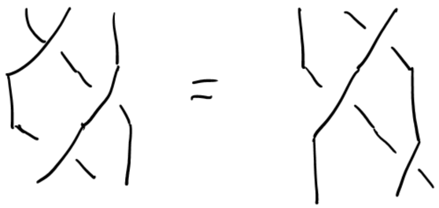
16 February 2022 15:12

1 Yang-Baxter equation:

- braid relation: in B_n braid group: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



: YB equation

- V finite dim vs $/\mathbb{C}$, $R \in \mathrm{End}(V \otimes V)$ is an R-matrix if
$(R \otimes id_V) \circ (id_V \otimes R) \circ (R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R) \in \mathrm{End}(V \otimes V \otimes V)$

$\Longleftrightarrow \boxed{R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}}$ $\quad R_{13} = \tau \circ \tilde{R}_{13}$

Why? → appears in maths physics.
→ rep. theory: braided category of rep.

Hopf algebra: (A, m, Δ, S) : $m: A \otimes A \longrightarrow A$
$\Delta: A \longrightarrow A \otimes A$

V, W some A-modules
$V \otimes W$ is A-mod. : $g \cdot (v \otimes w) = \Delta(g) \cdot (v \otimes w)$
V^* is A-mod : $g \cdot \phi = \phi \circ S(g)$

Example: $\mathfrak{g}$ Lie algebra, $A = U(\mathfrak{g})$ is a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$

cocommutative : $\boxed{\tau \circ \Delta = \Delta}$ (*) $\tau(x \otimes y) = y \otimes x$

Braided category: $\forall X, Y$, $\exists$ iso $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$.

Deformation of $\otimes$ by Universal R-matrix:

$R \in A \otimes A$, invertible st $R \Delta(x) R^{-1} = (\tau \circ \Delta)(x)$, $\forall x \in A$

(+ conditions) $\leftarrow (\Delta \otimes id) R = R_{13} R_{23}$

$\to$ the category A-mod is braided: $c_{u,w} = \tau \circ R$

$\to$ the R-matrix satisfies the YB equation.

2) Quantum groups:

$\mathfrak{g}$ simple Lie algebra $/ \mathbb{C}$ (KM algebra)

$q \in \mathbb{C}^\times$, not a root of 1

$U_q(\mathfrak{g})$: q-deformation of $U(\mathfrak{g}) = \langle E_i, F_i, \overbrace{K_i^{\pm 1}}^{\text{commutative}} \rangle_{1 \leq i \leq n}$

$C = (c_{ij})$ Cartan matrix of $\mathfrak{g}$:

$K_i^{\pm} E_j = q^{\pm c_{ij}} E_j K_i^{\pm}$

$K_i^{\pm} F_j = q^{\mp c_{ij}} F_j K_i^{\pm}$

+ q-Serre relations

$[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ ($q_i = q^{d_i}$, D sym. of C)

Example: • $U_q(sl_2)$: $\langle E, F, K^{\pm} \rangle$

$K^{\pm 1} E = q^{\pm 2} E K^{\pm}$, $[E, F] = \dfrac{K - K^{-1}}{q - q^{-1}}$

$K^{\pm} F = q^{\mp 2} F K^{\pm}$

$U_q(gl_2) = \langle E, F, K_1^{\pm}, K_2^{\pm} \rangle$

$K_1 E = q E K_1$, $K_1 F = q^{-1} F K_1$

$K_2 E = q^{-1} E K_2$, $K_2 F = q F K_2$

$U_q(sl_2) = U_q(gl_2) / \langle K_1 K_2 = 1 \rangle$

$U_q(\mathfrak{g})$ is a Hopf algebra

$\Delta(K_i) = K_i \otimes K_i$, $S(K_i) = K_i^{-1}$

$K \quad K_1 K^{-1}$

$K = k_1 k_2$

$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$, $S(E_i) = -K_i^{-1} E_i$

$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$, $S(F_i) = -F_i K_i$

Thm: [Drinfeld] $\exists R \in U_q(\mathfrak{g}) \hat{\otimes} U_q(\mathfrak{g})$, universal R-matrix

specializes to finite dimensional rep:

$\forall V, W$ fd rep: $R_{V,W} = (\rho_V \otimes \rho_W)(R) \in \mathrm{End}(V \otimes W)$

The category $\mathrm{Rep}_{fd}\, U_q(\mathfrak{g})$ is braided.

Example: $U_q(sl_2)$: $R = \sum_{r=0}^{+\infty} c_r(q)\, e^{\frac{1}{2} H \otimes H} F^r \otimes E^r$, where $K = q^H$.

$V = W = V_1 = \mathbb{C} v_0 \oplus \mathbb{C} v_1$

$K v_0 = q v_0$ | $E v_0 = 0$, $F v_0 = v_1$

$K v_1 = q^{-1} v_1$ | $E v_1 = v_0$, $F v_1 = 0$

$$R_{V,W} = \left(q^{-1/2}\right) \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

$(v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1)$

$F \otimes E (v_0 \otimes v_1) = v_1 \otimes v_0$

$F \otimes E (v_1 \otimes v_0) = 0$

Rk) R has multiplicative formula of the form: $R = \prod_{\beta \in \Phi_+} \exp(c_\beta\, F_\beta \otimes E_\beta)\, q^{H \otimes}$: also true for $\mathfrak{g}$ affine (twisted) Lie algebra.

3] RTT presentations:

V a fd $U_q(\mathfrak{g})$-module, $T \in \mathrm{End}(V) \otimes_{U_q(\mathfrak{g})} U_q(\mathfrak{g})$, it satisfies the RTT relation:

$$\boxed{R_{12} T_{13} T_{23} = T_{23} T_{13} R_{12}} \in \mathrm{End}(V^{\otimes 2}) \otimes U_q(\mathfrak{g}).$$

↯ ↳ consequence of YB.

<u>Hopf dual</u>: in general a R-matrix R ($\in \mathrm{End}(V\otimes V)$) : ($\dim V = n$)

$$A(R) = \langle t_{ij},\ 1\leq i,j\leq n \rangle + RTT \text{ relation, with } T = (t_{ij})_{1\leq i,j\leq n}.$$

↳ $A(R)$ is also a Hopf algebra:

<u>Example</u>: $R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$ $RTT = TTR$ gives: $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$

$$\left\{\begin{array}{l} t_{11}t_{12} = q\,t_{12}t_{11} \\ t_{12}t_{21} = t_{21}t_{12} \\ t_{21}t_{22} = q\,t_{22}t_{21} \end{array}\right. \qquad \left\{\begin{array}{l} t_{11}t_{21} = q\,t_{21}t_{11} \\ t_{12}t_{22} = q\,t_{22}t_{12} \\ t_{11}t_{22} - t_{22}t_{11} = (q-q^{-1})\,t_{12}t_{21} \end{array}\right.$$

(rel)

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik}\otimes t_{kj}.$$

<u>Example</u>: $V = \mathbb{C}^n$, $R = R_{V,V}$ for $U_q(sl_n)$

$A(R) = M_n(q)$ = quantized algebra of functions.

<u>$n=2$</u>: $M_2(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} / (\text{rel}) \}$: q-deformation of $M_2(\mathbb{C})$.

$\det_q = ad - q^{-1}bc$ central in $M_2(q)$

$GL_2(q) = M_2(q)[t] / (t\det_q - 1)$ $\quad SL_2(q) = M_2(q)/(\det_q - 1)$

- FRT presentation:

$U(R) = \langle (l_{ij}^{\pm})_{1\leq i,j\leq n} \rangle \subset A^*(R)$, with relations:

$$R L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R \in \mathrm{Mat}(V^{\otimes 2}, A^*(R))$$

$L = (l_{ij})$, $L_1 = L\otimes \mathrm{id}$, $L_2 = \mathrm{id}\otimes L$

<u>Then</u>: $U(R) \cong U_q(sl_n)$.

$\underline{n=2}$: $L^+ = \begin{pmatrix} K^{-1/2} & (q-q^{-1})E \\ 0 & K^{1/2} \end{pmatrix}$, $L^- = \begin{pmatrix} K^{1/2} & 0 \\ (q-q^{-1})F & K^{-1/2} \end{pmatrix}$

4) Quantum affine algebras :

YB with parameters: $R(z) \in (A \otimes A)((z))$

$$\boxed{R_{12}(z) R_{13}(wz) R_{23}(w) = R_{23}(w) R_{13}(zw) R_{12}(z)}$$

with a solution $R(z)$ $\to$ can construct $U_q(\hat{g})$.

same thing as presentation of $U_q(g)$, with g affine

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^{-1}(z-1)}{z-q^{-2}} & \frac{1-q^{-2}}{z-q^{-2}} & 0 \\ 0 & \frac{z(1-q^{-2})}{z-q^{-2}} & \frac{q^{-1}(z-1)}{z-q^{-2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

solution to YB (6 vertex model)

$\to$ rep of evaluation of $U_q(\hat{sl_2})$.

Take $\begin{cases} z = e^u, & u \to 0 \\ q = e^{h/2}, & h \to 0 \end{cases}$

"classical limit"

Obtain : $\boxed{\frac{1}{u+h}\begin{pmatrix} u & h \\ h & u \end{pmatrix}}$: R-matrix for $T^*\mathbb{P}^1$.

$R(z-q^{-2})\Big|_{q^{-2}} = \begin{pmatrix} 0 & & 0 \\ 0 & \square & \\ 0 & & 0 \end{pmatrix}$: not invertible.