Yangians

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For a f.d. simple Lie algebra $\mathfrak{g}$, Drinfeld defined $Y(\mathfrak{g})$, Yangian, which is a Hopf algebra (in some sense) canonical deformation of $\mathcal{U}(\mathfrak{g}[x])$
(more precisely, $\hbar \mapsto 1$)

I will describe the case $\mathfrak{g} = \mathfrak{gl}_n$

I am following Molev (see refs)

1. Definition of $Y(\mathfrak{gl}_n) = Y(n)$

Consider the stand. basis E_{ij} of $\mathfrak{gl}_n$ $\left(i : \begin{array}{c} j \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \right)$

$$[E_{ij}, E_{kl}] = \delta_{k,j} E_{il} - \delta_{il} E_{kj}.$$

$$E = (E_{ij}) \qquad \begin{pmatrix} E_{11} & E_{12} & \cdots \\ E_{21} & E_{22} & \\ \vdots & & \ddots \end{pmatrix}$$

Useful fact:

$$g_s = tr(E^s) \in Z(\mathcal{U}(\mathfrak{gl}_n))$$

Proof of centrality uses the following:

$$[E_{ij}, (E^s)_{kl}] = \delta_{kj} (E^s)_{il} - \delta_{il} (E^s)_{kj}.$$

Proposition:

$$[(E^{r+1})_{ij}, (E^s)_{kl}] - [(E^r)_{ij}, (E^{s+1})_{kl}] = (E^r)_{kj} (E^s)_{il} - (E^s)_{kj} (E^r)_{il}$$

$$(E^0)_{ij} = \delta_{ij}.$$

Definition: The Yangian $Y(\mathfrak{gl}_n) = Y(n)$ is unital assoc. algebra w. generators $t_{ij}^{(r)}$, $r \in \mathbb{Z}_{\geq 0}$, $i,j = 1, \dots, n$, $t_{ij}^{(0)} = \delta_{ij}$

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}.$$

Remarks: 1) $t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r} = \delta_{ij} + t_{ij}^{(1)} u^{-1} + \dots$

$$(u-v)[t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u)$$

2) There is a homomorphism

$Y(n) \longrightarrow \mathcal{U}(\mathfrak{gl}_n)$, $t_{ij}^{(r)} \mapsto (E^r)_{ij}$ (special to type A)!

$\mathcal{U}(\mathfrak{gl}_n) \hookrightarrow Y(n)$, $E_{ij} \mapsto t_{ij}^{(1)}$

2. Matrix form of relations

Write $T(u)$ for a matrix $(t_{ij}(u))$

$T(u) \in Y(n)[[u^{-1}]] \otimes \mathrm{End}(\mathbb{C}^n)$, $T(u) = \sum t_{ij} \otimes e_{ij}$

Define: $P = \sum e_{ij} \otimes e_{ji} \in \mathrm{End}(\mathbb{C}^n)^{\otimes 2}$ (~~Killing~~ form)

$$R(u) = \mathrm{Id} - P u^{-1} \quad \text{– Yang's } R\text{-matrix}$$

R satisfies

$$R_{12}(u) R_{13}(u+v) R_{23}(v) = R_{23}(v) R_{13}(u+v) R_{12}(u)$$

(QYBE w. spectral parameter)

Proposition: relations of $Y(n)$ are given by

$$R(u-v)\, T_1(u) T_2(v) = T_2(v)\, T_1(u)\, R(u-v)$$

$$T_i(u) \in Y(n)[[u^{-1}]] \otimes \mathrm{End}(\mathbb{C}^n)^{\otimes 2}$$

$$T_1(u) = \sum t_{ij}(u) \otimes e_{ij} \otimes 1$$

$$T_2(u) = \sum t_{ij}(u) \otimes 1 \otimes e_{ij}.$$

Exercise: 1) check what set of relations is obtained

2) a) $T^{-1}(u)$ also satisfies RTT relations ($T \mapsto T^{-1}$ gives an autom. of the Yangian) (up to sign changes)

b) $t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1}$ — gives another homomorphism $Y(n) \to U(\mathfrak{gl}_n)$.

3. $Y(n)$ is a Hopf algebra

Theorem: $Y(n)$ is a Hopf algebra

$$\Delta: t_{ij}(u) \mapsto \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u)$$

$$S: T(u) \mapsto T^{-1}(u)$$

$$\varepsilon: T(u) \longmapsto 1$$

Proof: will just prove Δ is a homomorphism.

$$\Delta(T(u)) = T_{[1]}(u)\, T_{[2]}(u) = \Big(\sum t_{ij} \otimes 1 \otimes e_{ij}\Big)\Big(\sum 1 \otimes t_{ij} \otimes e_{ij}\Big)$$

$$\cap \quad Y(n)^{\otimes 2}[[u^{-1}]] \otimes \mathrm{End}(\mathbb{C}^n)$$

$$\underset{i^{th}}{T_{[i]j}} \in Y(n)^{\otimes 2} \otimes \underset{j^{th}}{\mathrm{End}(\mathbb{C}^2)^{\otimes 2}}[[u^{-1}, v^{-1}]]$$

$$T_{[1]2}(u) = \sum t_{ij}(u) \otimes 1 \otimes 1 \otimes e_{ij}$$

Δ – homomorphism $\iff$

$$R(u-v)\, T_{[1]1}(u)\, T_{[2]1}(u)\;\; T_{[1]2}(v)\, T_{[2]2}(v)$$

$$= T_{[1]2}(v)\, T_{[2]2}(v)\;\; T_{[1]1}(u)\, T_{[2]1}(u)\, R(u-v)$$

This follows from RTT & $T_{[1]1}$ commutes w. $T_{[2]2}$

$T_{[1]2}$ —"— $T_{[2]1}$

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4. Let $\deg t_{ij}^{(r)} = r-1$ — defines a filtration on $Y(n)$.

Exercise: $\mathrm{gr}\, Y(n) \longrightarrow U(\mathfrak{gl}_n[x])$

$$\bar{t}_{ij}^{(r)} \longmapsto E_{ij} x^r$$

image of $t_{ij}^{(r)}$ in the $(r-1)^{st}$ graded piece

(PBW for $Y(n) \Rightarrow$ an isomorphism)

5. There is family of commutative subalgebras in $Y(n)$ indexed by $C \in GL(n)$ (called Bethe subalgebras)

$$\tau_k(u, C) = \mathrm{tr}\, \underbrace{A_n T_1(u) T_2(u-1) \cdots T_k(u-k+1) C_{k+1} \cdots C_n}_{\in\, Y(n) \otimes \mathrm{End}(\mathbb{C}^n)^{\otimes n}[[u^{-1}]]}$$

A_n is a projector to the sign rep. of S_n in $(\mathbb{C}^n)^{\otimes n}$.

Theorem: coeffs. of $\tau_k(u, C)$, $k = 1, \dots, n$ commute w. each other.

If C is regular semisimple, define maximal comm. subalg. of $Y(n)$.

What about $\mathfrak{g} \neq \mathfrak{g}/\mathfrak{h}$?

Motivation

Def $\mathfrak{g}$ - Lie algebra & co-Lie bracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$

$(\mathfrak{g}, \delta)$ is a <u>Lie bialgebra</u> if δ is a 1-cocycle, i.e.

$$\delta([X,Y]) = (ad_X \otimes 1 + 1 \otimes ad_X)\delta(Y) - (ad_Y \otimes 1 + 1 \otimes ad_Y)\delta X.$$

Example: 1) $\mathfrak{g}$ - semisimple $\quad \mathfrak{b}_+, \mathfrak{b}_- \subset \mathfrak{g}$ $\qquad \mathfrak{b}_+ \xrightarrow{\pi} \mathfrak{h} \xleftarrow{\pi} \mathfrak{b}_-$

$\mathfrak{g}' = \{(X,Y) : \pi(X) = -\pi(Y)\} \subset \mathfrak{b}_- \times \mathfrak{b}_+$ is a Lie subalgebra

& $\mathfrak{g}' \simeq \mathfrak{g}^*$ via $\langle (X,Y), Z \rangle = (Y - X, Z)$

$\rightsquigarrow$ compatible δ.

2) $\mathfrak{g}[u] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[u]$ $\qquad \delta : f \longmapsto (ad_{f(u)} \otimes id + id \otimes ad_{f(v)}) \left(\frac{t}{u-v}\right)$

Casimir element

Drinfeld expected (in 80s) that any $(\mathfrak{g}, \delta)$ admitted a <u>quantization</u>:

- $\to$ $\mathbb{C}[[\hbar]]$ - linear bialg. $U_\hbar(\mathfrak{g})$ s.t.
- $\to$ $U_\hbar(\mathfrak{g}) \simeq U(\mathfrak{g})[[\hbar]]$ as $\mathbb{C}[[\hbar]]$-modules
- $\to$ $\mu_\hbar = \mu$, $\Delta_\hbar = \Delta$ mod $\hbar = 0$.

Proved by Etingof-Kazhdan in the 90s. $\qquad$ Example 2 $\rightsquigarrow Y(\mathfrak{g})$.

1 | Two presentations

Def (J-presentation)

$U_\hbar(\mathfrak{g}[u])$ gen. by $x \in \mathfrak{g}$, $J(x)$, $x \in \mathfrak{g}$

modulo rels:
1) $\mathfrak{g}$ Lie subalgebra
2) J is $\mathbb{C}[[\hbar]]$-linear : $J(ax + by) = aJ(x) + bJ(y)$.
3) $[x, J(y)] = J([x,y])$.

4) $$[J(x), J([y,z])] + [J(z), J([x,y])] + [J(y), J([z,x])]$$
$$\| $$
$$\frac{1}{24}\hbar^2 \sum_{\lambda,\mu,\nu} \left([x, x_\lambda], [[y, x_\mu], [z, x_\nu]]\right) \{x_\lambda, x_\mu, x_\nu\}$$

$x_\lambda x_\mu x_\nu + x_\mu x_\lambda x_\nu + \ldots$ 6 terms

$\{x_\lambda\}$ - orthonorm. basis of $\mathfrak{g}$

Coproduct:
$$\Delta(x) = x \otimes 1 + 1 \otimes x$$
$$\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \tfrac{1}{2}\hbar [x \otimes 1, t]$$

Rmk 1) This quantization is unique.

2) $\mathcal{U}_\hbar \to \mathcal{U}_{\hbar'}$ $\quad$ $x \mapsto x$, $J(x) \mapsto \frac{\hbar'}{\hbar} J(x)$ $\quad$ is an iso for $\hbar, \hbar' \neq 0$ $\quad \Rightarrow \hbar = 1$.

<u>Def</u> (Current presentation)

$$\overset{\mathcal{U}_1(\mathfrak{g})}{\overset{\shortparallel}{Y(\mathfrak{g})}} \simeq \langle X^{\pm}_{i,r}, H_{i,r}, \; i=1,\dots,\mathrm{rk}\,\mathfrak{g}, \; r \in \mathbb{Z}_{\geq 0} \rangle / \text{relations}$$

$$X^{\pm}_i(u) = \sum_{r\geq 0} X^{\pm}_{i,r} u^{-r-1} \qquad H_i(u) = 1 + \sum_{r\geq 0} H_{i,r} u^{-r-1}$$

- $[H_i(u), H_j(v)] = 0$
- $[X^+_i(u), X^-_j(v)] = \delta_{ij} \dfrac{H_i(u) - H_i(v)}{v-u}$
- $[H_i(u), X^{\pm}_j(v)] = \pm \dfrac{c_{ij}}{2} \dfrac{\{H_i(u), X^{\pm}_j(u) - X^{\pm}_j(v)\}}{u-v}$ $\qquad \{u,v\} = uv + vu$

Cartan matrix

- $[X^{\pm}_i(u), X^{\pm}_j(v)] = \pm \dfrac{c_{ij}}{2} \dfrac{(X^{\pm}_j(u) - X^{\pm}_j(v))^2}{u-v}$ $\quad$ + Serre relations $\quad$ $[X_i(u),[X_i(u), \dots [X_i(u), X_j(v)] = 0$ $\quad {}_{1-a_{ij}}$

$$\varphi(E_i) = X^+_{i,0} \qquad \varphi(J(H_i)) = H_{i,1} + \dots$$
$$\varphi(F_i) = X^-_{i,0} \qquad \varphi(J(E_i)) = X^+_{i,1} + \dots$$
$$\varphi(H_i) = H_{i,0}$$

<u>Advantages</u>: $\quad J$: has coproduct (explicit)

Current : · better suited to study fin. dim reps

· easier to see PBW basis. (monomials in generators).

<u>Prop</u> $\quad \exists$ 1-param family of automorphisms of $Y(\mathfrak{g})$ given by

$$\tau_a (H_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} H_{i,s} \qquad a \in \mathbb{C}$$

$$\tau_a (X^{\pm}_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} X^{\pm}_{i,s}$$

<u>Pf</u> $\quad$ In J-presentation $\quad \tau_a(x) = x$, $\quad \tau_a(J(x)) = J(x) + ax$ $\quad$ is obviously an automorphism.

The formulas above are derived by induction, using expr. of $X^{\pm}_{i,r+1}$ in terms of $H_{i,r}$, $X^{\pm}_{i,r}$

& $H_{i,r+1} = [X^+_{i,r+1}, X^-_{i,0}]$. $\qquad \square$

<u>2| Quantum Yang-Baxter</u> (with spectral params)

$$R_{12}(\lambda_1-\lambda_2)\,R_{13}(\lambda_1-\lambda_3)\,R_{23}(\lambda_2-\lambda_3) = R_{23}(\lambda_2-\lambda_3)\,R_{13}(\lambda_1-\lambda_3)\,R_{12}(\lambda_1-\lambda_2). \qquad (*)$$

<u>Thm</u> 1 (Drinfeld) $\exists\, R(\lambda) = 1\otimes 1 + \frac{t}{\lambda} + \sum_{r\geq 1} R_r \lambda^{-r-1} \in Y(\mathfrak{g})\,\widehat{\otimes}\, Y(\mathfrak{g})[[\lambda^{-1}]]$

which satisfies $(*)$

<u>Thm 2</u> $R(\lambda)|_\rho$ is rational in λ. $\Rightarrow$ almost always makes sense.

<u>Prop</u> $\{V(\lambda)\}$ f.d. reps on the same v.space V

1) $\forall \lambda,\mu$ $\exists\, I(\lambda,\mu) : V(\lambda)\otimes V(\mu) \xrightarrow{\sim} V(\mu)\otimes V(\lambda)$

2) $\mathrm{Aut}(V(\lambda)\otimes V(\mu)\otimes V(\nu)) = $ scalars.

$\Rightarrow R = \mathrm{flip}\circ I$ satisfies

$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu) = \dots$

For Yangians, take $V(a) = \tau_a \circ \rho$ $\rightsquigarrow$ R-matrix compatible with Thm 1.

$\rho: Y(\mathfrak{g}) \to \mathrm{End}(V)$
$\uparrow$ fin.dim.