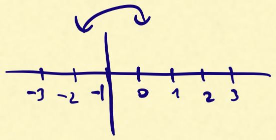
$\mathfrak{g}$ reductive cpx Lie algebra, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, $\lambda \in \mathfrak{h}^*$ $\rightsquigarrow$ $M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ – Verma

Category $\mathcal{O}$: smallest abelian subcategory of $\mathcal{U}(\mathfrak{g})$-modules s.t.
- $M(\lambda) \in \mathcal{O}$ $\forall \lambda \in \mathfrak{h}^*$
- closed under taking quotient
- closed under tensoring with fin. dim. reps

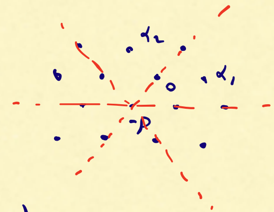
Rmk $\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/W} \mathcal{O}_\lambda$; $M(\mu) \in \mathcal{O}_\lambda \iff \mu \in W \cdot \lambda$

dot action (origin at $-\rho$)

Ex $\mathfrak{sl}_2$



$\mathfrak{sl}_3$



Tensor identity: $M(\lambda) \otimes V = (\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda) \otimes V \simeq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} (\mathbb{C}_\lambda \otimes V)$

Ex $\mathfrak{g} = \mathfrak{sl}_2$ $V = \mathbb{C}^2$ natural rep. $\rightsquigarrow$ as $\mathcal{U}(\mathfrak{b})$-modules, $\mathbb{C}_\lambda \otimes \mathbb{C}_1 \hookrightarrow \mathbb{C}_\lambda \otimes V \twoheadrightarrow \mathbb{C}_\lambda \otimes \mathbb{C}_{-1}$ s.e.s.

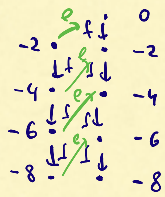
PBW $\Rightarrow$ tensoring $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} -$ is exact

$\rightsquigarrow M(\lambda+1) \hookrightarrow M(\lambda) \otimes V \twoheadrightarrow M(\lambda-1)$ s.e.s.

Does it split?

$\lambda = 0$: $M(0) \overset{-\otimes V}{\rightsquigarrow} \begin{matrix} M(-1) \\ \oplus \\ M(1) \end{matrix} \overset{-\otimes V}{\rightsquigarrow} \begin{matrix} M(-2) & \\ M(0) & M(0) \\ M(2) & \end{matrix}$ ← unclear. ← splits

$\lambda = -1$: $M(-1) \otimes V$



$e(1 \otimes 1 \otimes v_{-1}) = 1 \otimes e(1 \otimes v_{-1}) = 1 \otimes \cancel{e 1} \otimes v_{-1} + 1 \otimes 1 \otimes e v_{-1} = v_1$.

$\uparrow$

$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda \otimes V$

$\Rightarrow M(0) \to M(-1) \otimes V \to M(-2)$ doesn't split!!

Rmk Note that $V \otimes V = V(2) \oplus V(0)$.

Thm 1) $\mathcal{O}, \mathcal{O}_\lambda$ have enough projectives $\{$indec. proj$\} \longleftrightarrow \mathfrak{h}^*$

2) Simple obj. in $\mathcal{O}$: $L(\lambda)$, $\lambda \in \mathfrak{h}^*$

3) Every proj. is filtered by Vermas

Ex $\mathfrak{g} = \mathfrak{sl}_2$ $\quad \mathrm{End}_{\mathcal{O}} \begin{pmatrix} M(-2) \\ M(0) \end{pmatrix} = \mathrm{End}_{\mathcal{O}}(P(-2)) = \mathbb{C}[x]/_{x^2} = H^*(\mathbb{P}^1)$

In general, $M(0) \otimes V(n) = \begin{pmatrix} M(-n) \\ M(n-2) \end{pmatrix} \begin{pmatrix} M(-n+2) \\ M(n-4) \end{pmatrix} \cdots \begin{pmatrix} M(-2) \\ M(0) \end{pmatrix}$ or $M(-1)$ $\oplus M(n)$

$\Rightarrow$ endomorphism ring is commutative!

Thm (S) λ dominant integral, $P := P(x \cdot \lambda)$ TFAE: (1) $\mathrm{End}_{\mathcal{O}}(P)$ is commutative

(2) $(P : M(\mu)) \leq 1$ $\forall \mu \in \mathfrak{h}^*$

(3) multiplication gives a surj. $Z(\mathcal{U}(\mathfrak{g})) \twoheadrightarrow \mathrm{End}_{\mathcal{O}}(P)$

Conj In general, $Z(\mathcal{U}(\mathfrak{g})) \twoheadrightarrow Z(\mathrm{End}_{\mathcal{O}}(P))$

How to show (2) $\Rightarrow$ (1)? Deform! $\mathcal{U}(\mathfrak{h}) = S(\mathfrak{h})$, $T := S(\mathfrak{h})_0$, T' any T-algebra (commutative)

$$\mathcal{U}(\mathfrak{b}) \twoheadrightarrow \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) \to S(\mathfrak{h})_0 = T \hookrightarrow T'$$

$M_{T'}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} (\mathbb{C}_\lambda \otimes T')$

$\{T' = \mathbb{C}\}$ $\{T' = Q := \operatorname{Frac} T\}$

$M(\lambda)$ $\qquad$ $\mathcal{U}(\mathfrak{g}) \otimes Q$ - Verma

<u>Thm</u> (Soergel)

$$\operatorname{Hom}_{\mathcal{U}(\mathfrak{g}) \otimes T'}(P_{T'}(\lambda), P_{T'}(\mu)) \simeq \operatorname{Hom}_{\mathcal{U}(\mathfrak{g}) \otimes T}(P_T(\lambda), P_T(\mu)) \otimes T'.$$

Shon Type A big algebras & Bethe subalgebras of the Yangian

$\mathfrak{g}=\mathfrak{gl}_n$, $G=GL_n$, (π,V) $\mathfrak{gl}_n$-rep

<u>Kirillov algebra</u> $\quad \mathcal{C}(V) = \left(S(\mathfrak{gl}_n^*)\otimes \operatorname{End} V\right)^{GL_n}$ — "GL_n-equiv. End V-valued polynomial maps"

$$\begin{array}{c}\mathcal{C}(V) \\ \cup \\ S(\mathfrak{gl}_n^*)^{GL_n}\otimes \mathrm{Id}_V\end{array}$$

Denote E_{ij} basis of $\mathfrak{gl}_n$; $y_{ij}\in S(\mathfrak{gl}_n^*)$ - corr. coordinates.

<u>Kirillov-Wei operators</u> ["Introduction to family algebras"]

$$D_V : \mathcal{C}(V)\to\mathcal{C}(V) \quad , \quad [D_V(F)]\big(\underset{\mathfrak{gl}_n}{\overset{\in}{Y}}\big) = \sum_{i,j=1}^{n} \frac{\partial F}{\partial y_{ij}}(Y)\cdot \pi(E_{ji})$$

<u>Big algebras</u>: $1\le k\le n$, define $c_k\in S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ by $\det(I-tY) = 1-c_1(Y)t+c_2(Y)t^2-\dots+(-1)^n c_n(Y)t^n$

$$c_k(Y) = \sum_{|I|=k} \det Y_{II} \quad ; \quad Y=\begin{pmatrix} y_{11} & \cdots & y_{1n}\\ \vdots & \ddots & \vdots\\ y_{n1} & \cdots & y_{nn}\end{pmatrix}$$

<u>Rmk</u> This choice is important; other sym. polys (e.g. Tr ...) do not work!

<u>Def</u> <u>Big algebra</u> $\mathcal{B}(V) = \langle D^p(c_k) : \begin{smallmatrix}0\le p\le k\\ 1\le k\le n\end{smallmatrix}\rangle$

<u>Medium algebra</u> $\mathcal{M}(V) = \langle D^p(c_k) : \begin{smallmatrix}p=0,1\\ 1\le k\le n\end{smallmatrix}\rangle$

<u>Fact</u> $\mathcal{M}(V) = Z(\mathcal{C}(V))$

<u>Coordinate ring of $Mat(n,r)$</u>

$GL_n\times GL_r \curvearrowright Mat(n,r) \qquad (g,h)\cdot A = (g^{-1})^T A h$

$P(n,r) = \mathbb{C}[Mat(n,r)] = \mathbb{C}[x_{ij} : \begin{smallmatrix}1\le i\le n\\ 1\le j\le r\end{smallmatrix}] \curvearrowleft GL_n$

<u>Action of $\mathfrak{gl}_n$</u> : $L : U(\mathfrak{gl}_n)\longrightarrow PD(n,r)$ where $PD(n,r)$ = Weyl alg. gen. by $\langle x_{i\alpha},\partial_{i\alpha}\rangle_{\begin{smallmatrix}1\le i\le n\\ 1\le\alpha\le r\end{smallmatrix}}$

$$L(E_{ij}) = \sum_{\alpha=1}^{r} x_{i\alpha}\,\partial_{j\alpha} \leftarrow \frac{\partial}{\partial x_{j\alpha}}$$

Then $\mathcal{B}(P(n,r)) \subset S(\mathfrak{gl}_n^*)\otimes PD(n,r)$

<u>Explicit formula</u>:

For $p,q\ge 0$, define $M_{pq} = D_L^q(c_{p+q})$ — "big operators" (generators of $\mathcal{B}(P(n,r))$)

$$F_{pq}(Y) = \sum_{\begin{smallmatrix}|I_1|=|J_1|=p\\ |I_2|=|J_2|=q\\ I_1\cup I_2 = J_1\cup J_2=\{1\ldots n\}\end{smallmatrix}} \operatorname{sgn}\begin{pmatrix}I_1 & I_1\\ J_1 & J_2\end{pmatrix}\det(Y_{I_1J_1})\sum_{\begin{smallmatrix}|R|=q\\ R\subset\{1\ldots r\}\end{smallmatrix}} \det(X_{J_2R})\det(\mathcal{D}_{I_2R})$$

$\leftarrow$ related to Capelli identities

$X=(x_{ij}) \qquad \mathcal{D}=(\partial_{ij})$

$M_{pq} = q!\,F_{pq}$ + lin. comb of $\{F_{p0}\ldots F_{p\,q-1}\}$ $\Rightarrow$ enough to prove commutativity

<u>Yangians & Bethe subalgebras</u>

$Y(\mathfrak{gl}_n) = \langle t_{ij}^{(r)}\rangle / _{RTT=TTR}$ where $T=(t_{ij}(u))$, $t_{ij}(u) = \delta_{ij}+\sum_{r\ge1} t_{ij}^{(r)}u^{-r}$ $\qquad R_{12}(u-v) = \mathrm{Id}_{\mathbb{C}^n\otimes\mathbb{C}^n} - \frac{1}{u-v}\overset{\text{permutation}}{P_{12}}$

$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$ - equality in $\operatorname{End}\mathbb{C}^n\otimes\operatorname{End}\mathbb{C}^n\otimes Y(\mathfrak{gl}_n)[u^{-1},v^{-1}]$

Bethe subalgebra

$C \in \mathrm{End}(\mathbb{C}^n)$ $\qquad \sigma_k(u, C) = \frac{1}{n!} \mathrm{tr}\left(A_n T_1(u) T_2(u-1) \dots T_k(u-k+1) C_{k+1} \dots C_n\right) \in Y(\mathfrak{gl}_n)[u^{-1}]$

artisymmetizer $\in \mathbb{C}[S_n] \subset \mathrm{End}((\mathbb{C}^n)^{\otimes n})$

Fact If $\sigma_k(u, C) = \sum_{r \geq 0} \sigma_k^{(r)}(C) u^{-r}$,

then $\{\sigma_k^{(r)}(C)\} \subset Y(\mathfrak{gl}_n)$ form commutative subalg. in $Y(\mathfrak{gl}_n)$

Facts • $Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$, $T(u) \longmapsto 1 - u^{-1}E$; $E = \begin{pmatrix} E_{11} & & \\ & \ddots & \\ & & E_{nn} \end{pmatrix}$ - evaluation map.

• $(L \circ ev)(\sigma_{n-p}(u, Y^T)) = \sum_{l=0}^{n-p} \frac{1}{u(u-1)\dots(u-l+1)} F_{pl}(Y) \Rightarrow$ commutativity of big algebra.

Mischa: Minuscule big algebras

$\mathfrak{g}$ cpx semisimple Lie alg ; G connected, simply connected

V irrep of h.w. $\lambda \in \mathfrak{h}^*$, $\mathfrak{h}$ Cartan

$S(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{g}^*)^{\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{g}]^G = J$

$\mathcal{C} = \mathcal{C}(V) = (\mathbb{C}[\mathfrak{g}] \otimes \operatorname{End} V)^{\mathfrak{g}}$; J-algebra, finite rank free J-module; rank $= \dim \operatorname{End}_0(V)$ weight preserving

$\mathcal{B} \subseteq \mathcal{C}$ maximal (wce) commutative subalgebra "big algebra"

$\mathcal{M} = \mathcal{Z}(\mathcal{C})$ "medium algebra"

$\mathcal{C} \simeq \operatorname{Mor}_G(\mathfrak{g}, \operatorname{End} V) = \operatorname{Mor}_G(\mathfrak{g}^{reg}, \operatorname{End} V)$ $\qquad x \in \mathfrak{g}^{reg} \rightsquigarrow ev_x : \mathcal{C} \longrightarrow (\operatorname{End} V)^{G_x} \subseteq \operatorname{End} V$, $f \longmapsto f(x)$

<u>Thm</u> (Panyushev 2004 & 2002) ev_x is onto.

$\Rightarrow$ if $x \in \mathfrak{g}^{reg} \cap \mathfrak{h}$, then $(\operatorname{End} V)^{G_x} = (\operatorname{End} V)^{\mathfrak{g}_x} = \operatorname{End}_{\mathfrak{h}} V$

<u>Cor</u> (Kirillov 2000) $\mathcal{C}$ is commutative $\Leftrightarrow m_\mu = 1 \ \forall \mu$. $= \dim V_\mu$

<u>Def</u> λ is <u>minuscule</u> iff $wt(V^\lambda) = W \cdot \lambda$. $\Rightarrow$ $\mathcal{C}$ is commutative $\Rightarrow \mathcal{C} = \mathcal{B} = \mathcal{M}$.

<u>Fact</u> minuscule wts are fundamental; in type A reverse implication is true, but not in general.

For now, assume V is weight multiplicity free. $\rightsquigarrow$
$$\begin{array}{c} \operatorname{Spec} \mathcal{C} \\ \downarrow \pi \\ \operatorname{Spec} J = \mathfrak{g}/\!/G \end{array}$$

<u>Fibers</u>: $x \in \mathfrak{g}^{reg} \rightsquigarrow \pi^{-1}(x) = \operatorname{Spec}\left(\mathcal{C} \otimes_J J/\ker(J \to \mathbb{C})\right) \simeq ev_x(\mathcal{C}) = (\operatorname{End} V)^{\mathfrak{g}_x}$, $f \mapsto f(x)$

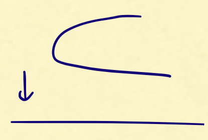
If $x \in \mathfrak{g}^{reg} \cap \mathfrak{h}$ then $\pi^{-1}(x) = \operatorname{Spec}(\operatorname{End}_{\mathfrak{h}} V) = \operatorname{Spec}(\mathbb{C}^{wt(V)}) = \{wt(V)\}$

If λ minuscule, $\mathcal{C} \xrightarrow[\sim]{res_{\mathfrak{h}}} (\mathbb{C}[\mathfrak{h}] \otimes \operatorname{End}_{\mathfrak{h}} V)^W = \left(\mathbb{C}[\mathfrak{h}] \otimes \left(\bigoplus_{\mu \in W \cdot \lambda} \operatorname{End}(V_\mu)\right)\right) = \mathbb{C}[\mathfrak{h}]^{W_\lambda}$
$$\begin{array}{c} \mathbb{C}[\mathfrak{h}]^{W_\lambda} \\ \uparrow \\ \mathbb{C}[\mathfrak{h}]^W \simeq J \end{array}$$

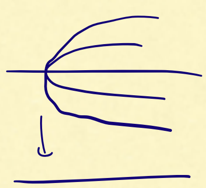
Geometrically, $\mathcal{C} = \mathcal{M} = H^*_{G^\vee}(Gr^{\leq \lambda}_{G^\vee}) = H^*_{G^\vee}(G^\vee/P_\lambda) = H^*_{P_\lambda}(pt) = \mathbb{C}[\mathfrak{h}]^{W_\lambda}$

<u>Examples</u>

1) $\mathfrak{g} = \mathfrak{sl}_2$, $\operatorname{Spec} J = \mathbb{A}^1$, $V = \mathbb{C}^2$ $\rightsquigarrow J = \mathbb{C}[\overbrace{x^2}^{c_2}] \hookrightarrow \mathbb{C}[x] = \mathbb{C}[\mathfrak{h}]$ where $\mathfrak{h} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \right\}$



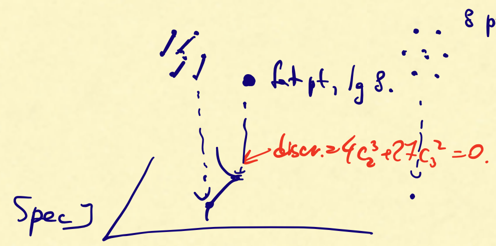
2) More generally, for $V = S^k \mathbb{C}^2$



$$\mathcal{C}(S^k\mathbb{C}^2) = \mathbb{C}[c_2][M_1] \Big/ \begin{array}{ll} M_1(M_1^2 - 2^2 c_2) \cdots (M_1^2 - k^2 c_2) & k \text{ even} \\ (M_1^2 - c_2)(M_1^2 - 3^2 c_2) \cdots (M_1^2 - k^2 c_2) & k \text{ odd} \end{array}$$

3) $\mathfrak{g} = \mathfrak{sl}_3$, $J = \mathbb{C}[c_2, c_3]$, $V = \mathfrak{sl}_3$ - adjoint rep - <u>not</u> weight multiplicity free!

$$\mathcal{B} = \mathbb{C}[c_2, c_3, M_1, N_1] \Big/ \begin{array}{l} M_1^2 + 3N_1^2 + 4c_2 = 0 \\ M_1(N_1^3 + c_2 N_1 + c_3) = 0 \end{array}$$



Daniel Endomorphismensatz

Thm 1 $\pi_0(M(-\rho)\otimes L(\rho)) = $ "$P(w_0\cdot 0)$" $\quad \rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha \qquad$ Then $\mathrm{End}_{U(\mathfrak{g})\text{-mod}}(P(w_0\cdot 0)) = S(\mathfrak{h})/S(\mathfrak{h})^W_+ = H^*(G/B)$

proj. to 0-block.

this lives in simple block $\Rightarrow M = L = P$

Thm 2

$$\begin{tikzcd} Z(U(\mathfrak{g})) \arrow[d, "\Phi_1"] \arrow[r, equal] & Z(U(\mathfrak{g})) \arrow[d, "\Phi_2"] \\ \mathrm{End}_{U(\mathfrak{g})\text{-mod}}(P(w_0\cdot 0)) & S(\mathfrak{h})/S(\mathfrak{h})^W_+ \end{tikzcd}$$

$Z(U(\mathfrak{g})) \xrightarrow{\sim} S(\mathfrak{h})^{W_0} \hookrightarrow S(\mathfrak{h}) \twoheadrightarrow S(\mathfrak{h})/S(\mathfrak{h})^W_+$

Then Φ_2 is surjective, and $\mathrm{Ker}\,\Phi_1 \hookrightarrow \mathrm{Ker}\,\Phi_2$, and $\dim \mathrm{End} = \dim$ coinv.

This implies Thm 1.

Harish-Chandra iso

Note that $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})n^+ + n^- U(\mathfrak{g}))$. If $z\in Z(U(\mathfrak{g}))$, then $z \in U(\mathfrak{h}) \oplus [U(\mathfrak{g})n^+ \cap n^- U(\mathfrak{g})]$

Def $Z(U(\mathfrak{g})) \xrightarrow{\xi} U(\mathfrak{h})$ ("discard things with e & f on the sides") — Harish-Chandra iso.

$$\begin{tikzcd} Z(U(\mathfrak{g})) \arrow[r, "\xi"] \arrow[dr, "\eta"'] & U(\mathfrak{h}) \\ & U(\mathfrak{h})^{W_0} \arrow[u, hook] \end{tikzcd}$$

Ex $\mathfrak{g} = \mathfrak{sl}_2 \qquad Z(U(\mathfrak{sl}_2)) = \langle h^2 + 2(ef+fe)\rangle = \langle \underbrace{h^2+2h+4fe}_{\Omega}\rangle \qquad \xi(\Omega) = h^2+2h$

Note : $(-h-2)^2 + 2(-h-2) = h^2+4h+4-2h-4 = h^2+2h$ ✓

Fact Assume $\lambda \le \mu$. Then $\chi_\lambda = \chi_\mu \iff M(\lambda) \hookrightarrow M(\mu)$ (or vice versa) $\iff \lambda \underset{W_0}{\sim} \mu$

Ex $\mathbb{C}_1 \hookrightarrow L(1) = \overset{\curvearrowright}{\underset{2}{\cdot}\ \underset{1}{\cdot}} \twoheadrightarrow \mathbb{C}_{-1} \qquad \rightsquigarrow M(0) \hookrightarrow P(-2) \twoheadrightarrow M(-2)$

$L(1) = \langle v_1, v_{-1}\rangle$

$\Omega(1\otimes v_1) = (4ef + h^2 + 2h)(1\otimes v_0) = 0$

$\Omega(1\otimes v_{-1}) = (\ ...\ )(1\otimes v_{-2}) = 4(1\otimes v_{-2})$

Alexis: Endomorphism algebra of $M \otimes V(1)^{\otimes n}$

Goal: $\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M \otimes V(1)^{\otimes n})$

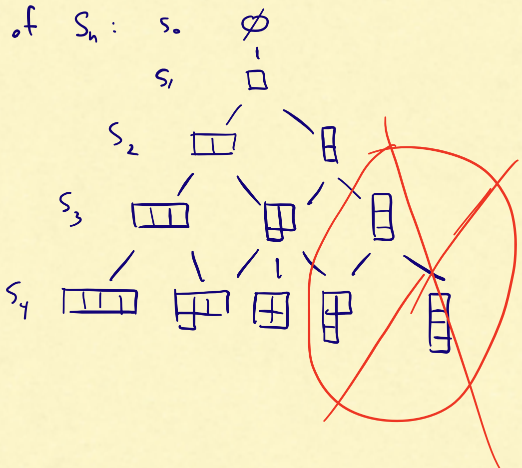
Sources: In general, Brundan–Stroppel III.
For the talk, Iohara–Lehrer–Zhang '21
Math physics: Martin Saleur '94 "blob algebra"

$\mathrm{End}_{U(\mathfrak{sl}_2)}(V(1)^{\otimes n}) \rightsquigarrow$ Temperley–Lieb

$U(\mathfrak{sl}_2) \curvearrowright (\mathbb{C}^2)^{\otimes n} \curvearrowleft S_n$ Schur–Weyl duality $\Rightarrow \mathbb{C}S_n \twoheadrightarrow \mathrm{End}_{\mathfrak{sl}_2}((\mathbb{C}^2)^{\otimes n})$

$$(\mathbb{C}^2)^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq 2}} W_\lambda \otimes V_\lambda$$

Irreps of S_n: S_0 $\varnothing$, S_1, S_2, S_3, S_4



$S_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1, \; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \; s_i s_j = s_j s_i, \; |i-j|>1 \rangle$

$TL_n(\underline{\underline{-2}}) = S_n \big/ \langle 1 - s_1 - s_2 + s_1 s_2 + s_2 s_1 - s_1 s_2 s_1 \rangle$

$t_i = s_i - 1$

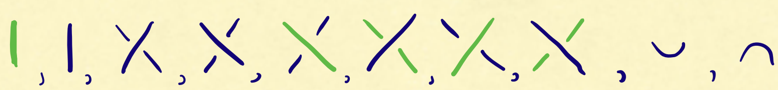
$\simeq \langle 1, t_1, \dots, t_{n-1} \mid t_i^2 = \underline{\underline{-2}} t_i, \; t_i t_{i\pm 1} t_i = t_i, \; t_i t_j = t_j t_i, \; |i-j|>1 \rangle$

Rmk You can check that $U_q(\mathfrak{sl}_2) \curvearrowright (\mathbb{C}^2)^{\otimes n} \curvearrowleft \mathcal{H}_q(S_n)$ semisimple over $\mathbb{C}(q)$
Then quotient of $\mathcal{H}_q$ is $TL_n(-q-q^{-1})$

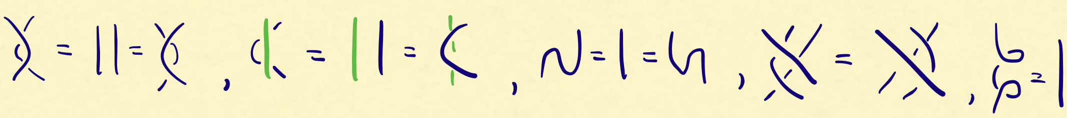
Now: back to $\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M \otimes V(1)^{\otimes n})$

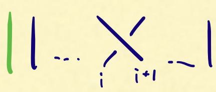
Plan
1) define category of tangles $\mathcal{RT}$
2) find subcategory TLB with correct endomorphisms
3) interpret tangles as $\mathfrak{sl}_2$-action
4) prove equivalence.

① **Obj** $(1, \mathbb{N}_{\geq 0})$ **Morph**:



Relations:



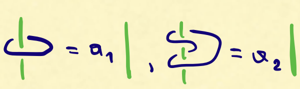
Denote $\xi_1 =$  , $\sigma_i :$ 

We have relations: $\xi_1 \sigma_1 \xi_1 \sigma_1 = \sigma_1 \xi_1 \sigma_1 \xi_1$ $\qquad \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1$
$\xi_1 \sigma_i = \sigma_i \xi_1 \quad i \neq 1$ $\qquad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Add skein relations:

$\Rightarrow$ relation on $\times$

$MTL = \mathcal{RT}/\text{skein}$,

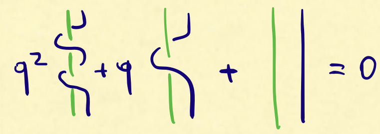


where $a_1 = -(\Omega + \Omega^{-1})$, $\quad a_2 = -q^{-1}((\Omega + \Omega^{-1})^2 + q^{-1})$, $\quad \Omega$ – formal parameter.

② TLB :

1) only allow objects with one | on the left
2) all loops simplified
3) no self-tangles
4) each white strand passes at most once around |

Lemma



③ $V = V(1) = \langle v_{-1}, v_1 \rangle$

$\check{c} : \mathbb{C} \to V \otimes V \qquad 1 \longmapsto c_0 = -q v_1 \otimes v_{-1} + v_{-1} \otimes v_1$

$\hat{c} : V \otimes V \to \mathbb{C} \qquad v \otimes w \longmapsto \langle v, w \rangle$

R-matrix: $\check{R}_{VV} = q + \check{c}\hat{c}$

Assign: $| \mapsto id_V \qquad |\text{(green)} \mapsto id_M \qquad \cup \longmapsto \check{c}\ , \quad \cap \longmapsto \hat{c}\ ,$

$\times \longmapsto q^{\frac{1}{2}} \check{R}_{V,V}\ , \quad \times \longmapsto q^{-\frac{1}{2}} \check{R}_{V,V}^{-1}$

$\times \longmapsto \check{R}_{VM}$ etc ..

restrict to objects in TLB

$$\begin{tikzcd} \text{"RT"} \arrow[r] & \mathcal{O} \\ \text{MTL} \arrow[u, hook] \arrow[ur, "F"'] & \end{tikzcd}$$

F restricts to $F' : \text{TLB} \to T$ ← full subcat of $\mathcal{O}$ containing $M \otimes V^{\otimes r}$

It is an equivalence of cats

$\text{End}_{\text{TLB}}(|\ m) = \text{TLB}_m(q, \Omega)$ ← TL of type B

Till — Chern classes, where are you?

Main actors: $T^*\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$ $\quad T = \underset{\curvearrowright (\mathbb{C}^*)^3}{T_1} \times \underset{\cong \mathbb{C}^*}{T_2}$ $\quad S$ - taut. bundle over $\mathbb{P}^2$

Q - univ. quot. bundle over $\mathbb{P}^2$.

By def., $\overset{\text{quantum cohomology}}{qH_T^*(T^*\mathbb{P}^2)}$ is the $\underbrace{\mathbb{Q}[t_1,t_2,t_3,\hbar]\otimes\mathbb{Q}[[q]]}_{=:R}$-module $H_T^*(T^*\mathbb{P}^2)\otimes\mathbb{Q}[[q]]$ endowed with "quantum product"

Thm (GRTV '13) $\exists!$ iso of R-modules $\psi: R[x, y_1+y_2, y_1y_2]/\langle w_0, w_1, w_2\rangle \xrightarrow{\sim} qH_T^*(T^*\mathbb{P}^2)$ s.t. $x \mapsto c_1(S)$

Here, $w_0, w_1, w_2 \in R[x, y_1, y_2]^{S_2}$ are defined via

$$u^3 + w_2u^2 + w_1u + w_0 = \underbrace{(u-x)(u-y_1)(u-y_2) - (u-t_1)(u-t_2)(u-t_3)}_{\text{"non-quantum" part}} + \frac{q}{1-q}\Big((u-x)(u-y_1)(u-y_2) - (u-y_1+\hbar)(u-y_2+\hbar)(u-x-\hbar)\Big)$$

Question: What are $\psi^{-1}(c_1(Q))$, $\psi^{-1}(c_2(Q))$? How are they related to

$c_1(Q) = t_1+t_2+t_3 - c_1(S)$ $\quad ; \quad w_2 = -x - y_1 - y_2 - \hbar\frac{q}{1-q} + t_1 + t_2 + t_3$

ψ is R-linear $\Rightarrow$ $\psi^{-1}(c_1(Q)) = t_1+t_2+t_3 - \psi^{-1}(c_1(S)) = t_1+t_2+t_3 - x = y_1 + y_2 + \hbar\frac{q}{1-q}$ $\quad$ quantum correction!

$c_2(Q)$ First, we compute $c_1(S) * c_1(S)$

Bases of equivariant cohomology:

1) Fix $\sigma: \mathbb{C}^* \to T_1$, $t \mapsto (t, t^2, t^3)$; $(T^*\mathbb{P}^2)^{T_1} = \{p_1, p_2, p_3\}$; $\begin{array}{l} p_1 = [1:0:0] \\ p_2 = [0:1:0] \\ p_3 = [0:0:1] \end{array}$; $\mathfrak{S} = (\mathrm{Stab}_\sigma(p_1), \mathrm{Stab}_\sigma(p_2), \mathrm{Stab}_\sigma(p_3))$ $\quad$ stable envelope basis

2) Let $L_{23} = \{[0:a:b] : (a,b) \neq 0\}$

$$G_\emptyset = [\pi^{-1}(\mathbb{P}^2)] = [T^*\mathbb{P}^2]$$
$$G_\square = [\pi^{-1}(L_{23})]$$
$$G_{\boxminus} = [\pi^{-1}(p_3)]$$

Note: $G_\square = t_1 - c_1(S)$ $\quad ; \quad c_2(Q) = G_{\boxminus} + t_3 G_\square + t_2t_3 G_\emptyset$

Base change:

Fact
$$\mathrm{Stab}_\sigma(p_1) = (c - t_2 + \hbar)(c - t_3 + \hbar)$$
$$\mathrm{Stab}_\sigma(p_2) = (c - t_1)(c - t_3 + \hbar) \qquad \text{where } c = c_1(S)$$
$$\mathrm{Stab}_\sigma(p_3) = (c - t_1)(c - t_2)$$

$$G_\emptyset = 1$$
$$G_\square = t_1 - c$$
$$G_{\boxminus} = (t_1 - c)(t_2 - c)$$

$\Rightarrow$
$$\mathrm{Stab}_\sigma(p_3) = G_{\boxminus}$$
$$\mathrm{Stab}_\sigma(p_2) = G_{\boxminus} + (t_3 - t_2 - \hbar)G_\square$$
$$\mathrm{Stab}_\sigma(p_1) = G_{\boxminus} + (t_3 - t_1 - 2\hbar)G_\square + \big((t_1-t_2)(t_1-t_3) + \hbar(t_1 - t_2 + t_1 - t_3 + \hbar)\big)G_\emptyset$$

A = bases change matrix

Thm (Maulik-Okounkov) In the basis $\mathfrak{S}$, $c_1(S) * -$ is given by

$$\underbrace{\begin{pmatrix} t_1 & & \\ \hbar & t_2 & \\ \hbar & \hbar & t_3 \end{pmatrix}}_{B} + \hbar\frac{q}{1-q}\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{C}$$

Finally, $c_1(S) * c_1(S) = A\left(B + \hbar\frac{q}{1-q}C\right)A^{-1}\begin{pmatrix} t_1 \\ 1 \\ 0 \end{pmatrix}$. $\quad ACA^{-1} = \begin{pmatrix} 0 & 0 & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \Rightarrow c_1(S) * c_1(S) = c_1(S)c_1(S)$!

Doing further computation, we see that $c_2(Q) \neq y_1y_2$.

Tanguy Specializing quantum cohomology rings

$Fl_\lambda = Gr(2,4) \qquad R = \mathbb{C}[a_1, \dots, a_4, \hbar, \frac{q}{1-q}]$

$QH^\bullet_T(T^*Gr(2,4)) \simeq R\left[\begin{matrix}\gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22}\end{matrix}\right]^{S_2 \times S_2} \Big/ W^q(u) = (1-q)(u-a_1)\dots(u-a_4),$

γ_{11}, γ_{12} Chern roots of $\overset{\text{taut}}{\check{}}V \subset \mathbb{C}^4$

γ_{21}, γ_{22} — " — $\mathbb{C}^4/V$.

Rmk No conflict w/ last talk: the mismatch disappears for $Gr(k, 2k)$!

where $W^q(u) = (u-\gamma_{11})(u-\gamma_{12})(u-\gamma_{21})(u-\gamma_{22}) - q(u-\gamma_{11}-\hbar)(u-\gamma_{12}-\hbar)(u-\gamma_{21}+\hbar)(u-\gamma_{22}+\hbar)$

$QH^\bullet_T(Gr(2,4)) \longrightarrow QH^\bullet_T(Spe)$ ← want to define!

- set $q=1$
- kill $Ann(\hbar)$

Define $QH^\bullet_T(pol) \subseteq QH^\bullet_T(Gr(2,4))$: subalg. generated by
- $H^\bullet_T(Gr(2,4)) \subseteq H^\bullet_T(Gr(2,4)) \otimes \mathbb{C}(q)$
- $\mathbb{C}[q]$

Then, we can set $q=1$.

<u>Coeff of u^3 in $W^q(u)$</u> : $-(\gamma_{11}+\gamma_{12}+\gamma_{21}+\gamma_{22}) + q(\gamma_{11}+\gamma_{12}+\gamma_{21}+\gamma_{22})$

$\Rightarrow (q-1)(\gamma_{11}+\gamma_{12}+\gamma_{21}+\gamma_{22}) = (q-1)(a_1+a_2+a_3+a_4) \quad \Rightarrow \quad \gamma_{11}+\gamma_{12}+\gamma_{21}+\gamma_{22} = a_1+a_2+a_3+a_4$ survives

<u>Coeff of u^2 in $W^q(u)$</u>: $(q-1)(\gamma_{11}\gamma_{12}+\gamma_{11}\gamma_{21}+\gamma_{11}\gamma_{22}+\gamma_{12}\gamma_{21}+\gamma_{12}\gamma_{22}+\gamma_{21}\gamma_{22}) - \hbar(\gamma_{11}+\gamma_{12}-\gamma_{21}-\gamma_{22}) - 2\hbar^2 = (1-q)(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)$

$\Rightarrow \hbar(\gamma_{11}+\gamma_{12}-\gamma_{21}-\gamma_{22}) + 2\hbar^2 = 0$ survives.

Same behaviour happens for other coefficients.

$\rightsquigarrow QH^\bullet_T(pol)\big|_{q=1} = R'\left[\begin{matrix}\sigma_1 & \pi_1 \\ \sigma_2 & \pi_2\end{matrix}\right] \Big/ \begin{matrix} \hbar(\sigma_1\pi_2 - \sigma_2\pi_1) + \hbar^2(\pi_1+\pi_2-\sigma_1\sigma_2) + \hbar^3(\sigma_1-\sigma_2) + \hbar^4 & u^0 \\ \sigma_1\pi_2+\sigma_2\pi_1 - 2\hbar(\pi_1-\pi_2) - \hbar^2(\sigma_1+\sigma_2) & u^1 \\ \hbar(\sigma_1-\sigma_2) + 2\hbar^2 & u^2 \\ \sigma_1+\sigma_2 = a_1+a_2+a_3+a_4 & u^3 \end{matrix}$

$R' = \mathbb{C}[a_1, \dots, a_4, \hbar]$

$\begin{cases} \sigma_i = \gamma_{i1}+\gamma_{i2} \\ \pi_i = \gamma_{i1}\gamma_{i2} \end{cases}$

<u>Kill $Ann(\hbar)$</u> :

$QH^\bullet_T(Spe) = R'\left[\begin{matrix}\sigma_1 & \pi_1 \\ \sigma_2 & \pi_2\end{matrix}\right] \Big/ \begin{matrix} (\sigma_1\pi_2 - \sigma_2\pi_1) + \hbar(\pi_1+\pi_2-\sigma_1\sigma_2) + \hbar^2(\sigma_1-\sigma_2) + \hbar^3 \\ \sigma_1\pi_2+\sigma_2\pi_1 - 2\hbar(\pi_1-\pi_2) - \hbar^2(\sigma_1+\sigma_2) \\ (\sigma_1-\sigma_2) + 2\hbar \\ \sigma_1+\sigma_2 = a_1+a_2+a_3+a_4 \end{matrix} \quad \simeq R'$ ← not the expectation!

Expect $rk_{R'} QH(spe) = \dim IH(\overline{\mathbb{O}}) = 2$, not 1

$N=2$
$n=4$

$V_4 = (\mathbb{C}^2)^{\otimes 4} \underset{\mathbb{C}}{\otimes} R' = \underset{(4,0)}{H^\bullet_T(pt)} \oplus \underset{(3,1)}{H^\bullet_T(T^*\mathbb{P}^3)} \oplus \underset{(2,2)}{H^\bullet_T(T^*Gr(2,4))} \oplus \underset{(1,3)}{H^\bullet_T(T^*\mathbb{P}^3)} \oplus \underset{(0,4)}{H^\bullet_T(\mathbb{P}^4)}$

$IH^\bullet_T(\overline{\mathbb{O}}_{(2,2)}) \overset{?}{\subseteq} H^\bullet_T(T^*Gr(2,4))$

$V_4 = V(\ydiagram{4}) \oplus V(\ydiagram{3,1})^{\oplus 3} \oplus \underbrace{V(\ydiagram{2,2})^{\oplus 2}}_{IH^\bullet_T(\overline{\mathbb{O}}_{(2,2)})}$

<u>Geometric pt of view</u>

$Spec\, QH^\bullet_T(T^*Gr(2,4)) \subseteq B \times \mathbb{A}^4$

$\downarrow$

$B = Spec(R = \mathbb{C}[a_i, \hbar, q, \frac{1}{1-q}]) \subset \overline{B} = Spec(\mathbb{C}[a_i, \hbar, q])$

Then $Spec(QH^\bullet_T(pol)) = \overline{Spec(QH^\bullet_T(T^*Gr(2,4)))}$ ← schematic closure inside $\overline{B} \times \mathbb{A}^4$.

<u>From $T^*Gr(2,4)$ to $Gr(2,4)$</u> :

Shiyu Involution on big algebra

① Setup

G simple, simply connected $/\mathbb{C}$, μ dominant weight, $S = e + C_{\mathfrak{g}}(f)$

$$\begin{array}{l} C^\mu := (S(\mathfrak{g}^*) \otimes \mathrm{End}(V^\mu))^G = \mathrm{Map}(\mathfrak{g}, \mathrm{End}(V^\mu))^G \quad \text{– Kirillov algebra} \\ \cup\!\shortmid \\ B^\mu \\ \cup\!\shortmid \quad \leftarrow \text{finite free of rank } \dim V^\mu \\ \mathbb{C}[\mathfrak{g}]^G \end{array}$$

$$B^\mu \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \otimes V^\mu = \mathrm{Map}(S, V^\mu)$$

$$f \longmapsto (s \in S \longmapsto f(s)(v_{lw}))$$ lowest weight vector in V^μ.

Principal grading on V^μ : $V^\mu = \bigoplus V^\mu_i$ lowest weight $\rightsquigarrow 0$, simple roots $\rightsquigarrow 1$

Poincaré polynomial of V^μ : $\displaystyle\prod_{\alpha \in \Phi^+} \frac{1-q^{\langle \mu+\rho, \alpha^\vee\rangle}}{1-q^{\langle \rho, \alpha^\vee\rangle}} =: D^\mu$

Involution $\iota : B^\mu \to B^\mu$ by $(-1)^{\deg}$ $\subset C^\mu = \mathrm{Map}(\mathfrak{g}, \mathrm{End}\, V^\mu) \circlearrowleft (-1)$

We want to study $(\mathrm{Spec}\, B^\mu)^\iota \longrightarrow (\mathfrak{g}/\!/G)^\iota$

$$\begin{array}{c} \| \\ \mathrm{Spec}\, \underbrace{B^\mu_\iota} \end{array} \leftarrow = B^\mu / \langle \iota(f) - f \rangle$$

<u>Ex</u> $G = SL_4$, ω_1, ω_2

$H^*(Gr)$

<u>ω_1</u> $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[a_2, a_3, a_4]$ $\quad B^{\omega_1} = \mathbb{C}[b_1, c_1, c_2, c_3, a_2, a_3, a_4] / (y+b_1)(y^3+c_1y^2+c_2y+c_3) = y^4 + a_2y^2 + a_3y + a_4$

$$\begin{array}{ccc} \cup\!\shortmid & & \cup\!\shortmid \\ \mathbb{C}[\mathfrak{g}]^G_\iota = \mathbb{C}[a_2, a_4] & \xrightarrow{a_4 = 0.} & B^{\omega_1}_\iota = \mathbb{C}[a_2] \end{array}$$

<u>ω_2</u> $B^{\omega_2} = \mathbb{C}[b_1, b_2, c_1, c_2, a_2, a_3, a_4] / (y^2+b_1y+b_2)(y^2+c_1y+c_2) = y^4 + a_2y^2 + a_3y + a_4$

$\cup\!\shortmid$

$B^{\omega_2}_\iota = \mathbb{C}[b_2, c_2, a_2, a_4] / (y^2+b_2)(y^2+c_2) = y^4 + a_2y^2 + a_4$

$(\mathrm{Spec}\, B^\mu_\iota) \xrightarrow{\pi} (\mathfrak{g}/\!/G)^\iota$

① $a \in \mathfrak{t}^{rs}$ $\rightsquigarrow \#\pi^{-1}(a) = \mathrm{tr}(\iota) = D^\mu(-1)$

② if $w_0(a) = -a$, ι acts on weights by w_0.

② Work of Stembridge on $D^\mu(-1)$

$\Phi_{\rho+\mu} = \{\alpha \in \Phi : \langle \rho+\mu, \alpha^\vee\rangle \in 2\mathbb{Z}\}$ $\qquad \Phi(2) := \Phi_\rho = \{\alpha : \alpha^\vee \text{ has even height}\}$

<u>Thm 1</u> TFAE :

1) $D^\mu(-1) \neq 0$
2) $\#\Phi_{\rho+\mu} = \#\Phi(2)$
3) $\Phi_{\rho+\mu} \simeq \Phi(2)$ as root systems
4) $\exists\, w \in W$ s.t. $w(\rho+\mu) - \rho \in 2\Lambda$ ← weight lattice

<u>Rmk</u> $w(\rho+\mu) - \rho \in 2\Lambda \iff w \cdot \Phi_{\rho+\mu} = \Phi(2)$.

<u>Ex</u> A_{2n-1} : $\Phi(2) = \{e_i - e_j : i,j \text{ odd}\} \cup \{e_i - e_j : i,j \text{ even}\} = A_{n-1} \oplus A_{n-1}$; $\mu \rightsquigarrow \mu + (2n-1, 2n-2, \ldots, 1, 0)$.

<u>Thm 2</u> Assume conditions in thm 1. $\exists\, w \in W$ & dominant wt γ for $\Phi(2)$ s.t.

$$\left\langle \frac{w(\mu+\rho)}{2}, \alpha^\vee \right\rangle = \langle \gamma + \underbrace{\rho(2)}_{\rho \text{ for } \Phi(2)}, \alpha^\vee \rangle \quad \forall \alpha \in \Phi(2)$$

$$D^\mu(-1) = \frac{d(\gamma)}{d(\gamma_0)}$$

$d(\gamma)$ dim of $\Phi(2)$ irrep V^γ

γ_0: γ for $\mu = 0$.

Ex A_{2n-1} $\gamma_0 = 0$

$$\mu = (6,4,2,1,1,0)$$

$$\downarrow \mu + (5,4,3,2,1,0)$$

$$(11,8,5,3,2,0)$$

even $\swarrow$ $\searrow$ odd

$$(8,2,0) \qquad (11,5,3)$$

$$\downarrow -(4,2,0) \qquad \downarrow -(5,3,1)$$

$$(4,0,0) \qquad (6,2,2)$$

$$\downarrow \tfrac{1}{2} \qquad \downarrow \tfrac{1}{2}$$

$$(2,0,0) \qquad (3,1,1)$$

$$D^\mu(-1) = d(\gamma_{even}) \cdot d(\gamma_{odd})$$

Schur function

Rmk $D^\mu(-1) = \pm S_\mu(1,-1,1,-1,\dots)$

$$\pm S_\mu(x_1,-x_1,x_2,-x_2,\dots,x_n,-x_n) = S_{\nu_{even}}(x_1^2\dots x_n^2)\, S_{\nu_{odd}}(x_1^2\dots x_n^2)$$

Upshot: $\iota \curvearrowright$ Higgs ; $(E,\Phi) \longmapsto (E,-\Phi)$

$\mathrm{Higgs}^{\iota}_{GL_{2n}} = \bigcup$ Higgs moduli for real forms of GL_{2n}

$\mathrm{Spec}\, B^\mu \rightsquigarrow \mathrm{Spec}\, B^\mu_\iota$

Upward flows

$\simeq$

Hecke modifications of Hitchin section $\rightsquigarrow$ Hecke mods of Hitchin section in real Hitchin moduli space.

Kamil: Computing the zero schemes

$B \subset G$

Borel, reductive

Vector fields on $\mathbb{P}^n$ $\quad V\setminus\{0\} \xrightarrow{\pi} \mathbb{P}(V)$ $\quad x \in V\setminus\{0\} \rightsquigarrow \pi_x : T_xV \to T_{[x]}\mathbb{P}(V)$, $\ker \pi_* = \mathbb{C}\cdot x$.

Let $v \in T_xV$, $w \in T_{tx}V$ ($t \in \mathbb{C}^*$). Then $\pi_*(v) = \pi_*(w)$ iff $w = tv$.

$$0 \to \mathrm{Hom}(\mathbb{C}x, \mathbb{C}x) \to \mathrm{Hom}(\mathbb{C}x, V) \to T_{\pi(x)}\mathbb{P}(V) \to 0 \quad \rightsquigarrow \quad 0 \to \mathcal{O} \to \mathcal{O}(1)\otimes V \to T\mathbb{P}(V) \to 0$$

Euler sequence on $\mathbb{P}(V)$

$$\rightsquigarrow 0 \to H^0(\mathcal{O}) \to \underbrace{H^0(\mathcal{O}(1))\otimes V}_{\mathrm{Hom}(V,V)} \to \underbrace{H^0(T\mathbb{P}(V))}_{\text{vector}} \to 0 \quad \Rightarrow \text{vector fields on } \mathbb{P}(V) = \mathrm{End}\,V/\mathbb{C}\,\mathrm{Id}.$$

Now $V = \mathbb{C}^{n+1}$, $M \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{C})$ – coordinates of the vector field?

$U_0 \subset \mathbb{P}^{n+1}$, $U_0 = \{x_0 \neq 0\} = \{[1:x_1:\dots:x_n] : x_i \in \mathbb{C}\}$

$$M\begin{pmatrix}1\\x_1\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}y_0\\y_1\\\vdots\\y_n\end{pmatrix} =: \underline{y}$$

Assume $y_0 = 0$, then $\underline{y}$ maps to $(y_1,\dots,y_n) \in T_{[x]}\mathbb{P}^n$

Otherwise, $\pi_*\underline{y} = \pi_*(\underline{y} - y_0(1,x_1,\dots,x_n)) = (0, y_1 - y_0x_1, \dots, y_n - y_0x_n)$

it's in the kernel of π_*!

Ex 1 $\quad GL_{n+1} \curvearrowright \mathbb{P}^n$

Vector field for $B \curvearrowright \mathbb{P}^n$ is $e + t$
$\qquad\qquad\qquad G \curvearrowright \mathbb{P}^n$ $\quad e + c_g(f)$

$$e = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & 1 \\ & & & & 0\end{pmatrix} \quad f = \begin{pmatrix} 0 & & & & \\ n\cdot 1 & 0 & & & \\ & 2(n-2) & 0 & & \\ & & 3(n-3) & \ddots & \\ & & & (1-n) & 0\end{pmatrix} \quad h = \begin{pmatrix} n & & & \\ & n-2 & & \\ & & n-4 & \\ & & & \ddots \\ & & & & -n\end{pmatrix}$$

$$h \in \mathfrak{t} = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \Rightarrow \text{1-dim torus } \begin{pmatrix} t^n & & & \\ & t^{n-2} & & \\ & & \ddots & \\ & & & t^{-n}\end{pmatrix}$$

Rmk The zero schemes for both B & G lie in $S\times U_0$.
$\Rightarrow$ enough to compute in U_0.

$$e\begin{pmatrix}1\\x_1\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\\vdots\\x_n\\0\end{pmatrix} \rightsquigarrow \begin{pmatrix}0\\x_2 - x_1x_1\\x_3 - x_1x_2\\\vdots\\x_n - x_1x_{n-1}\\-x_1x_n\end{pmatrix} \Rightarrow \mathbb{C}[Z] = \mathbb{C}[U_0]/_{x_2 - x_1^2,\ x_3 - x_1x_2,\ \dots,\ x_n - x_1x_{n-1},\ -x_1x_n}$$

$$= \mathbb{C}[x_1]/_{x_1^{n+1}} = H^*(\mathbb{P}^n, \mathbb{C})$$

Thm (C-L, '70s) $\quad V$ v. field on a smooth proj. variety X, $Z \subset X$ zero scheme. s.t. $\dim Z = 0$.
Then $\exists$ $F_\cdot$ ascending on $\mathbb{C}[Z]$ s.t. $H^*(X,\mathbb{C}) \simeq \mathrm{Gr}_F(\mathbb{C}[Z])$

Rmk Kumar–Kaveh computed this for toric varieties.

Rmk The proof uses the spectral sequence $E^1_{pq} = H^q(X, \Omega^p) \Rightarrow \mathrm{gr}\, H^{p+q}(Z, \mathcal{O}_Z) \overset{\dim Z = 0!}{=} \mathrm{gr}\,\mathbb{C}[Z]$.

Thm (Akyildiz–Carrell, '80s) Assumptions as before $+ \mathbb{C}^* \curvearrowright X$ s.t. $t_*V = t^kV$ for some $k \neq 0$
Then $\mathbb{C}^*$ preserves Z, the weights on $\mathbb{C}[Z]$ are divisible by k, and $F_i = \bigoplus_{j \leq i} A_{kj}$; $\mathbb{C}[Z] = \bigoplus_{i \geq 0} A_{ki}$

Let's move to the equivariant setting.

*) $B(SL_2) \curvearrowright \mathbb{P}^n$ $\qquad S = e + \mathfrak{t} = \{e + vh : h \in \mathbb{C}\}$ $\quad Z \subset S \times \mathbb{P}^1$

$$\begin{pmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & -n \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} n \\ (n-2)x_1 \\ \vdots \\ -n x_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ x_2 - (2v + x_1)x_1 \\ x_3 - (4v + x_1)x_2 \\ \vdots \\ x_n - (2(n-1)v + x_1)x_{n-1} \\ -(2nv + x_1)x_n \end{pmatrix} \Rightarrow$$

$$\mathbb{C}[Z] = \mathbb{C}[U_0] \big/ x_2 - (2v+x_1)x_1,\ x_3 - (4v+x_1)x_2,\ \dots\ x_n - (2(n-1)v + x_1)x_{n-1},\ -(2nv+x_1)x_n$$
$$= \mathbb{C}[x_1, v] \big/ x_1(x_1+2v)(x_1+4v)\dots(x_1+2nv)$$
$$\cong H^*_B(\mathbb{P}^n, \mathbb{C})$$

*) $B(GL_{n+1}) \curvearrowright \mathbb{P}^n$ $\qquad S = e + \mathfrak{t} = \begin{pmatrix} v_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & v_n \end{pmatrix}$

A similar computation gives $\mathbb{C}[Z] = \mathbb{C}[U_0] \big/ x_2 - (x_1 - (v_1 - v_0))x_1,\ x_3 - (x_1 - (v_2 - v_0))x_2, \dots, -x_n(x_1 - (v_n - v_0))$
$$= \mathbb{C}[x_1, v_0 \dots v_n] \big/ x_1(x_1 - (v_1 - v_0))(x_1 - (v_2 - v_0)) \dots (x_1 - (v_n - v_0))$$
$$= [e = x_1 + v_0] = \mathbb{C}[e, v_0 \dots v_n] \Big/ \prod_{i=0}^{n} (e - v_i).$$

*) $SL_2 \curvearrowright \mathbb{P}^n$ $\qquad S = e + C_{\mathfrak{sl}_2}(f) = \{e + wf \mid w \in \mathbb{C}\}$

$$e\big|_{[1:x_1:\dots:x_n]} = (x_2 - x_1^2,\ x_3 - x_1x_2,\ \dots,\ x_n - x_1x_{n-1},\ -x_1x_n)$$

Let $\underline{n=3}$. $\quad f = \begin{pmatrix} 0 & & & \\ 3 & 0 & & \\ & 4 & 0 & \\ & & 3 & 0 \end{pmatrix}$; $\quad \begin{pmatrix} 0 & & & \\ 3 & 0 & & \\ & 4 & 0 & \\ & & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4x_1 \\ 3x_2 \end{pmatrix}$

$$\rightsquigarrow \quad e + wf\big|_{[1:x_1:\dots:x_n]} = (x_2 - x_1^2 + 3w,\ x_3 - (x_2 - 4w)x_1,\ -x_1x_3 + 3wx_2)$$

$$\begin{cases} x_2 = x_1^2 - 3w \\ x_3 = (x_2 - 4w)x_1 = x_1^3 - 7wx_1 \\ 0 = 3wx_2 - x_1x_3 = 3wx_1^2 - 9w^2 - x_1^4 + 7wx_1^2 = -x_1^4 + 10wx_1^2 - 9w^2 = -(x_1^2 - w)(x_1^2 - 9w) \end{cases}$$

if you add "loop rotation", generically parabolas intersect in different point $\Rightarrow$ get idempotents?

$$\Rightarrow \mathbb{C}[Z] = \mathbb{C}[x_1, w] \big/ (x_1^2 - w)(x_1^2 - 9w)$$
$$= \mathcal{C}^3_{SL_2} \quad \text{(Kirillov algebra)}$$

$\rightsquigarrow$ 

$\underline{n=4}$ $\quad f = \begin{pmatrix} 0 & & & & \\ 4 & 0 & & & \\ & 6 & 0 & & \\ & & 6 & 0 & \\ & & & 4 & 0 \end{pmatrix} \Rightarrow \quad e + wf = (x_2 - x_1^2 + 4w,\ x_3 - x_1x_2 + 6wx_1,\ x_4 - x_1x_3 + 6wx_2,\ -x_1x_4 + 4wx_3)$

$$\begin{cases} x_2 = x_1^2 - 4w \\ x_3 = x_1x_2 - 6wx_1 = x_1^3 - 10wx_1 \\ x_4 = x_1x_3 - 6wx_2 = x_1^4 - 10wx_1^2 - 6wx_1^2 + 24w^2 = x_1^4 - 16wx_1^2 + 24w^2 \\ 0 = 4wx_3 - x_1x_4 = 4wx_1^3 - 40w^2x_1 - x_1^5 + 16wx_1^3 - 24w^2x_1 = -x_1^5 + 20wx_1^3 - 64w^2x_1 = -x_1(x_1^2 - 4w)(x_1^2 - 16w) \end{cases}$$

$$\Rightarrow \mathbb{C}[Z] = \mathbb{C}[x_1, w] \big/ x_1(x_1^2 - 4w)(x_1^2 - 16w)$$