STABLE ENVELOPES: GENERALIZATIONS

PAVEL SAFRONOV

1. Assumptions

We work over the complex numbers. Let A be a torus and $\mathsf{T} = \mathsf{A} \times \mathbf{C}^\times$. Let $\Lambda_{\mathsf{A}} = \mathrm{Hom}(\mathsf{A}, \mathbf{C}^\times)$ be the weight lattice of A. Let $\mathfrak{a}_{\mathbf{R}} = \mathrm{Hom}(\mathbf{C}^\times, \mathsf{A}) \otimes_{\mathbf{Z}} \mathbf{R}$ be the real locus of the Lie algebra of A.

Let X be a smooth quasi-projective variety with a T-action.

Definition 1.1. An ***equivariant root*** $\alpha \in \Lambda_{\mathsf{A}}$ is a weight for the A-action of the normal bundle of the fixed point locus $X^{\mathsf{A}} \subset X$. A ***root hyperplane*** is the hyperplane $\{\alpha = 0\} \subset \mathfrak{a}_{\mathbf{R}}$. A ***chamber*** $\mathfrak{C} \subset \mathfrak{a}_{\mathbf{R}}$ is a connected component of the complement of root hyperplanes.

Definition 1.2. Let $\mathfrak{C}$ be a chamber. Given a subvariety $Y \subset X^{\mathsf{A}}$ the ***attracting variety of*** Y $\mathrm{Attr}_{\mathfrak{C}}(Y) \subset X$ consists of all points $x \in X$ such that the limit $\lim_{z\to 0} \sigma(z) \cdot x \in Y$ exists for some (equivalently, all) cocharacters $\sigma \in \mathfrak{C}$. The ***attracting variety*** $\mathrm{Attr}_{\mathfrak{C}} \subset X^{\mathsf{A}} \times X$ is the subset of pairs (y, x), such that $\lim_{x\to 0} \sigma(z) \cdot x = y$.

We have a correspondence

$$\begin{array}{ccccc} & & \mathrm{Attr} & & \\ & \swarrow & & \searrow & \\ X^{\mathsf{A}} & & & & X \end{array}$$

Let

$$X^{\mathsf{A}} = \coprod_i F_i$$

be the decomposition of the fixed point locus into connected components. Each F_i is a smooth variety. There is a partial order on the set of components of X^{A} defined as follows.

Definition 1.3. Let $\mathfrak{C}$ be a chamber. $F_i \geq F_j$ if $F_j \subset \overline{\mathrm{Attr}_{\mathfrak{C}}(F_i)}$. The ***full attracting variety*** is

$$\mathrm{Attr}^f_{\mathfrak{C}}(F_i) = \coprod_{F_j \leq F_i} \mathrm{Attr}_{\mathfrak{C}}(F_j).$$

We will assume the following on the above data.

Assumption 1.4. *For any connected component $F_i \subset X^{\mathsf{A}}$ the full attracting variety $\mathrm{Attr}^f_{\mathfrak{C}}(F_i) \subset X$ is closed.*

Remark 1.5. If X is an A-equivariant symplectic resolution, then the assumption is satisfied, see [MO19, Lemma 3.2.7].

The following is our main example.

Example 1.6. Consider $\mathsf{A} = \mathbf{C}^\times \times \mathbf{C}^\times$ acting on $\mathbf{P}^1$ via $(t_1, t_2) \cdot [x : y] = [t_1 x : t_2 y]$. Consider its canonical extension to a symplectic A-action on $X = \mathrm{T}^*\mathbf{P}^1$. Consider the $\mathbf{C}^\times$-action on X given by scaling the fibers so that the symplectic structure has weight 2. This defines the $\mathsf{T} = \mathsf{A} \times \mathbf{C}^\times$-action on X.

The fixed point locus X^{A} is $\{0, \infty\}$. The two equivariant roots are $(t_1, t_2) \mapsto (t_1/t_2)^{\pm 1}$. Let $\mathfrak{C}_+$ be the chamber containing the cocharacter $t \mapsto (t, 1)$ and $\mathfrak{C}_-$ the opposite chamber.

From now on the chamber $\mathfrak{C}_+$ will be implicit. The attracting varieties are

$$\mathrm{Attr}(\infty) = \mathrm{T}^*_\infty \mathbf{P}^1, \qquad \mathrm{Attr}(0) = \mathbf{P}^1 \setminus \{\infty\}.$$

As $\{\infty\} \in \overline{\mathrm{Attr}(0)}$, the partial order is $\{0\} > \{\infty\}$.

2. Cohomological stable envelopes

Fix the data $(\mathsf{T}, X, \mathfrak{C})$ as before. The (cohomological) stable envelope will be a certain map

$$\mathrm{Stab}\colon \mathrm{H}^\bullet_\mathsf{T}(X^\mathsf{A}) \longrightarrow \mathrm{H}^\bullet_\mathsf{T}(X)$$

of $\mathrm{H}^\bullet_\mathsf{T}(\mathrm{pt})$-modules. To define it, we need extra data. For an A-equivariant vector bundle $V \to X$ we denote by $e^\mathsf{A}(V) \in \mathrm{H}^{2\dim(V)}_\mathsf{A}(X)$ the equivariant Euler class (equal to the equivariant top Chern class of V).

For a connected component $F_i \subset X^\mathsf{A}$ we have a decomposition

$$\mathrm{T}_X|_{F_i} = \mathrm{T}_{F_i} \oplus \mathrm{N}_+ \oplus \mathrm{N}_-,$$

where T_{F_i} is fixed by A, N_+ has positive and N_- has negative A-weights (with respect to the chamber $\mathfrak{C}$).

To define stable bases, one has to fix some signs. It is useful to assume one has a polarization of X. In the following we consider T-equivariant K-theory of X and denote $\mathrm{K}_{\mathbf{C}^\times}(\mathrm{pt}) = \mathbf{C}[q^{1/2}, q^{-1/2}]$.

Definition 2.1. A ***polarization*** of X is a choice of a virtual bundle $\mathrm{T}^{1/2}_X \in \mathrm{K}_\mathsf{T}(X)$ with an equality $\mathrm{T}^{1/2}_X + q^{-1}(\mathrm{T}^{1/2}_X)^* = \mathrm{T}_X$ in $\mathrm{K}_\mathsf{T}(X)$.

Proposition 2.2. *Suppose X is a symplectic variety with a T-action so that $\mathbf{C}^\times \subset \mathsf{T}$ scales the symplectic structure with weight 2. If $\mathrm{T}^{1/2}_X$ is a Lagrangian subbundle in T_X, then it defines a polarization of X.*

Proof. We have an exact sequence of T-equivariant vector bundles

$$0 \longrightarrow \mathrm{T}^{1/2}_X \longrightarrow \mathrm{T}_X \longrightarrow q^{-1}(\mathrm{T}^{1/2}_X)^* \longrightarrow 0,$$

where we have used that the symplectic structure has weight 2 to identify $q(\mathrm{T}_X/\mathrm{T}^{1/2}_X) \cong (\mathrm{T}^{1/2}_X)^*$. In K-theory this gives the equality

$$\mathrm{T}_X = \mathrm{T}^{1/2}_X + q^{-1}(\mathrm{T}^{1/2}_X)^* \in \mathrm{K}_\mathsf{T}(X).$$

□

Example 2.3. Suppose Y is a smooth variety with an A-action and consider the induced symplectic A-action on $X = \mathrm{T}^*Y$. Extend it to a T-action so that $\mathbf{C}^\times \subset \mathsf{T}$ acts on fibers with weight 2. Then the vertical tangent bundle of $X \to Y$ defines a Lagrangian subbundle and hence a polarization.

Given a polarization of X, we get a polarization of each component $F_i \subset X^\mathsf{A}$, i.e. we have a splitting

$$\mathrm{N}_{F_i} = \mathrm{N}^{1/2} + q^{-1}(\mathrm{N}^{1/2})^* \in \mathrm{K}_\mathsf{T}(F_i).$$

An element of $\mathrm{H}^\bullet_\mathsf{A}(\mathrm{pt}) \cong \mathbf{C}[\mathfrak{a}]$ is a polynomial and we denote by $\deg_\mathsf{A}$ its degree. Since A acts trivially on X^A we have

$$\mathrm{H}^\bullet_\mathsf{T}(X^\mathsf{A}) \cong \mathrm{H}^\bullet_{\mathbf{C}^\times}(X^\mathsf{A}) \otimes \mathbf{C}[\mathfrak{a}]$$

and so the degree $\deg_\mathsf{A}$ of elements of $\mathrm{H}^\bullet_\mathsf{T}(X^\mathsf{A})$ is also well-defined.

Example 2.4. Consider the A-equivariant Euler class $e^\mathsf{A}(\mathrm{N}_-)$ of the negative normal bundle N_- to F_i. We have

$$\deg_\mathsf{A} e^\mathsf{A}(\mathrm{N}_-) = \dim \mathrm{N}_- = \mathrm{codim} F_i/2.$$

Theorem 2.5. *Assume that $\dim(X^\mathsf{A}) = 0$ for simplicity. There is a unique map*

$$\mathrm{Stab}\colon \mathrm{H}^\bullet_\mathsf{T}(X^\mathsf{A}) \longrightarrow \mathrm{H}^\bullet_\mathsf{T}(X)$$

of $\mathrm{H}^\bullet_\mathsf{T}(\mathrm{pt})$-modules, such that for any component $F_i \subset X^\mathsf{A}$ the stable envelope $\mathrm{Stab}(i) = \mathrm{Stab}(1_{F_i})$ satisfies

1. $\mathrm{supp}\,\mathrm{Stab}(i) \subset \mathrm{Attr}^f(F_i)$,
2. $\mathrm{Stab}(i)|_{F_i} = (-1)^{\dim \mathrm{N}^{1/2}_+} e^\mathsf{T}(\mathrm{N}_-) \cdot \gamma$,
3. $\deg_\mathsf{A}(\mathrm{Stab}(i)|_{F_j}) < \mathrm{codim} F_j/2 = \deg_\mathsf{A}(\mathrm{Stab}(j)|_{F_j})$ *for any $F_j < F_i$.*

Uniqueness is shown in [MO19, Theorem 3.3.4] and existence (under the more general assumptions stated here) is shown in [Oko21].

Example 2.6. Consider $X = \mathrm{T}^*\mathbf{P}^1$ and the T-action on X from example 1.6. Fix the chamber $\mathfrak{C}_+$ and the polarization from example 2.3.

We claim that

$$\mathrm{Stab}(0) = [\mathbf{P}^1] + [\mathrm{T}^*_\infty \mathbf{P}^1], \qquad \mathrm{Stab}(\infty) = -[\mathrm{T}^*_\infty \mathbf{P}^1]$$

satisfy the axioms.

Indeed, the support axiom is obvious. Next, identify

$$\mathrm{H}^\bullet_{\mathsf{T}}(\mathrm{pt}) = \mathbf{C}[u_1, u_2, \hbar].$$

Then

$$\begin{aligned}
[\mathbf{P}^1]|_0 &= u_2 - u_1 - \hbar \\
[\mathbf{P}^1]|_\infty &= u_1 - u_2 - \hbar \\
[\mathrm{T}^*_\infty \mathbf{P}^1]|_0 &= 0 \\
[\mathrm{T}^*_\infty \mathbf{P}^1]|_\infty &= u_2 - u_1,
\end{aligned}$$

which can be computed from the excess intersection formula. E.g. $[\mathbf{P}^1]|_0 = e^{\mathsf{T}}(\mathrm{T}^*_0\mathbf{P}^1)$ and $[\mathrm{T}^*_\infty \mathbf{P}^1]|_\infty = e^{\mathsf{T}}(\mathrm{T}_\infty \mathbf{P}^1)$.

We have

$$e^{\mathsf{T}}(\mathrm{N}_-) = u_2 - u_1 - \hbar, \qquad \mathrm{N}_+^{1/2} = 0$$

at $\{0\}$ and

$$e^{\mathsf{T}}(\mathrm{N}_-) = u_2 - u_1, \qquad \mathrm{N}_+^{1/2} = \mathrm{T}^*_\infty \mathbf{P}^1$$

at $\{\infty\}$ which proves the second axiom.

Finally, we have

$$\mathrm{Stab}(0)|_\infty = u_1 - u_2 - \hbar + u_2 - u_1 = -\hbar$$

which has A-degree $0 < (\mathrm{codim}\{0\})/2 = 1$, which verifies the last axiom.

The pullback along the zero section induces an isomorphism

$$\mathrm{H}^\bullet_{\mathsf{T}}(\mathrm{T}^*\mathbf{P}^1) \cong \mathbf{C}[x, u_1, u_2, \hbar]/(x + u_1)(x + u_2).$$

The restriction to 0 is $x \mapsto -u_2$ and the restriction to ∞ is $x \mapsto -u_1$. Under this isomorphism $[\mathrm{T}^*_\infty \mathbf{P}^1]$ goes to $x + u_2$ and the stable envelopes are

$$\mathrm{Stab}(0) = -u_1 - x - \hbar, \qquad \mathrm{Stab}(\infty) = -u_2 - x.$$

3. General stable envelopes

For a smooth complex variety X with an action of a torus T we can consider the following generalized cohomology theories:

- Equivariant cohomology $\mathrm{H}^\bullet_{\mathsf{T}}(X)$.
- Equivariant K-theory $\mathrm{K}_{\mathsf{T}}(X)$.
- Equivariant elliptic cohomology $\mathrm{Ell}_{\mathsf{T}}(X)$ which depends on an elliptic curve E.

Correspondingly, there are stable envelope constructions in all of these three settings: the K-theoretic stable envelopes are defined in [Oko17] and the elliptic stable envelopes are defined in [AO21].

In the case of Nakajima quiver varieties the stable envelopes allow one to construct R-matrices for the following quantum groups:

- Yangian $Y(\mathfrak{g})$ from cohomological stable envelopes.
- Quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ from K-theoretic stable envelopes.
- Elliptic quantum groups $E_{\tau,\eta}(\mathfrak{g})$ from elliptic stable envelopes.

These also admit categorifications. Namely, equivariant K-theory $\mathrm{K}_{\mathsf{T}}(X)$ can be *categorified* to the equivariant derived category $\mathrm{D^bCoh}_{\mathsf{T}}(X)$. Correspondingly, there are expected to be stable envelopes in this setting which would be *functors*

$$\mathrm{D^bCoh}_{\mathsf{T}}(X^{\mathsf{A}}) \longrightarrow \mathrm{D^bCoh}_{\mathsf{T}}(X).$$

This is currently a work in progress by Halpern-Leistner, Maulik and Okounkov.

The elliptic cohomology of X over the Tate elliptic curve $E = \mathbf{C}^\times/q^{\mathbf{Z}}$ defined over $\mathbf{C}(\!(q)\!)$ is closely related to the "Tate K-theory" [KM07] which is the $\mathbf{C}^\times$-equivariant K-theory of the algebraic loop space $LX = \mathrm{Map}(\mathrm{Spec}\,\mathbf{C}(\!(t)\!), X)$, where $\mathbf{C}^\times$ acts on LX by scaling t. More precisely, for the Tate elliptic curve we should have

$$\mathrm{Ell}(X) \cong \mathrm{K}_{\mathbf{C}^\times}(X) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{C}(\!(q)\!).$$

This again can be categorified. We refer to [MSY20] for some developments in this direction.

4. K-theoretic stable envelopes

In this section we explain stable envelopes in the setting of equivariant K-theory $\mathrm{K}_{\mathsf{T}}(X)$. Let us recall some basic facts about (equivariant) K-theory:

- For a torus T we have
$$\mathrm{K}_{\mathsf{T}}(\mathrm{pt}) = \mathbf{Z}[\Lambda],$$
the group algebra of the character lattice $\Lambda = \mathrm{Hom}(\mathsf{T}, \mathbf{C}^\times)$.
- If X is a T-variety, then $\mathrm{K}_{\mathsf{T}}(X)$ is a $\mathrm{K}_{\mathsf{T}}(\mathrm{pt})$-algebra.
- If E is a coherent sheaf over a smooth variety X, then it defines a class $[E] \in \mathrm{K}(X)$ and similarly for the equivariant version.
- If E is a vector bundle over X, then it has the Euler class
$$e(E) = \sum_{i=0}^{\dim E} (-1)^i \left[\bigwedge^i E^*\right].$$
- For a T-equivariant morphism $f\colon X \to Y$ of smooth varieties we have the pullback morphism
$$f^*\colon \mathrm{K}_{\mathsf{T}}(Y) \longrightarrow \mathrm{K}_{\mathsf{T}}(X).$$
If f is proper, then we also have the pushforward morphism
$$f_*\colon \mathrm{K}_{\mathsf{T}}(X) \longrightarrow \mathrm{K}_{\mathsf{T}}(Y).$$
- Suppose
$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array}$$
is a Cartesian diagram of quasi-compact quasi-separated schemes, where i and i' are regular closed immersions, so that there is an exact sequence of vector bundles
$$0 \longrightarrow \mathrm{N}_{Y'/X'} \longrightarrow (f')^*\mathrm{N}_{Y/X} \longrightarrow \Delta \longrightarrow 0.$$
Then there is an excess intersection formula (see [Tho93, Théorème 3.1])
$$f^* i_*(-) = i'_*(e(\Delta) \cdot (f')^*(-))$$
of maps $\mathrm{K}(Y) \to \mathrm{K}(X')$. An analogous formula holds in the equivariant context (see [Kha21, Corollary 0.2]).
- If $p\colon V \to X$ is a vector bundle with $i\colon X \to V$ the zero section, then i^* and p^* induce isomorphisms
$$\mathrm{K}(X) \cong \mathrm{K}(V).$$
An analogous formula holds in the equivariant context.

Example 4.1. If $i\colon Y \hookrightarrow X$ is a regular closed immersion, then the excess intersection formula gives $i^* i_*(-) = e(N_{Y/X}) \cdot (-)$.

Remark 4.2. If E is a vector bundle of rank r over X, then

$$\mathrm{ch}(\wedge^\bullet E^*) = c_r(E).$$

In particular, the K-theoretic Euler class goes to the cohomological Euler class under the Chern character.

We continue with the previous assumptions on X. In particular, we assume that it comes with a polarization $\mathrm{T}^{1/2}_X \in \mathrm{K}_{\mathsf{T}}(X)$. Given a component $F_i \subset X^{\mathsf{A}}$ recall that the polarization gives elements $\mathrm{N}^{1/2} \in \mathrm{K}_{\mathsf{T}}(F_i)$ such that

$$\mathrm{N}_{F_i} = \mathrm{N}^{1/2} + q^{-1}(\mathrm{N}^{1/2})^*.$$

Further splitting these into positive and negative A-weights (with respect to $\mathfrak{C}$) we get an equality

$$\mathrm{N}_- = \mathrm{N}^{1/2}_- + q^{-1}(\mathrm{N}^{1/2}_+)^*$$

and similarly for positive weights. In particular,

$$\mathrm{N}_- - \mathrm{N}^{1/2} = q^{-1}(\mathrm{N}^{1/2}_+)^* - \mathrm{N}^{1/2}_+.$$

Its determinant is

$$\frac{\det \mathrm{N}_-}{\det \mathrm{N}^{1/2}} = q^{-\dim \mathrm{N}^{1/2}_+} \left(\det \mathrm{N}^{1/2}_+\right)^{\otimes(-2)}.$$

In particular, it has a canonical square root (recall that $q^{1/2} \in \mathrm{K}_{\mathsf{T}}(X)$)

$$\left(\frac{\det \mathrm{N}_-}{\det \mathrm{N}^{1/2}}\right)^{1/2} = q^{-\dim \mathrm{N}^{1/2}_+/2}(\det \mathrm{N}^{1/2}_+)^{-1}.$$

To define K-theoretic stable envelopes we also have to choose a ***slope*** parameter

$$\mathcal{L} \in \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}.$$

Just like in the case of equivariant parameters, there are ***Kähler roots*** which define walls (*affine* hyperplanes) in $\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$. We say a slope $\mathcal{L} \in \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ is ***generic*** if it does not lie on a wall. Note that we may choose a lift of the slope to a class in $\mathrm{Pic}_{\mathsf{A}}(X) \otimes_{\mathbf{Z}} \mathbf{R}$; the choice of a lift will not matter.

Theorem 4.3. *Assume that* $\dim(X^{\mathsf{A}}) = 0$ *for simplicity. For a* generic *slope* $\mathcal{L}$ *there is a unique map*

$$\mathrm{Stab}\colon \mathrm{K}_{\mathsf{T}}(X^{\mathsf{A}}) \longrightarrow \mathrm{K}_{\mathsf{T}}(X)$$

of $\mathrm{K}_{\mathsf{T}}(\mathrm{pt})$*-modules, such that for any component* $F_i \subset X^{\mathsf{A}}$ *the stable envelope* $\mathrm{Stab}(i) = \mathrm{Stab}(1_{F_i})$ *satisfies*

1. $\mathrm{supp}\,\mathrm{Stab}(i) \subset \mathrm{Attr}^f(F_i)$,
2. $\mathrm{Stab}(i)|_{F_i} = (-1)^{\dim \mathrm{N}^{1/2}_+} \left(\frac{\det \mathrm{N}_-}{\det \mathrm{N}^{1/2}|_{F_i}}\right)^{1/2} e^{\mathsf{T}}(N_-)$,
3. $\deg_{\mathsf{A}}(\mathrm{Stab}(i)|_{F_j}) \subset \deg_{\mathsf{A}}(\mathrm{Stab}(j)|_{F_j}) + \mathcal{L}|_{F_j} - \mathcal{L}|_{F_i}$ *for any* $F_j < F_i$.

Remark 4.4. Call the complement of the walls in $\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ an ***alcove***. For any two slopes lying in the same alcove the K-theoretic stable envelopes are the same.

Remark 4.5. Stable envelopes also exist for non-generic slopes. But if a slope lies on a wall, uniqueness fails: the stable envelopes for the adjacent alcoves satisfy the axioms.

Example 4.6. Consider $X = \mathrm{T}^*\mathbf{P}^1$ and the T-action on X from example 1.6. Fix the chamber $\mathfrak{C}_+$ and the polarization from example 2.3. We have $\mathrm{Pic}(X) = \mathbf{Z}$ generated by $\mathcal{O}_{\mathbf{P}^1}(1)$. Consider the slope $\mathcal{L} = \mathcal{O}(-\epsilon) \in \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ for $0 < \epsilon < 1$. Let

$$\mathrm{K}_{\mathsf{T}}(\mathrm{pt}) = \mathbf{Z}[u_1^{\pm 1}, u_2^{\pm 1}, q^{\pm 1/2}].$$

We claim that

$$\mathrm{Stab}(0) = [\mathcal{O}_{\mathbf{P}^1}] + q u_2 u_1^{-1}[\mathcal{O}_{\mathrm{T}^*_\infty \mathbf{P}^1}], \qquad \mathrm{Stab}(\infty) = -q^{1/2} u_2 u_1^{-1}[\mathcal{O}_{\mathrm{T}^*_\infty \mathbf{P}^1}].$$

The support axiom is obvious. By the excess intersection formula

$$\begin{aligned}
[\mathcal{O}_{\mathbf{P}^1}]|_0 &= 1 - u_1 u_2^{-1} q \\
[\mathcal{O}_{\mathbf{P}^1}]|_\infty &= 1 - u_2 u_1^{-1} q \\
[\mathcal{O}_{\mathrm{T}^*_\infty \mathbf{P}^1}]|_0 &= 0 \\
[\mathcal{O}_{\mathrm{T}^*_\infty \mathbf{P}^1}]|_\infty &= 1 - u_1 u_2^{-1}
\end{aligned}$$

At $\{0\}$ we have $\mathrm{N}^{1/2}_+ = 0$, so the normalization axiom is

$$\mathrm{Stab}(0)|_0 = 1 - q u_1 u_2^{-1}$$

which is obviously satisfied. At $\{\infty\}$ we have $\mathrm{N}_+^{1/2} = \mathrm{T}^*_\infty \mathbf{P}^1$ which has class $u_1 u_2^{-1} q^{-1}$ in K-theory, so the normalization axiom is

$$\mathrm{Stab}(\infty)|_\infty = \frac{-q^{-1/2}}{u_1 u_2^{-1} q^{-1}}(1 - u_1 u_2^{-1})$$

which is again satisfied.

Finally, we have to check the degree axiom for $\infty < 0$. Since all expressions involve $a = u_2 u_1^{-1}$, it will be convenient to consider the degree with respect to this variable. We have $\deg(\mathrm{Stab}(\infty)|_\infty) = [0, 1]$. Similarly, since

$$\mathrm{Stab}(0)|_\infty = 1 - q$$

we have $\deg_A(\mathrm{Stab}(0)_\infty) = [0, 0]$.

$\mathcal{O}_{\mathbf{P}^1}(2) = \mathrm{T}_{\mathbf{P}^1}$ lifts to an A-equivariant line bundle with $\mathcal{O}_{\mathbf{P}^1}(2)|_\infty$ having class $u_2 u_1^{-1}$ and $\mathcal{O}_{\mathbf{P}^1}(2)|_0$ having class $u_1 u_2^{-1}$ in equivariant K-theory. Therefore, the degree axiom is that $\deg(\mathrm{Stab}(0)|_\infty) = [0, 0]$ is contained in $[0, 1] - \epsilon = [-\epsilon, 1 - \epsilon]$. So, it is also satisfied.

The pullback along the zero section induces an isomorphism

$$\mathrm{K}_{\mathsf{T}}(\mathrm{T}^*\mathbf{P}^1) \cong \mathbf{Z}[x^{\pm 1}, u_1^{\pm 1}, u_2^{\pm 1}, q^{\pm 1/2}]/(1 - x u_1)(1 - x u_2).$$

The restriction to 0 is $x \mapsto u_2^{-1}$ and the restriction to ∞ is $x \mapsto u_1^{-1}$. Under this isomorphism $[\mathcal{O}_{\mathrm{T}^*_\infty \mathbf{P}^1}]$ goes to $1 - x^{-1} u_2^{-1}$ and the stable envelopes are

$$\mathrm{Stab}(0) = 1 - u_1 u_2 x^2 q + q u_2 u_1^{-1}(1 - x^{-1} u_2^{-1}), \qquad \mathrm{Stab}(\infty) = -q^{1/2} u_2 u_1^{-1}(1 - x^{-1} u_2^{-1}).$$

Example 4.7. Consider the previous example with the trivial slope $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}$. Then

$$\mathrm{Stab}(0) = [\mathcal{O}_{\mathbf{P}^1}] + u_2 u_1^{-1}[\mathcal{O}_{\mathrm{T}^*_\infty \mathbf{P}^1}], \qquad \mathrm{Stab}(\infty) = -q^{1/2} u_2 u_1^{-1}[\mathcal{O}_{\mathrm{T}^*_\infty \mathbf{P}^1}].$$

also satisfies the stable envelope axioms. To see that it is different from the previous expression observe that

$$\mathrm{Stab}(0)|_\infty = u_2 u_1^{-1}(1 - q).$$

Thus, for the trivial slope the uniqueness of stable envelopes fails. As explained before, it is the stable envelope for the slope $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}(\epsilon)$ for $0 < \epsilon < 1$.

5. K-theoretic stable envelopes for cotangent bundles of flag varieties

Let G be a connected simply-connected semisimple complex algebraic group, $B \subset G$ a Borel subgroup and $\mathcal{B} = G/B$ the flag variety. In this section we will consider $X = \mathrm{T}^*\mathcal{B}$ with its $G \times \mathbf{C}^\times$-action, where G acts on $\mathcal{B}$ in the obvious way and $\mathbf{C}^\times$ scales the fibers with weight 2. We let $\mathsf{A} \subset G$ be the maximal torus. Let W be the Weyl group, $\Lambda = \mathrm{Hom}(\mathsf{A}, \mathbf{C}^\times)$ the weight lattice. For $s, t \in W$ denote by $n_{s,t}$ the order of $st \in W$.

Let $\mathcal{N} \subset \mathfrak{g}^*$ be the nilpotent cone with $\mathrm{T}^*\mathcal{B} \to \mathcal{N}$ the moment map for the G-action and $\mathrm{St} = \mathrm{T}^*\mathcal{B} \times_{\mathcal{N}} \mathrm{T}^*\mathcal{B}$ the Steinberg variety. Recall the affine Hecke algebra.

Definition 5.1. The ***affine Hecke algebra*** $\mathbf{H}$ is the $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-algebra with generators T_s (for s a simple reflection) and θ_x (for $x \in \Lambda$ a character) with the relations

$$\begin{aligned} \underbrace{T_s T_t \dots}_{n_{st}} &= \underbrace{T_t T_s \dots}_{n_{st}} \\ \theta_x \theta_y &= \theta_{x+y} \\ T_s \theta_x &= \theta_x T_s, \qquad s(x) = x \\ \theta_x &= T_{s_\alpha} \theta_{x-\alpha} T_{s_\alpha} \\ (T_s + q^{-1/2})(T_s - q^{1/2}) &= 0. \end{aligned}$$

For $w \in W$ we denote by $T_w \in \mathbf{H}$ the element obtained by writing a reduced expression of w. There is a natural algebra structure on $\mathrm{K}_{G \times \mathbf{C}^\times}(\mathrm{St})$ given by convolution. Moreover, there is a natural $\mathrm{K}_{G \times \mathbf{C}^\times}(\mathrm{St})$-module structure on $\mathrm{K}_{\mathsf{T}}(\mathrm{T}^*\mathcal{B})$.

Theorem 5.2 (Kazhdan–Lusztig, Ginzburg)**.** *There is an isomorphism of algebras*

$$\mathrm{K}_{G\times\mathbf{C}^\times}(\mathrm{St}) \cong \mathbf{H}.$$

The equivariant roots in this case coincide with the usual roots for G. So, as the chamber $\mathfrak{C}$ we may take the chamber of positive roots with respect to the Borel B. We may identify the fixed points with

$$X^{\mathsf{A}} \cong W,$$

where $w = e \in W$ corresponds to $B \in \mathcal{B}$. Consider the polarization from example 2.3. We may identify

$$\mathrm{Pic}(X) \cong \Lambda.$$

We define the ***negative fundamental alcove*** of $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ to consist of weights λ such that

$$-1 < (\lambda, \alpha^\vee) < 0$$

for all positive roots α. From now on the slope $\mathcal{L} \in \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ is an arbitrary element of the negative fundamental alcove. The stable envelopes will define K-theoretic classes

$$\mathrm{Stab}(w) \in \mathrm{K}_{\mathsf{T}}(\mathrm{T}^*\mathcal{B}).$$

The following is shown in [SZZ20].

Theorem 5.3.

(1) $\mathrm{Stab}(e) = [\mathcal{O}_{\mathrm{T}^*_e\mathcal{B}}]$.
(2) *For any $w \in W$ one has* $\mathrm{Stab}(w) = T_{w^{-1}}(\mathrm{Stab}(e))$.

Remark 5.4. In the case when $\mathfrak{C}$ is the *negative* Weyl chamber one has

$$\mathrm{Stab}(w_0) = (-q^{1/2})^{\dim(G/B)} e^{2\rho} [\mathcal{O}_{\mathrm{T}^*_{w_0}\mathcal{B}}].$$

This coincides with the calculation of $\mathrm{Stab}(\infty)$ in example 4.6 for $G = \mathrm{SL}_2$.

6. Deformation quantization in positive characteristic and stable envelopes

In this section we assume that X is a conical A-equivariant symplectic resolution. In particular, it implies that $\mathrm{H}^i(X, \mathcal{O}) = 0$ for $i > 0$. In this case

$$\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C} \cong \mathrm{H}^2(X; \mathbf{C})$$

by taking the first Chern class.

According to [BK04] filtered deformation quantizations of X are parametrized by $\lambda \in \mathrm{H}^2(X; \mathbf{C})$ which can be made A-equivariant. It is also expected that one can quantize the Lagrangian correspondence $X \leftarrow \mathrm{Attr} \rightarrow X^{\mathsf{A}}$ to define a *parabolic induction* functor from deformation quanzation modules on X^{A} to deformation quantization modules on X. It is expected (the ongoing work of Bezrukavnikov and Okounkov) that it is related to K-theoretic stable envelopes and their categorifications via reductions mod p which we will briefly recall.

If X is a symplectic variety over a field of positive characteristic, its deformation quantization A_λ often has a large center, so that A localizes to a sheaf of Azumaya algebras $\mathcal{A}_\lambda$ over the Frobenius twist $X^{(1)}$ of X. In certain cases $\mathcal{A}_\lambda$ might be canonically split, i.e. it is isomorphic to the endomorphism algebra of a vector bundle on $X^{(1)}$. In this case we get an equivalence

$$\mathrm{D}^{\mathrm{b}}_{\mathsf{T}}(A_\lambda - \mathrm{mod}) \cong \mathrm{D}^{\mathrm{b}}\mathrm{Coh}_{\mathsf{T}}(X^{(1)})$$

of the corresponding derived categories. This is expected to happen for quantization parameters

$$\mathcal{L} \in \frac{1}{p}\mathrm{Pic}(X) \subset \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

which lie away from the same affine hyperplanes that appear in the definition of K-theoretic stable envelopes. We refer to [Los17] for some results in this direction.

So, the parabolic induction functor has a "classical shadow" given by a functor

$$\mathrm{D}^{\mathrm{b}}\mathrm{Coh}_{\mathsf{T}}(X^{\mathsf{A}}) \longrightarrow \mathrm{D}^{\mathrm{b}}\mathrm{Coh}_{\mathsf{T}}(X)$$

which categorifies the K-theoretic stable envelope. In this section we describe how this works for $X = \mathrm{T}^*\mathcal{B}$, the cotangent bundle of the flag variety, following [SZZ21].

Let ρ be the Weyl vector and consider the affine hyperplanes

$$H^p_{\alpha^\vee,n} = \{\lambda' \in \Lambda \otimes_{\mathbf{Z}} \mathbf{Q} \mid \langle \alpha^\vee, \lambda' + \rho \rangle = np\}.$$

Consider the fundamental alcove containing $(\epsilon - 1)\rho$ for a small positive ϵ. We say $\lambda' \in \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$ is ***regular*** if it does not lie on the affine hyperplanes.

Remark 6.1. Let

$$\lambda = -\frac{\lambda' + \rho}{p}.$$

Then λ' is *regular* in the above sense if, and only if, the slope $\lambda \in \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$ is *generic* as in the definition of K-theoretic stable envelopes. Moreover, λ' lies in the fundamental alcove if, and only if, λ lies in the negative fundamental alcove, i.e.

$$-1 < \langle \alpha^\vee, \lambda \rangle < 0$$

for every positive root α.

For a regular integral λ' the papers [BMR06; BMR08] have established a localization result identifying a certain derived category of $\mathfrak{g}$-representations over an algebraically closed field k of positive characteristic p and $\mathrm{D^bCoh}_{\mathsf{T}_k}(\mathrm{T}^*\mathcal{B}^{(1)})$. We denote this equivalence by $\gamma^{\lambda'}$.

Now consider the Verma module $Z(w(\lambda' + \rho) + \rho)$ with the corresponding highest weight over $\mathfrak{g}$. In particular,

$$\gamma^{\lambda'} Z(w(\lambda' + \rho) + \rho) \in \mathrm{D^bCoh}_{\mathsf{T}_k}(\mathrm{T}^*\mathcal{B}^{(1)}).$$

Let $\mathcal{B}_{\mathbf{Z}}$ be the $\mathbf{Z}$-form of the flag variety. The following is [SZZ21, Theorem 1.2].

Theorem 6.2. *Suppose p is greater than the Coxeter number of G, $\lambda' \in \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$ is regular and integral and let*

$$\lambda = -\frac{\lambda' + \rho}{p}.$$

There are objects $\mathrm{Stab}^\lambda_{\mathbf{Z}}(w) \in \mathrm{D^bCoh}_{\mathsf{T}_{\mathbf{Z}}}(\mathrm{T}^*\mathcal{B}_{\mathbf{Z}})$ with the following properties:

- *We have an equality*

$$[\mathrm{Stab}^\lambda_{\mathbf{Z}}(w) \otimes_{\mathbf{Z}} \mathbf{C}] = e^{-\rho}\mathrm{Stab}(w) \in \mathrm{K}_{\mathsf{T}}(\mathrm{T}^*\mathcal{B}),$$

  where we use λ as the slope in the definition of K-theoretic stable envelopes.
- *We have an isomorphism*

$$\mathrm{Stab}^\lambda_{\mathbf{Z}}(w) \otimes_{\mathbf{Z}} k \cong \mathcal{L}_{-\rho} \otimes \gamma^{\lambda'} Z(w(\lambda' + \rho) + \rho) \in \mathrm{D^bCoh}_{\mathsf{T}_k}(\mathrm{T}^*\mathcal{B}^{(1)}),$$

  where k is an algebraically closed field of characteristic p and $\mathcal{L}_{-\rho}$ is the corresponding $G \times \mathbf{G}_{\mathrm{m}}$-equivariant line bundle on $\mathrm{T}^*\mathcal{B}$.

Quantizing $\mathrm{T}^*\mathcal{B}$ in positive characteristic we get the representation category of $\mathfrak{g}$ by the Beilinson–Bernstein localization and the corresponding parabolic induction functor quantizing

$$W \longleftarrow \mathrm{Attr} \longrightarrow \mathrm{T}^*\mathcal{B}$$

sends w to the corresponding Verma module. The above theorem asserts a compatibility between quantization in positive characteristic and stable envelopes.

7. Stable envelopes for cotangent bundles

Consider the setting of example 2.3, so that Y is a smooth variety with an A-action and $X = \mathrm{T}^*Y$ with a $\mathsf{T} = \mathsf{A} \times \mathbf{C}^\times$-action. We fix a chamber $\mathfrak{C}$. Let Attr^X be the attracting variety in X and Attr^Y the attracting variety in Y. In this case the Lagrangian correspondence

$$X^{\mathsf{A}} \longleftarrow \mathrm{Attr}^X \longrightarrow X$$

becomes

$$\begin{array}{ccc} & \mathrm{N}^*\mathrm{Attr}^Y & \\ \swarrow & & \searrow \\ \mathrm{T}^*Y^{\mathsf{A}} & & \mathrm{T}^*Y \end{array}$$

which is the conormal bundle of the correspondence

$$\begin{array}{ccc} & \mathrm{Attr}^Y & \\ {}^{p}\swarrow & & \searrow^{\eta} \\ Y^{\mathsf{A}} & & Y \end{array}$$

Let $\mathrm{MHM}(Y)$ be the category of A-eqvuiariant mixed Hodge modules on Y and $\mathrm{D}^{\mathrm{b}}\mathrm{MHM}_{\mathsf{A}}(Y)$ the corresponding equivariant derived category which admits a six-functor formalism [Ach13]. For us it is important to know that a mixed Hodge module has an underlying filtered D-module. In particular, there is a functor

$$\mathrm{gr} \circ \mathrm{DR} \colon \mathrm{D}^{\mathrm{b}}\mathrm{MHM}_{\mathsf{A}}(Y) \longrightarrow \mathrm{D}^{\mathrm{b}}\mathrm{Coh}_{\mathsf{T}}(Y)$$

passing to the associated graded of the filtration on the de Rham complex.

We will be interested in the functor

$$\eta_! p^* \colon \mathrm{D}^{\mathrm{b}}\mathrm{MHM}_{\mathsf{A}}(Y^{\mathsf{A}}) \longrightarrow \mathrm{D}^{\mathrm{b}}\mathrm{MHM}_{\mathsf{A}}(Y).$$

For $\mathcal{L} \in \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ we may also consider the category of twisted mixed Hodge modules which we denote by $\mathrm{MHM}^{\mathcal{L}}(Y)$.

Theorem 7.1. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}\mathrm{MHM}^{\mathcal{L}}_{\mathsf{A}}(Y^{\mathsf{A}})) & \xrightarrow{\eta_! p^*} & \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}\mathrm{MHM}^{\mathcal{L}}_{\mathsf{A}}(Y)) \\ \downarrow{\scriptstyle \mathrm{gr}\circ\mathrm{DR}} & & \downarrow{\scriptstyle \mathrm{gr}\circ\mathrm{DR}} \\ \mathrm{K}_{\mathsf{T}}(Y^{\mathsf{A}}) & & \mathrm{K}_{\mathsf{T}}(Y) \\ \sim\downarrow{\scriptstyle \pi^*} & & \sim\downarrow{\scriptstyle \pi^*} \\ \mathrm{K}_{\mathsf{T}}(X^{\mathsf{A}}) & \xrightarrow{\mathrm{Stab}} & \mathrm{K}_{\mathsf{T}}(X) \end{array}$$

Here the vertical isomorphisms on K-theory are induced by pullbacks under the projections $\pi \colon \mathrm{T}^*Y \to Y$ and $\pi \colon \mathrm{T}^*Y^{\mathsf{A}} \to Y^{\mathsf{A}}$ and the K-theoretic stable envelope at the bottom involves $\mathcal{L}$ as the slope parameter.

Remark 7.2. The above theorem is a combination of the following results:

1. For a subvariety $i \colon U \hookrightarrow Y$ the class in K-theory of the mixed Hodge module $i_! \mathbf{Q}_U$ is the equivariant motivic Chern class. Motivic Chern classes (along with the relationship to mixed Hodge modules) were introduced in [BSY10]; their equivariant version was introduced in [FRW21] and their relationship to equivariant mixed Hodge modules was explained in [DM20].
2. The K-theoretic stable envelope for the cotangent bundle coincides with the equivariant motivic Chern class of the corresponding attracting set [FRW21].

References

[Ach13] P. Achar. "Equivariant mixed Hodge modules". Lecture notes from the Clay Mathematics Institute workshop on Mixed Hodge Modules and Applications. 2013. URL: https://www.math.lsu.edu/~pramod/docs/emhm.pdf.

[AO21] M. Aganagic and A. Okounkov. "Elliptic stable envelopes". *J. Amer. Math. Soc.* 34.1 (2021), pp. 79–133. arXiv: 1604.00423 [math.AG].

[BK04] R. Bezrukavnikov and D. Kaledin. "Fedosov quantization in algebraic context". *Mosc. Math. J.* 4.3 (2004), pp. 559–592, 782. arXiv: math/0309290.

[BMR06] R. Bezrukavnikov, I. Mirković, and D. Rumynin. "Singular localization and intertwining functors for reductive Lie algebras in prime characteristic". *Nagoya Math. J.* 184 (2006), pp. 1–55. arXiv: math/0602075.

[BMR08] R. Bezrukavnikov, I. Mirković, and D. Rumynin. "Localization of modules for a semisimple Lie algebra in prime characteristic". *Ann. of Math. (2)* 167.3 (2008). With an appendix by Bezrukavnikov and Simon Riche, pp. 945–991. arXiv: math/0205144.

[BSY10] J.-P. Brasselet, J. Schürmann, and S. Yokura. "Hirzebruch classes and motivic Chern classes for singular spaces". *J. Topol. Anal.* 2.1 (2010), pp. 1–55. arXiv: math/0503492.

[DM20] B. Dirks and M. Mustaţă. "The Hilbert series of Hodge ideals of hyperplane arrangements". *J. Singul.* 20 (2020), pp. 232–250. arXiv: 2003.11681 [math.AG].

[FRW21] L. M. Fehér, R. Rimányi, and A. Weber. "Motivic Chern classes and K-theoretic stable envelopes". *Proc. Lond. Math. Soc. (3)* 122.1 (2021), pp. 153–189. arXiv: 1802.01503 [math.AG].

[Kha21] A. A. Khan. "Virtual excess intersection theory". *Ann. K-Theory* 6.3 (2021), pp. 559–570. arXiv: 1909.13829 [math.AG].

[KM07] N. Kitchloo and J. Morava. "Thom prospectra for loopgroup representations". *Elliptic cohomology*. Vol. 342. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2007, pp. 214–238. arXiv: math/0404541.

[Los17] I. Losev. "On modular categories $\mathcal{O}$ for quantized symplectic resolutions" (2017). arXiv: 1712.07726 [math.RT].

[MO19] D. Maulik and A. Okounkov. "Quantum groups and quantum cohomology". *Astérisque* 408 (2019), pp. ix+209. arXiv: 1211.1287 [math.AG].

[MSY20] M. McBreen, A. Sheshmani, and S.-T. Yau. "Elliptic stable envelopes and hypertoric loop spaces" (2020). arXiv: 2010.00670 [math.AG].

[Oko17] A. Okounkov. "Lectures on K-theoretic computations in enumerative geometry". *Geometry of moduli spaces and representation theory*. Vol. 24. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2017, pp. 251–380. arXiv: 1512.07363 [math.AG].

[Oko21] A. Okounkov. "Inductive construction of stable envelopes". *Lett. Math. Phys.* 111.6 (2021), Paper No. 141, 56. arXiv: 2007.09094 [math.AG].

[SZZ20] C. Su, G. Zhao, and C. Zhong. "On the K-theory stable bases of the Springer resolution". *Ann. Sci. Éc. Norm. Supér. (4)* 53.3 (2020), pp. 663–711. arXiv: 1708.08013 [math.AG].

[SZZ21] C. Su, G. Zhao, and C. Zhong. "Wall-crossings and a categorification of K-theory stable bases of the Springer resolution". *Compos. Math.* 157.11 (2021), pp. 2341–2376. arXiv: 1904.03769 [math.RT].

[Tho93] R. W. Thomason. "Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire". *Invent. Math.* 112.1 (1993), pp. 195–215.