Stable envelopes continued

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Recall setup from Sebastian's talk:

$$A \subset T \subset G \curvearrowright (X,\omega) \xrightarrow{\text{proper}} X_0$$

tori; smooth symplectic; affine

G-action scales ω by $\hbar \in \mathrm{Hom}(G,\mathbb{G}_m) \subset \mathfrak{g}^*$

A-action fixes ω.

Thm: for any chamber $\mathfrak{C} \subset \mathfrak{a}_{\mathbb{R}}$ ($\cong X_*(A)\otimes\mathbb{R}$) and polarisation ε the <u>stable envelope</u> map is the unique map

$$\mathrm{Stab}_{\mathfrak{C}} : H_T^\bullet(X^A) \longrightarrow H_T^\bullet(X)$$

such that for $Z \in \pi_0(X^A)$ and $\gamma \in H_T^\bullet(Z) \subset H_T^\bullet(X^A)$

(i) $\mathrm{Supp}\,\mathrm{Stab}_{\mathfrak{C}}(\gamma) \subset \mathrm{Attr}^f_{\mathfrak{C}}(Z)$

(ii) $\mathrm{Stab}_{\mathfrak{C}}(\gamma)|_Z = \pm e(N_-) \cup \gamma$

sign det by ε; $N_{Z/X} = N_+ \oplus N_-$; A weights >0 on $\mathfrak{C}$

(iii) $\mathrm{Stab}_{\mathfrak{C}}(\gamma)|_{Z'}$ has A-degree $< \frac{1}{2}\operatorname{codim} Z'$ for $Z' \prec Z$

<u>partial order generated by</u> $Z' \cap \mathrm{Attr}_{\mathfrak{C}}(Z) \neq \emptyset \Rightarrow Z' \prec Z$.

Sebastian: proof of uniqueness

Today:
- proof of existence
- R matrices
- Symplectic resolutions
- Examples

§ Construction:

<u>Idea</u>: Suppose we are given a T-invariant cycle $L \subset X \times Y$ ($\mathbb{Q}$-linear combination of subvarieties) such that $\mathrm{Supp}\,L$ is proper over X.

Then we get a map

$$\Theta_L : H_T^\bullet(Y) \longrightarrow H_T^\bullet(\mathrm{Supp}\,L) \longrightarrow H_T^{BM}(\mathrm{Supp}\,L) \longrightarrow H_T^\bullet(X)$$

$$\alpha \longmapsto pr_2^*\alpha \longmapsto pr_2^*\alpha \cap [L] \longmapsto pr_{1*}(pr_2^*\alpha \cap [L])$$

<u>Prop^n</u>: There exists a Lagrangian cycle $\mathcal{L}_{\mathfrak{C}} \subset X \times X^A$ proper over X such that

(i) For any $Z \in \pi_0(X^A)$, $\mathcal{L}_{\mathfrak{C}}|_{X\times Z}$ is supported on $\mathrm{Attr}^f_{\mathfrak{C}}(Z)\times Z$

(ii) $[\mathcal{L}_{\mathfrak{C}}]|_{Z\times Z} = \pm e(N_-) \cap [\Delta]$

(iii) if $Z' \prec Z$ then $\deg_A [\mathcal{L}_{\mathfrak{C}}]|_{Z'\times Z} < \frac{1}{2}\operatorname{codim} Z'$.

view in $H_A^\bullet(Z'\times Z)$; $H^\bullet(Z'\times Z)\otimes H_A^\bullet(pt)$

Lagrangian cycle: all cpts have $\frac{1}{2}$ dimension and $\omega|_{\text{smooth locus}} = 0$.

Prop^n $\Rightarrow$ Th^m: Set $\mathrm{Stab}_{\mathfrak{C}} = \Theta_{\mathcal{L}_{\mathfrak{C}}}$.

Aside: $X^{\mathfrak{G}}$ is symplectic

How to construct $\mathcal{L}_{\mathfrak{C}}$?

Key lemma: Let $L \subset X$ be a T-invariant Lagrangian and $i: Z \longrightarrow X$ inclusion of $Z \in \pi_0(X^A)$.
Then $\exists!$ Lagrangian cycle $\mathrm{Res}_Z L \subset Z$ supported on $Z \cap L$ such that

$$i^*[L] = \varepsilon [\mathrm{Res}_Z L] + \text{terms of lower } A\text{-degree} \in H_T^{\cdot}(Z)$$

where $\varepsilon = \pm e(N_-)|_{pt} \in H_A^{\cdot}(pt)$ is the polarisation.

Pf: ... local calculation.

Proof of proposition:

Order components $\pi_0(X^A) = \{Z_1, \dots, Z_n\}$ so that $i < j \Rightarrow Z_i \prec Z_j$.

For each i set $\mathcal{L}_{i,i} = \pm \overline{(\lim_{\mathfrak{C}} \times \mathrm{id})^{-1}(\Delta)} \subset X \times Z_i$

where $\lim_{\mathfrak{C}} \times \mathrm{id}: \mathrm{Attr}_{\mathfrak{C}}(Z_i) \times Z_i \longrightarrow Z_i \times Z_i$

Then set inductively

$$\mathcal{L}_{i,j} = \mathcal{L}_{i,j+1} - \overline{(\lim_{\mathfrak{C}} \times \mathrm{id})^{-1}(\pm \mathrm{Res}_{Z_j \times Z_i} \mathcal{L}_{i,j+1})}$$

for $j < i$.

$\mathcal{L}_{\mathfrak{C}} = \sum_i \mathcal{L}_{i,1}$ has right properties. $\square$

Example: $X = T^*\mathbb{P}^1$, $T = \mathbb{C}_u^\times \times \mathbb{C}_\hbar^\times$, $A = \mathbb{C}_u^\times$.

$\mathbb{C}_u^\times$ acts by $a \cdot [x, y] = [ax, y]$

$\mathbb{C}_\hbar^\times$ acts by -1 scaling on fibres

$\mathfrak{a}_{\mathbb{R}} = \mathbb{R}$, chambers $\mathfrak{C}_+ = \mathbb{R}_{>0}$, $\mathfrak{C}_- = \mathbb{R}_{<0}$

$X^A = \{\underset{[0,1]}{0}, \underset{[1,0]}{\infty}\}$ $\qquad \varepsilon = \pm e(N_-)$

Polarisation: $\varepsilon_0 = -u$, $\varepsilon_\infty = u$

Take $\mathfrak{C} = \mathfrak{C}_+$:

$\mathrm{Attr}_{\mathfrak{C}_+}(0) = \mathbb{P}^1 - \{\infty\}$, $\mathrm{Attr}_{\mathfrak{C}_+}(\infty) = T^*_\infty \mathbb{P}^1$

Label $Z_1 = \{\infty\}$, $Z_2 = \{0\}$.

$\mathcal{L}_{1,1} = -T^*_\infty \mathbb{P}^1 \times \infty$

$\mathcal{L}_{2,2} = \mathbb{P}^1 \times 0$

$\mathrm{Res}_{Z_1 \times Z_2}(\mathcal{L}_{2,2}) = ?$

$$i_\infty^* [\mathcal{L}_{2,2}] = \underbrace{u \cdot 1 = \varepsilon_\infty \cdot 1}_{\mathrm{Res}_{Z_1 \times Z_2}(\mathcal{L}_{2,2})}$$

$$\mathcal{L}_{2,1} = \mathcal{L}_{2,2} - \overline{(\lim_{\mathfrak{C}} \times \mathrm{id})^{-1}(-\mathrm{Res}_{Z_1 \times Z_2}(\mathcal{L}_{2,2}))}$$
$$= \mathbb{P}^1 \times 0 + T^*_\infty \mathbb{P}^1 \times 0$$

$$\mathcal{L}_{\mathfrak{C}} = \mathbb{P}^1 \times 0 + T^*_\infty \mathbb{P}^1 \times 0 - T^*_\infty \mathbb{P}^1 \times \infty.$$

§ R-matrices: Suppose $\mathfrak{C}', \mathfrak{C}^- \subset \mathfrak{a}_{\mathbb{R}}$ are chambers.

The R-matrix is

$$R_{\mathfrak{C}', \mathfrak{C}} = \mathrm{Stab}_{\mathfrak{C}'}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}} \in \mathrm{End}\left(H_T^\bullet(X^A)\right) \otimes \mathbb{Q}(Z)$$

Example: For $T^*\mathbb{P}^1$: $\mathcal{L}_{\mathfrak{C}_-} = \mathbb{P}^1 \times \infty + T_0^*\mathbb{P}^1 \times \infty - T_0^*\mathbb{P}^1 \times 0$

$$H_T^\bullet(X) \otimes \mathbb{Q}(Z) \xrightarrow[\substack{\text{restrict} \\ \text{to fixed pts}}]{\sim} \mathbb{Q}(u,\hbar)_0 \oplus \mathbb{Q}(u,\hbar)_\infty$$

$$[\mathbb{P}^1] \longmapsto \begin{pmatrix} -u-\hbar \\ u-\hbar \end{pmatrix}$$

$$[T_\infty^*\mathbb{P}^1] \longmapsto \begin{pmatrix} 0 \\ -u \end{pmatrix}$$

$$[T_0^*\mathbb{P}^1] \longmapsto \begin{pmatrix} u \\ 0 \end{pmatrix}$$

$\mathbb{C}_u^\times \circlearrowright \mathbb{P}^1$

$\mathbb{C}_\hbar^\times \circlearrowright T^*$

$$\mathrm{Stab}_{\mathfrak{C}_+} = \begin{pmatrix} -u-\hbar & 0 \\ -\hbar & u \end{pmatrix}, \quad \mathrm{Stab}_{\mathfrak{C}_-} = \begin{pmatrix} -u & -\hbar \\ 0 & u-\hbar \end{pmatrix}$$

$$R_{\mathfrak{C}_+, \mathfrak{C}_-} = \begin{pmatrix} -u-\hbar & 0 \\ -\hbar & u \end{pmatrix}^{-1} \begin{pmatrix} -u & -\hbar \\ 0 & u-\hbar \end{pmatrix} = \frac{1}{u+\hbar} \begin{pmatrix} u & \hbar \\ \hbar & u \end{pmatrix}.$$

Yang's R-matrix for $\mathfrak{sl}_2$

cf. Léa's talk.

§ Symplectic resolutions

Assume X is a symplectic resolution

i.e.,

$$X \longrightarrow \underset{\substack{\| \\ X_0}}{\mathrm{Spec}\, H^0(X, \mathcal{O}_X)} \quad \text{is proper and birational}$$

E.g. smooth Nakajima quiver varieties, T^*G/P.

Then (X, ω) has a semi-universal deformation

(X', ω') lives over $[\omega'] \in H^2(X', \mathbb{C}) = H^2(X, \mathbb{C})$.

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d] & \tilde{X} \arrow[d, "\phi"] \\ \mathrm{pt} \arrow[r, hook] & B = H^2(X, \mathbb{C}) \circlearrowleft G \text{ with weight } \hbar. \end{tikzcd}$$

E.g.
$$\begin{tikzcd} T^*G/B \arrow[r] \arrow[d] & \tilde{\mathfrak{g}} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathfrak{h} \end{tikzcd}$$
Grothendieck-Springer resolution

For $b \in B$, $(X', \omega') = \phi^{-1}(b)$ non-affine $\iff \exists\, \alpha \in H_2(X', \mathbb{Z}) = H_2(X, \mathbb{Z})$ effective curve class
$\Rightarrow \langle b, \alpha \rangle = \int_\alpha \omega' = 0$

So $\exists\, B^0 \subset B$ complement of 'coroot' hyperplanes such that $\tilde{X}^0 = \phi^{-1}(B^0) \to B^0$ is affine.

Construction: Let $\Delta^0 \subset (\tilde{X}^0)^A \times_{B^0} (\tilde{X}^0)^A$ be the diagonal

$\mathfrak{sl}_2$

singular quadric $\rightsquigarrow$ smooth quadric

By above $\mathrm{Attr}_{\mathfrak{C}}(\Delta^0) \subset \tilde{X}^0 \times_{B^0} (\tilde{X}^0)^A$ is closed

Define $\tilde{\mathcal{L}}_{\mathfrak{C}} = \overline{\mathrm{Attr}_{\mathfrak{C}}(\Delta^0)} \subset \tilde{X} \times_B \tilde{X}^A$.

Thm: $\mathcal{L}_{\mathfrak{C}}$ is the specialisation of $\tilde{\mathcal{L}}$ to X

i.e. $[\mathcal{L}_{\mathfrak{C}}] = i_0^*[\tilde{\mathcal{L}}_{\mathfrak{C}}] \in H_T^{BM}(X \times X^A)$.

Steinberg correspondences:

A <u>Steinberg correspondence</u> is a Lagrangian correspondence $L \subset X \times Y$ such that $\exists$ proper equivariant maps

$$
\begin{tikzcd}
X \arrow[r, "\pi_X"] & V & Y \arrow[l, "\pi_Y"'] \\
& \text{affine} \arrow[u] &
\end{tikzcd}
$$

with $L \subset X \times_V Y$.

<u>Prop^n</u>: If $b \in B$, then all top dimensional cpts of
$$(\vec{\mathfrak{Z}}_{\mathfrak{g}})^A \cap \phi^{-1}(b) \subset \phi^{-1}(b)^A \times \phi^{-1}(b)^A$$
are Steinberg correspondences.

$$
\begin{tikzcd}
& T^*G/N \circlearrowleft H \arrow[dl, "\mathrm{mod}\,H"'] \arrow[d, "\text{moment map}"] \\
\tilde{\mathfrak{g}} \arrow[r] & \mathfrak{h}
\end{tikzcd}
$$

Steinberg $\longleftrightarrow$ Bethe subalgebras