SFRG Talk 3

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STABLE ENVELOPES : Part 1

Recap: $\mathrm{Stab}_{\mathfrak{C}} : H_T(X^A) \longrightarrow H_T(X)$

upshot : $R_{\mathfrak{C}_2,\mathfrak{C}_1} := \mathrm{Stab}_{\mathfrak{C}_2}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}_1} \in \mathrm{End}(H_T(X^A)) \otimes [\hbar^{-1}]$

Plan
1. Setup
2. Chambers and attracting loci
3. Order and full attracting loci
4. Definition of stable envelopes.

$T^*\mathbb{P}^1$

1 Setup

X smth quasi proj var $/\mathbb{C}$

$\omega \in H^0(\Omega^2_X)$ nondegen, closed.

$$\begin{array}{ccc} \mathfrak{g} & & G \curvearrowright X \xrightarrow[\text{proper}]{\pi} X_0 \\ \cup & & \cup \\ \mathfrak{t} & & T \\ \cup & & \cup \\ \mathfrak{a} & & A \end{array}$$

G-equivariant.

tori

$\leftarrow$ affine.

$\hbar : G \xrightarrow[\text{regular}]{\text{grp hom}} \mathbb{C}^\times$

$g\omega = \hbar(g)\omega$

Assumption ω G-eigenvector

$\hbar \in \mathfrak{g}^*$ its weight.

A fixes ω : $\ker(\hbar) \supset A$.

$X^A \subset X$ fixed locus.

2 Chambers and attracting sets

$\cong \mathbb{Z}^{\mathrm{rk}(A)}$

Example $T^*\mathbb{P}^1$

$T := \underbrace{\mathbb{C}^\times_u}_{=:A} \times \mathbb{C}^\times_\hbar \rightsquigarrow T^*\mathbb{P}^1 \curvearrowleft A$

$A \curvearrowright \mathbb{P}^1$ $t[x:y] = [tx:y]$

$\mathbb{C}^\times_\hbar \curvearrowright T^*\mathbb{P}^1$ by scaling fibers by $\hbar$

$$(T^*\mathbb{P}^1)^A = \{0, \infty\}$$

$0 = [0:1]$, $\infty = [1:0]$

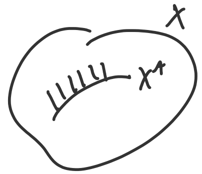
$X = T^*\mathbb{P}^1$

$\downarrow$

$X_0 = \mathcal{N}_{\mathfrak{sl}_2} = \mathrm{Spec}(\Gamma(\mathcal{O}_{T^*\mathbb{P}^1}))$

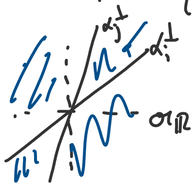
$X_*(A) = \{ \sigma : \mathbb{C}^* \longrightarrow A \} = \mathbb{Z}$
cocharacter

$\mathfrak{a}_\mathbb{R} := X_*(A) \otimes_\mathbb{Z} \mathbb{R} \hookrightarrow \mathfrak{a}$



<u>Def</u> the torus roots of $A \curvearrowright X$
the weights $\{\alpha_i\}$ appearing
in $A \curvearrowright N_{X^A/X}$

$\alpha_i^\perp := \{ a \in \mathfrak{a}_\mathbb{R} \mid \alpha_i(a) = 0 \} \subset \mathfrak{a}_\mathbb{R}$ <u>walls</u>



$\mathfrak{C}$ chambers are the connected
components of $\mathfrak{a}_\mathbb{R} \setminus (\bigcup \alpha_i^\perp)$
$= \bigsqcup \mathfrak{C}$

<u>Facts</u> $\sigma, \sigma' \in \mathfrak{C}$, $x \in X$, $Z \in \pi_0(X^A)$

$\mathrm{Attr}_\mathfrak{C}(Z) = \{ x \in X \mid \lim_{t \to 0} \sigma(t) x \in Z \} = \{ x \in X \mid \lim_{t \to 0} \sigma'(t) x \in Z \}$

$x \in \mathrm{Attr}_\mathfrak{C}(Z)$, $\lim_{t \to 0} \sigma(t) x = \lim_{t \to 0} \sigma'(t) x$

<u>Option 1</u> $\{ \sigma_1 \in \mathfrak{C} \mid \lim_{t \to 0} \sigma_1(t) x = \lim_{t \to 0} \sigma(t) x \}$
is closed and open. Exercise.

<u>Fact:</u> $\lim_\mathfrak{C} : \mathrm{Attr}_\mathfrak{C}(Z) \longrightarrow Z$ affine bundle
$x \longmapsto \lim_\mathfrak{C} x$ T-equivariant.

$\mathbb{A}^N \times U \longrightarrow U \subset Z$
open.

③ Order and full attracting loci

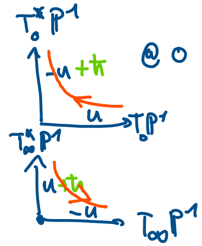
define partial order $\pi_0(X^A)$ $\preccurlyeq_\mathfrak{C}$
$Z' \preccurlyeq Z$ if $\overline{\mathrm{Attr}_\mathfrak{C}(Z)} \cap Z' \neq \emptyset$

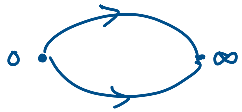
$A = \mathbb{C}^\times_u$

$(T^*\mathbb{P}^1)^A = \{0, \infty\}$

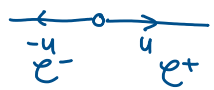
torus roots $\{u, -u\}$

$N_{X^A/X}\big|_0 = T_0 X$





$\mathfrak{a}_\mathbb{R}$



$\mathrm{Attr}_{\mathfrak{C}^+}(0) = \mathbb{P}^1 \setminus \infty$ $\qquad$ $\mathrm{Attr}_{\mathfrak{C}^-}(0) = T_0^*\mathbb{P}^1$

$\mathrm{Attr}_{\mathfrak{C}^+}(\infty) = T_\infty^*\mathbb{P}^1$ $\qquad$ $\mathrm{Attr}_{\mathfrak{C}^-}(\infty) = \mathbb{P}^1 \setminus 0$

$\mathrm{Attr}^f_{\mathfrak{C}^+}(0) = \mathbb{P}^1 \cup T_\infty^*\mathbb{P}^1$ $\qquad$ $\mathrm{Attr}^f_{\mathfrak{C}^-} = T_0^*\mathbb{P}^1$

$\mathrm{Attr}^f_{\mathfrak{C}^+}(\infty) = T_\infty^*\mathbb{P}^1$ $\qquad$ $\mathrm{Attr}^f_{\mathfrak{C}^-} = \mathbb{P}^1 \cup T_0^*\mathbb{P}^1$

$\mathrm{Stab}_{\mathfrak{C}^+} : \underset{\text{ll}s}{H_T(\{0, \infty\})} \longrightarrow H_T(T^*\mathbb{P}^1)$

$z' \preceq z$ & $z \preceq z' \implies z = z'$

uniqueness part in BB

Lemma $\mathrm{Attr}^f_{\mathfrak{C}}(z) := \bigsqcup_{z' \preceq z} \mathrm{Attr}_{\mathfrak{C}}(z')$ is closed.

full attracting locus of z

Proof uses properness of $\pi : X \longrightarrow X_0$

④ Definition of stable envelopes !

$\gamma \in H_T(X)$ $Y \subset X$ T-invariant, closed.

$\mathrm{supp}(\gamma) \subset Y$ $\Leftrightarrow$ $\gamma|_{X \setminus Y} = 0$

$\gamma \cap [X] = i_* \alpha$

A-degree of $\gamma \in H_T(X^A)$

$$H_T(X^A) = H_{T/A}(X^A) \otimes_{\underbrace{\mathbb{C}[\mathfrak{t}/\mathfrak{a}]}_{= H_{T/A}(pt)}} \underbrace{\mathbb{C}[\mathfrak{t}]}_{= H_T(pt)}$$

$\gamma \in$

$$\cong H_{T/A}(X^A) \otimes_{\mathbb{C}[\mathfrak{t}/\mathfrak{a}]} (\mathbb{C}[\mathfrak{t}/\mathfrak{a}] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{a}])$$

$$\cong H_{T/A}(X^A) \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{a}]$$

$\deg_A(\gamma) = \deg(\alpha)$ $\quad \gamma_{T/A} \otimes \alpha$

Polarization is a choice of sign for every $z \in \pi_0(X^A)$
(depends on $\mathfrak{C}$) $\quad \mathrm{sign}(z) \in \{\pm 1\}$

Theorem - Definition

For every chamber $\mathfrak{C} \subset \mathfrak{a}_{\mathbb{R}}$ (and every polarization sign)
there exists a unique $H_T(pt)$-module homomorphism

$\mathrm{Stab}_{\mathfrak{C}} : H_T(X^A) \longrightarrow H_T(X)$

$H_T(pt)0 \oplus H_T(pt)\infty$

$\mathrm{Stab}_{\mathfrak{C}+}(0) = [\mathbb{P}^1] + [T^*_\infty \mathbb{P}^1]$

$\mathrm{Stab}_{\mathfrak{C}+}(\infty) = -[T^*_\infty \mathbb{P}^1]$

$\mathrm{sign}(\infty) = 1$

$\mathrm{sign}(0) = -1$

$[\mathbb{P}^1]|_0 = e(N_{\mathbb{P}^1/T^*\mathbb{P}^1})|_0$

$= -u + \hbar$ (by self-int.)

satisfies (i) ✓ (ii) ✓

$[\mathbb{P}^1]|_\infty = e(N_{\mathbb{P}^1/T^*\mathbb{P}^1})|_\infty$

$= \underbrace{u + \hbar}_{\deg_A = 1 \geq 1}$ $\quad u \in \mathfrak{a}$

$[T^*_\infty \mathbb{P}^1]|_\infty = e^T(N_{T^*_\infty \mathbb{P}^1/T^*\mathbb{P}^1})|_\infty$

$[T^*_\infty \mathbb{P}^1] = i_{T^*_\infty *} 1$ $\quad = -u$

$\deg_A(u + \hbar - u) = 0$

$\mathrm{Stab}_{\mathfrak{C}(,\mathrm{sign})}$

s.t.

$\forall z \in \pi_0(X^A)\quad \forall \gamma \in H_{T/A}(z)\qquad \Gamma := \mathrm{Stab}_{\mathfrak{C}}(\gamma)$

satisfies (i) $\mathrm{supp}(\Gamma) \subset \mathrm{Attr}^f_{\mathfrak{C}}(z)$

(ii) $\Gamma|_z = (-1)^{\frac{\mathrm{codim}(z)}{2}}\, \mathrm{sign}(z)\, e^T(N_z^-) \cup \gamma$

(iii) $\deg_A(\Gamma|_{z'}) < \dfrac{\mathrm{codim}(z')}{2}$ for all $z' \prec z$

positive & negative weights w.r.t. $\mathfrak{C}$

$N_z = N_z^+ \oplus N_z^-$

via ω $\quad (N_z^+)^\vee = N_z^- \otimes \hbar \in K_T(z)$

<u>Pf of uniqueness</u> Apply the following lemma $\Gamma - \Gamma'$

<u>Lemma</u> $\gamma \in H_T(X) \quad \mathrm{supp}(\gamma) \subset \mathrm{Attr}^f_{\mathfrak{C}}(z)$

s.t. $\deg_A(\gamma|_{z'}) < \dfrac{\mathrm{codim}(z')}{2} \quad \forall z' \in \pi_0(X^A)$

$\Rightarrow \gamma = 0$

<u>Proof</u>

$$\begin{tikzcd}
z \arrow[r, hook, "\iota"] \arrow[d, hook, "\iota_1"', "\text{regular}"] & & X \\
\mathrm{Attr}_{\mathfrak{C}}(z) \arrow[r, hook, "\text{open}", "\iota_2"'] & \mathrm{Attr}^f_{\mathfrak{C}}(z) \arrow[ur, hook, "\iota_3"'] &
\end{tikzcd}$$

$U \subset X$ open T-invariant s.t. $U \cap \mathrm{Attr}^f_{\mathfrak{C}}(z) = \mathrm{Attr}_{\mathfrak{C}}(z)$

take $U = X \setminus (\mathrm{Attr}^f_{\mathfrak{C}} \setminus \mathrm{Attr}_{\mathfrak{C}})$

$\alpha \in H^{BM,T}(\mathrm{Attr}^f_{\mathfrak{C}}(z))$

s.t. $\gamma \cap [X] = \iota_{3*}\alpha$

$\Rightarrow \iota^*\gamma \cap [z] = \iota_1^* \iota_2^* \iota_3^* \iota_{3*} \alpha$

$\left(= \iota_1^* \iota_2^*\, e(N_{\mathrm{Attr}^f_{\mathfrak{C}}}) \cap \iota_1^* \iota_2^* \alpha \right)$

$= \iota_1^*\, e(N_{\mathrm{Attr}_{\mathfrak{C}}}) \cap \iota_1^* \iota_2^* \alpha$

$= e(N_z^-) \cap \iota_1^* \iota_2^* \alpha$

N_z^- has nonzero A-weights $\quad e(N_z^-) \cap$ is injective.

$$\deg_A(e(N_Z^-)) = \frac{\mathrm{codim}(Z)}{2}$$

i_1 homotopy eq. $\Rightarrow \deg_A(e(N_Z^-) \cap i_1^* i_2^* \alpha) \geq \frac{\mathrm{codim}(Z)}{2}$

if $i_2^* \alpha \neq 0$

$\Rightarrow i_2^* \alpha = 0$

$\Rightarrow \mathrm{supp}(\alpha) \subset \mathrm{Attr}^f_{\mathcal{C}}(Z) \setminus \mathrm{Attr}_{\mathcal{C}}(Z)$

induction using $\preceq$ $\Rightarrow \gamma = 0$ $\square$