STABLE ENVELOPES READING GROUP

The following plan is for 12 talks, which means that the reading group will run roughly from the end of January to the end of April.

The goal of the reading group is to get some working knowledge of stable envelopes. We are interested the most in Maulik-Okounkov Yangians, and its natural BPS Lie subalgebra. As a consequence, we will skip everything related to quantum cohomology, and instead try to have a look at more recent articles, dealing with various generalizations of stable envelopes.

1. **Intro.** Go through Maulik-Okounkov's motivation in the introduction of [MO19]. Expain what we want to achieve with this reading group (many examples, with a view towards non-quiver examples).

2. **Localization theorem.** Definition of equivariant cohomology. Formality. First localization theorem and Atiyah-Bott formula, with proofs. Examples: integration over $\mathbb{P}^1$, $\mathbb{P}^n$. Non-compact version. Explain that integrals over compact classes don't have poles. Second localization theorem. Example: Grassmanians.
References: [And12, AF].

3. **Stable envelopes.** General setup, pay very close attention to axioms (replace by ones in [Oko21a]?). Attracting manifolds, order depending on chamber. Define stable envelopes (Theorem 3.3.4). Discuss the example of $T^*\mathbb{P}^1$ in detail, compute R-matrix. Sketch the proof of uniqueness.
Reference: [MO19, Chapter 3].

4. **Stable envelopes continued.** Introduce Lagrangian correspondences and Lagrangian cycles. Prove the existence of stable envelopes. Discuss the special case of symplectic resolutions, Steinberg correspondences. Show that in this case the stable envelope is the specialization of the attractor (Theorem 3.7.4). Discuss the example of $T^*\mathbb{P}^1$. If time permits, talk about stable envelopes for Springer resolution [BMO10].
Reference: [MO19, Chapter 3].

5. **Quantum groups.** Explain the definition of $U_q(\mathfrak{g})$. Introduce R-matrices, Yang–Baxter equation. Explain $RTT = TTR$ construction. Perform it explicitly for $U_q(\mathfrak{gl}_2)$.
Reference: [Kas95, Chapter 8].

6. **Yangians.** Introduce the Yangian $Y(\mathfrak{gl}_n)$, or more generally $Y(\mathfrak{g})$. Talk about different realizations of $Y(\mathfrak{gl}_n)$ (RTT, Drinfeld's new, via adjoint representation). Study its basic properties (deforls $U(\mathfrak{g}[u])$, has a family of automorphisms). Possibly say something about finite dimensional representations.
References: [Mol02, Section 2], [CP, Chapter 12].

7. **Nakajima quiver varieties.** Discuss the general construction. Show how various examples, e.g. $T^*\mathbb{P}^n$, $T^*\mathrm{Gr}$, cotangents to flag varieties, resolutions of ADE singularities etc (not necessarily all of them) arise as quiver varieties. Another example: Hilbert scheme of points on $\mathbb{C}^2$, ADHM description [Nak99]. Describe torus fixed points, in general (Proposition 2.3.1, Section 2.4.1) and in various examples. If time permits, talk about moduli of framed sheaves $\mathcal{M}(r, n)$.
References: [MO19, Chapter 2], [Oko17, Section 4], [Gin09].

8. **MO Yangian for Grassmanians.** Work out the Yangian in detail for $Q = \mathrm{pt}$, that is for cotangents to Grassmanians.
Reference: [MO19, Chapter 11].

Date: December 30, 2021.

9. **MO Yangian in general.** Introduce R-matrices. For Nakajima varieties, all R-matrices are compositions of root R-matrices. Define Yangian. RTT relations. Yangian acts on the cohomology of quiver varieties. Lie algebra $\mathfrak{g}_Q$. Structure of the Yangian (Theorem 5.5.1).
Reference: [MO19, Chapters 4, 5].

10. **MO Yangian for Hilbert schemes.** Recall Hilbert schemes, moduli spaces $\mathcal{M}(r, n)$, ADHM construction. Identify $\mathfrak{g}_Q$ with Heisenberg algebra. Do a quick refresher on symmetric functions vs torus fixed points in the Hilbert scheme. Describe the stable basis.
Reference: [MO19, Chapters 12, 17, 18], [She13, Chapter 6] and references therein.

11. **Generalizations.** Explain K-theoretic stable envelope, in particular how it acquires a dependence on slope parameter [Oko17, Section 9]. After this, either talk about elliptic version, and how dependence on slope becomes meromorphic there [AO21], or about "non-abelian" stable envelopes (K-theoretic version) and their relation to "interpolation problems" for GIT quotients [Oko21b].

12. **Relation to CoHas.** Introduce the Yangians coming from CoHAs, and their relation to Maulik–Okounkov version. Possibly, say something about the case of moduli of framed sheaves on more general surfaces.

References

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