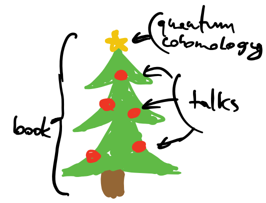
Introduction

In this talk I will tell you about some of the motivations of MO, and explain what await us ahead.



§1 | Quantum cohomology

Let us start by sketching one of the motivations.

Let X be a smooth projective variety, $H^*(X)$ its cohomology ring. H turns out cup product admits a (formal) family of associative (super)commutative deformations.

Def An *n-pointed stable map* of genus 0 is a connected marked curve $(C, p_1, .., p_n)$ + $f: C \to X$ satisfying:

1) C only has double point singularities;
2) $p_1, .., p_n$ are distinct, smooth
3) if $C_i \subset C$ irr. component, $C_i \cong \mathbb{P}^1$, f constant on C_i $\Rightarrow$ C_i contains 3 special (nodal/marked) points
4) arithmetic genus of C is 0.

Def Let $\beta \in H_2(X, \mathbb{Z})$. $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is the moduli (DM) stack of n-pointed stable maps of genus 0 & $[C] = \beta$.

Thm $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is proper.

Note that we have evaluation maps $p_i : \overline{\mathcal{M}}_{0,n}(X, \beta) \to X$. For any $\gamma_1, .., \gamma_n \in H^*(X)$, define

$$\langle \gamma_1, \dots, \gamma_n \rangle_\beta := \int_{\overline{\mathcal{M}}_{0,n}(X,\beta)} p_1^*(\gamma_1) \cup \dots \cup p_n^*(\gamma_n)$$

need to define "virtual fundamental class" in order to make sense of this.

Def The (small) quantum product $*$ on $H^*(X)$ is given by:

$$(\gamma_1 * \gamma_2, \gamma_3) := \sum_{\beta \geq 0} q^\beta \langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta \quad , \quad \text{where } (-,-) \text{ is the Poincaré pairing}$$

q^β = formal expression, can be evaluated against $\omega \in H^2(X, \mathbb{C})$.

Prop This product is associative, supercommutative, unital, and recovers the usual product when $q = 0$.

Rmk $\gamma * -$, $\gamma \in H^*(X)$ is a maximal commutative subalgebra of $\mathrm{End}(H^*(X))$.

Example For $X = \mathbb{P}^n$, we have $H^*(\mathbb{P}^n) = \mathbb{C}[u]/_{u^{n+1}}$, $(H^*(\mathbb{P}^n), *) = \mathbb{C}[u,q]/_{u^{n+1} - q}$.

Taking Spec, it looks like

Rmk If X is quasi-projective, then the definition of quantum cohomology doesn't make sense, since $\overline{\mathcal{M}}_{0,n}(X, \beta)$ fails to be proper. However, if $T \curvearrowright X$, T alg. torus, with X^T proper, we can define equivariant version of $\langle\rangle$ via localization (2, Lucien) and thus quantize cup product on $H^*_T(X)$.

For general X, quantum cohomology ring is not known. However, for some varieties it can be approached via rep. th. methods.

§2 | Yangians & Baxter subalgebras

Let $\{F_i\}_{i \in I}$ be a collection of vector spaces. An *R-matrix* is a collection of operator valued functions $R_{F_i, F_j}(u) \in \mathrm{End}(F_i \otimes F_j)$, satisfying Yang-Baxter equation: $R_{12}(u) R_{13}(u+v) R_{23}(v) = R_{23}(v) R_{13}(u+v) R_{12}(u)$.

For any $m \in \mathrm{End}\, F$, $W \in \{F_i\}$, consider $T_F(m, u) = \mathrm{tr}_F\big((m \otimes 1) R_{F,W}(u)\big) \in \mathrm{End}\, W$.

The algebra generated by those operators inside $\bigoplus \mathrm{End}\, W$ is called *Yangian* (this is Faddeev-Reshetikhin-Takhtajan formalism, see 5. Léa, 6. Kostya), denoted Y.

Further, let $\mathcal{G} \subset \prod GL(F_i)$ be the centralizer of all R-matrices. Take $g \in \mathcal{G}$. Then $[T_{F_1}(g, u_1), T_{F_2}(g, u_2)] = 0$.

Indeed, $R_{12}\big((g \otimes g \otimes 1) R_{13} R_{23}\big) R_{12}^{-1} \simeq (g \otimes g \otimes 1) R_{12} R_{13} R_{23} R_{12}^{-1} = (g \otimes g \otimes 1) R_{23} R_{13}$ $\Rightarrow$ have equality after taking traces.

Thus for each $g\in\mathfrak{G}$, we get a huge commutative subalgebra of the Yangian Y. It is called Baxter/Bethe subalgebra.

The main idea in MO is that if we find nice R-matrix on $H^*(X)$, then Baxter subalgebras $\longleftrightarrow$ quantum product subalgebras.

§3 | Stable envelopes

Let X be a non-singular symplectic G-variety (G reductive group), together with a proper map $X\to X_0$ to an affine variety.
Let $A\subset T\subset G$ be tori, s.t. A fixes symplectic form, and X is a formal T-variety.

Under such hypotheses, MO define stable envelope maps (3. Sebastian) $\mathrm{Stab}_{\mathfrak{C}}: H^*_T(X^A)\to H^*_T(X)$, where $\mathfrak{C}$ is a chamber in $\mathfrak{a}_{\mathbb{R}}$. needed for positivity

Very roughly speaking, each class on a conn. component C of fixed locus gets sent to a class on the full attracting locus $Attr(C)$, which is just the minimal closed subvariety in X, containing C & closed under taking limits $a\to 0$, a 1-param sg in $\mathfrak{C}$.

Ex $X=T^*\mathbb{P}^1$, $A=\mathbb{C}^*$. Then $X^A=\{0,\infty\}$, chambers are $\{t>0\},\{t<0\}$ $t>0$: $Attr(\infty)=T^*_\infty$, $Attr(0)=\mathbb{P}^1\cup T^*_\infty$.
$t<0$: $Attr(0)=T^*_0$, $Attr(\infty)=\mathbb{P}^1\cup T^*_0$.

(8. Shivang)

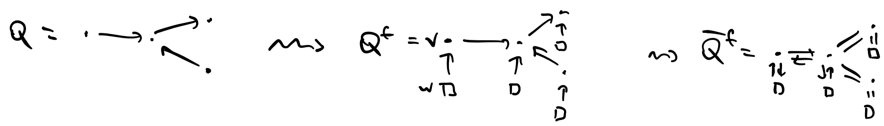
Definition $R_{\mathfrak{C},\mathfrak{C}'}(u) := \mathrm{Stab}_{\mathfrak{C}'}^{-1}\circ \mathrm{Stab}_{\mathfrak{C}}$.

As before, the collection of these R-matrices produces a Yangian, acting on $H^*_G(X)$. (9. Sasha S)

Conj Baxter algebra of this Yangian = quantum product algebra.

§4 | Quiver varieties

An example of varieties to which this theory can be applied is Nakajima quiver varieties. (7. Samnas)



$M(v,w)$ = stable points in $T^*\mathrm{Rep}\,Q^f \simeq$ some reps of $\bar{Q}^f$.

Rmk QH algebras are param by $H^2(X)$, Baxter by $\mathfrak{G}/Z(\mathfrak{G})$. Need to match up these params first.

I'm not sure if conj. above is supposed to hold in the generality above; in MO, they only consider quiver varieties.

They break the conjecture in two parts: 1) show equality for elements of degree 2
2) show that quantum cohomology is generated by elts of degree 2.

Part 2 is basically a computation, which MO do for $Q = \circlearrowleft\!\cdot$. Here, $M(v,w)$ become moduli of torsion-free sheaves on $\mathbb{P}^2$, and Yangian is related to Heisenberg Lie algebra. (talk 10)

Part 1 is more involved. Recall that quantum product lives in a formal family. It turns out that in many cases (conjecturally, for all symplectic resolutions) the series defining quantum product is actually a rational function.
Then we have an honest family of products over an open $U\subset H^2(X,\mathbb{C})$. $\frac{d}{d\lambda}q^p = (\lambda,p)q^p$
Next, $*$ can be used to produce a flat connection on $U\times H^*_G(X)$ by $\nabla_\lambda = \frac{d}{d\lambda} - \lambda *-$.
The proof of part 1 basically reduces to matching this connection with "trigonometric Casimir connection". proof goes by constructing a bunch of commuting connections.

Now, having a connection, we can ask about its monodromy, i.e. corr. representation of $\pi_1(U)$. In various 'classical' situations (i.e. Q Dynkin or affine), this gives rise to a representation of generalized braid group/affine Hecke algebra etc.

Passing from cohomology to K-theory, elliptic cohomology, category of coherent sheaves, we can hope to obtain modules of (double) Hecke alg, or, categorically, action of braid group on $D^bCoh(X/G)$. This circle of ideas might be touched upon in (11. Pavel)

§5 | CoHAs

In this reading group, we will not talk about quantum cohomology, and will instead care about the Yangian.

More classically, it is known that $\bigoplus_v H^*(M(v,w))$ admits an action of the Kac-Moody algebra $\mathfrak{g}_Q$ by correspondences

This action is contained in the MO Yangian. More precisely, all R-matrices they consider are of the form

$$R(u) = 1 + \frac{\hbar}{u} r + O(u^{-2})$$ ← weight of ω under T

, and matrix elements of r's span a Lie algebra $\tilde{\mathfrak{g}}_Q \subset Y_Q$.

This algebra contains $\mathfrak{g}_Q$, and moreover there is a filtration of Y_Q such that $\mathrm{gr}\, Y_Q \simeq U(\tilde{\mathfrak{g}}_Q[t])$.

$\tilde{\mathfrak{g}}_Q$ has a nice triangular decomposition, and its action on $H^*(M(w))$ is also given by certain correspondences. (4, Dongel)

On the other hand, there is a similar extension of the action $\mathfrak{g} \curvearrowright H^*(M(w))$ by means of cohomological Hall algebras

The difference is small: for MO, the correspondences live in $M(v+\alpha,w) \times M(v,w) \times M(\alpha,w_0)$,

for CoHAs, in $M(v+\alpha,w) \times M(v,w) \times \mathrm{Rep}\,\Pi_Q(\alpha)$ ← $w=0$, forget stability

It is yet unclear if the two algebras one obtains are the same. (talk 12)

The two points of view have different advantages:

- → for MO Yangians, the existence of $\tilde{\mathfrak{g}}_Q$ is more or less an automatic property, while for CoHAs it's a nontrivial theorem. Moreover, for CoHAs $\tilde{\mathfrak{g}}_Q$ is not explicit at all;
- → CoHA construction can be generalized to non-symplectic situation (moduli of sheaves on surfaces).

My personal motivation to study MO is the hope to be able to extend their methods to compute BPS Lie algebra for the moduli of Higgs bundles.