Nakajima Quiver varieties
16 March 2022 14:58

Following: Ginzburg notes

Quiver set up

Q directed graph $\qquad$ Q_0 vertex set
$\qquad\qquad\qquad\qquad\qquad\quad$ Q_1 arrow set

$$\begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

Representations:

$$V = \bigoplus_{i \in Q_0} V_i \qquad i \to j \in Q_1$$
$$V_i \to V_j$$

dimension vector $v = (v_i)_{i \in Q_0}$ $\quad v \in \mathbb{Z}_{\geq 0}^{Q_0}$

$Q = \bullet \to \bullet$ $\quad v = (n, m)$ $\quad \mathrm{Rep}(Q, v) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$

$$G_v = \prod GL_{v_i}$$

$$G_v \circlearrowright \mathrm{Rep}(Q, v)$$

$$Q = \begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop right] \end{tikzcd} \qquad G_v = GL_n \times GL_m$$

$v = (n, m)$

$$(g_1, g_2) \cdot \left(\begin{tikzcd} \mathbb{C}^n \arrow[r, shift left, "M_1"] & \mathbb{C}^m \arrow[l, shift left, "M_2"] \arrow[loop right, "M_3"] \end{tikzcd} \right) =$$

$$= \left(\mathbb{C}^n \underset{g_1 M_2 g_2^{-1}}{\overset{g_2 M_1 g_1^{-1}}{\rightleftarrows}} \mathbb{C}^m \circlearrowleft g_2 M_3 g_2^{-1} \right)$$

$\mathrm{Rep}(Q, v) / G_v$ gives isomorphism classes of quiver representations

Symplectic structure

X smooth variety / mfld

T^*X has a symplectic structure

$\omega = -d\alpha$

$X = V$

$T^*V = V \oplus V^*$

$\omega((x_1, y_1), (x_2, y_2)) = \langle y_1, x_2 \rangle - \langle y_2, x_1 \rangle$

$G \circlearrowright X$

$G \circlearrowright T^*X$

Moment map:

$\mu: T^*X \to \mathfrak{g}^*$

$\circlearrowleft$ $\quad\quad\quad\quad\quad\quad$ $\circlearrowleft$ coadjoint

$\circlearrowleft$ $\qquad\qquad$ $\circlearrowleft$ coadjoint
G $\qquad\qquad$ G

$$(x,\lambda) \mapsto \begin{array}{l} \mathfrak{g} \to K \\ a \mapsto \langle \lambda, \xi_a^{(x)} \rangle \end{array}$$

Double Quiver

$$T^*\mathrm{Rep}(Q,v) = \mathrm{Rep}(Q,v) \oplus \mathrm{Rep}(Q,v)^*$$

Trace form:

$$\mathrm{Hom}(V,W) \times \mathrm{Hom}(W,V) \to K$$
$$(f,f') \mapsto \mathrm{tr}(f \circ f')$$

perfect pairing so get

$$\mathrm{Hom}(V,W) \to \mathrm{Hom}(W,V)^*$$
$$f \mapsto \mathrm{tr}(- \circ f)$$

$$T^*\mathrm{Rep}(Q,v) = \mathrm{Rep}(Q,v) \oplus \mathrm{Rep}(Q^{op},v)$$

Double quiver $\overline{Q}$ $\qquad$ same vertex set
arrows from Q and Q^{op}

$Q \quad \bullet \longrightarrow \bullet \quad \rightsquigarrow \quad \overline{Q} = \bullet \rightleftarrows \bullet$

$$T^*\mathrm{Rep}(Q,v) = \mathrm{Rep}(\overline{Q},v)$$

$$T^* \mathrm{Rep}(Q, v) = \mathrm{Rep}(\bar{Q}, v)$$

Hamiltonian reduction

$G \underset{\text{free}}{\circlearrowright} X$ smooth affine variety

$G \circlearrowright T^* X$

$\mu : T^* X \to \mathfrak{g}^*$

Prop:
1) $\mu^{-1}(\lambda)/G$ this is smooth and symplectic
$\lambda \in (\mathfrak{g}^*)^G$

2) $T^*(X/G) \cong \mu^{-1}(0)/G$

Will have to use stability and GIT

Impose a stability condition:

$\chi : G \to \mathbb{C}^\times$

$$\begin{tikzcd}
X^{ss} \arrow[r] \arrow[d, hook] & X/\!/_{\chi} G \arrow[d, "\text{projective morphism}"] \\
X \arrow[r] & X/G
\end{tikzcd}$$

In quiver case

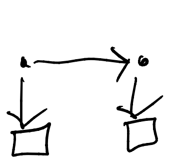
$\chi \leftrightarrow \Theta \in \mathbb{Z}^{Q_0}$

Stability for framed/doubled quivers

Framings

$Q^\heartsuit$ vertex set Q_0 extra vertex i' for each $i \in Q_0$

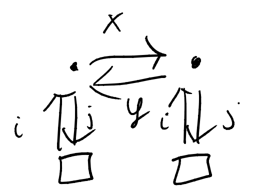
arrows Q_1 extra arrow $i \to i'$

$Q = \bullet \longrightarrow \bullet$ $\qquad Q^\heartsuit =$ 

v dim vector of original vertices

w dim vector of framing

$\mathrm{Rep}(Q^\heartsuit, v, w)$

$\overline{Q^\heartsuit} =$ 

$$\begin{array}{cccc} (x, & y, & i, & j) \\ Q & Q^{op} & Q^{op} & Q \\ & & \text{framing} & \text{framing} \end{array}$$

$G_v \curvearrowright \mathrm{Rep}(\overline{Q^\heartsuit}, v, w)$

$g \cdot (x, y, i, j) = (gxg^{-1}, gyg^{-1}, gi, jg^{-1})$

$\mu : \mathrm{Rep}(\overline{Q^\heartsuit}, v, w) \to \mathfrak{g}^* \underset{\text{trace form}}{\cong} \mathfrak{g}$ $\qquad G_v = \prod GL_{v_i}$, $\quad \mathfrak{g} = \bigoplus \mathfrak{gl}_{v_i}$

$(x, y, i, j) \mapsto [x, y] - ij$

Stability for framed quivers

Pick $\Theta \in \mathbb{Z}^{Q_0}$ $\quad (X, Y, i, j)$ Θ-semistable

$Q = \bullet$

$\dim S \leq 0$

$\Rightarrow \ker j = 0$

$\iff \quad \forall\, S \subset V$

$S \subset \ker j \Rightarrow \Theta(\dim S) \leq 0$

$S \supset \operatorname{im} i \Rightarrow \Theta(\dim S) \leq \Theta(\dim\ )$

Definition

Q quiver, $\quad V, W$ dim vector $\quad \Theta \in \mathbb{Z}^{Q_0}$ $\quad \lambda \in (\mathfrak{g})^{G_V}$

$\lambda \in \mathbb{C}^{Q_0}$

$$\mathcal{M}_{\Theta,\lambda}(v,w) = \mu^{-1}(\lambda) /\!/_{\Theta}\, G_V$$

Set $\lambda = 0$ from now on and denote by $\mathcal{M}_\Theta(v,w)$

Example

$Q = \bullet$ $\qquad \overline{Q^\heartsuit} = \begin{tikzcd} \bullet \arrow[d, shift left] \\ \square \arrow[u, shift left] \end{tikzcd}$ $\qquad G_V = GL_V$

$V, W \in \mathbb{Z}_{\geq 0}$

$$\operatorname{Rep}(\overline{Q^\heartsuit}, v, w) \to \mathfrak{gl}_V$$

$$(i,j) \mapsto -ij$$

$$\mu^{-1}(0) = \{(i,j) \mid ij = 0\}$$

$\theta = 1$ gives j is injective

$$\begin{tikzcd} V \arrow[d, shift left, "j"] \\ W \arrow[u, shift left, "i"] \end{tikzcd}$$

$ij = 0$

$$i|_V = 0$$

get element $\mathrm{Hom}(W/V, V)$

$$\underset{v}{V} \hookrightarrow \underset{w}{W}$$

get a point in $T^*Gr(v, w)$

if $v > w$ then $\mathcal{M}_\theta(v, w) = \emptyset$

$v = 1 \quad w = 2$

$\theta = 1 \quad T^*\mathbb{P}^1$

$\theta = 0$

$\mathcal{M}_0(1,2) \cong \mathcal{N}$ $\quad 2\times 2$ nilpotent matrices

$\mathcal{M}_1(1,2) \to \mathcal{M}_0(1,2)$

$T^*\mathbb{P}^1 \to \mathcal{N}$ this gives springer resolution for sl_2

This generalises

$Q = A_n$

$$\overline{Q}^{\heartsuit} = \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left] \arrow[d, shift left] & \bullet \arrow[l, shift left] \arrow[d, shift left] \arrow[r, phantom, "\cdots"] & {} \arrow[r, shift left] & \overset{n}{\bullet} \arrow[l, shift left] \arrow[d, shift left] \\ \square \arrow[u, shift left] & \square \arrow[u, shift left] & & \square \arrow[u, shift left] \end{tikzcd}$$

$v = (v_1, \cdots v_n)$
$w = (0, \ldots, r)$

$\theta^+ = (1, \cdots 1)$

(X, y, i, j) is θ-ss.

any chain of compositions

$j_n \circ X_n \circ \cdots X_u \quad u \geq 1$

is injective

This defines a partial flag

The moment map equation gives

a map $y : \mathbb{C}^r \to \mathbb{C}^r$

$y(V_k) \subset V_{k-1}$

$\mathcal{M}_{\theta^+}(v, w) = \{F, y\} \cong T^* FL_p$

$v = (1, \cdots n)$
$w = (0, \ldots \ldots n+1)$

Then we get the Springer resolution for sl_n

Generic θ this is defined using euler form for Q

θ^+, θ^- are generic

Prop : $\mathcal{M}_\theta(v, w) \neq \emptyset$

1) G_v acts freely on $\mu^{-1}(0)^s$

$\mu^{-1}(0)^s = \mu^{-1}(0)^{ss}$

$\mathcal{M}_\theta(v,w)$ is smooth and symplectic

2) $\mathcal{M}_\theta(v,w) \to \mathcal{M}_0(v,w)$

is a symplectic resolution

Hilbert scheme of pts

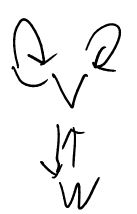
$X = \mathbb{C}^2$

$$\mathrm{Hilb}^n(X) = \left\{ I \triangleleft \mathbb{C}[x,y] \;\middle|\; \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \right\}$$

equivalently

$$\left\{ (M,v) \;\middle|\; \begin{array}{l} M \text{ is a } \mathbb{C}[x,y] \text{ module} \\ v \text{ is a cyclic vector} \end{array} \right\} / \text{iso}$$

$$S^n\mathbb{C}^2 = \mathrm{Spec}\left(\mathbb{C}[x,y]^{\otimes n}\right)^{S_n}$$

$Q =$  $\overline{Q^\heartsuit} =$  

dim n

dim 1

$V = n \quad W = 1$

$\theta = -1$

Prop :

We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_{-1}(n,1) \arrow[r] \arrow[d, "\cong"'] & \mathcal{M}_0(n,1) \arrow[d, "\cong"] \\
\mathrm{Hilb}^n(\mathbb{C}^2) \arrow[r] & S^n\mathbb{C}^2
\end{tikzcd}$$

Hilbert-Chow morphism

Will only explain why $\mathcal{M}_{-1}(n,1) \cong \mathrm{Hilb}^n(\mathbb{C}^2)$

$\mu(X,Y,i,j) = [X,Y] - ij$

Claim:

$$\mu^{-1}(0)^{ss} = \left\{ (X,Y,i,j) \;\middle|\; \begin{array}{l} i \text{ is a cyclic vector for } V \text{ under the action of } \mathbb{C}\langle x,y\rangle \\ j = 0 \\ [X,Y] = 0 \end{array} \right\}$$

$[X,Y] = ij$

<u>Lemma</u>:

If X, Y are such that $\mathrm{rank}[X,Y] \leq 1$ then they can simultaneously be put into upper triangular form

1) i generates:

$\theta = -1$

$S \subset V$

$\mathrm{im}\, i \subset S$ then $S = V$

$\mathbb{C}\langle x,y\rangle i \neq V$

so we must have $V = \mathbb{C}\langle x,y\rangle i$

2) $j = 0$

$v \in V$

$v = \sum_K M_K i$ $\qquad$ M_K is a product of x, y

Let's consider $\quad jM_K i = tr(jM_K i) = tr(M i j)$

$= tr(M_K [x,y]) = 0$

$\Rightarrow jM_K i = 0$

$j = 0$

$\mu^{-1}(0)^{ss} \to Hilb^n(\mathbb{C}^2)$

$(x, y, i, j) \mapsto (V, i)$ $\qquad$ $[x,y] = 0$

If we quotient out by G_v action will get a bijection.

Torus fixed points

Fix Q $\quad v, w$ $\quad \theta$ - generic

Consider G_w action

Fix a splitting of w

$w = w' + w''$

$Q = \bullet$

$Q^{\heartsuit} = \begin{array}{c} \bullet \\ \downarrow\uparrow \\ \square \end{array} \quad \begin{array}{c} GL_v \\ \\ GL_w \end{array}$

$w = w' + w''$

θ - $\bullet \longrightarrow \bullet$

$w = w' + w''$

$\mathbb{C}^\times \cong A \subset G_w$

acts on $W = \bigoplus \mathbb{C}^{w_i} = \bigoplus \mathbb{C}^{w_i'} \oplus \mathbb{C}^{w_i''}$

$\mathbb{C}^{w_i'}$ acts by mult t

$\mathbb{C}^{w_i''}$ acts by identity

$Q = \bullet \longrightarrow \bullet$

$W = (w_1, w_2)$

$w'(w_1', w_2')$ $\quad w_1' + w_1'' = w_1$

$w''(w_1'', w_2'')$ $\quad w_1'' + w_2'' = w_2$

<u>Claim</u>: $\displaystyle M_\theta^A(v,w) = \bigsqcup_{v'+v''=v} M_\theta(v',w') \times M_\theta(v'',w'')$

$x \in M_\theta^A(v,w)$ $\qquad \hat{x} \in \mathrm{Rep}(\overline{Q}^{\heartsuit}, v, w)$

$\hat{x} \in X$

define $G^x \subset G_V \times G_W$ as the subgroup

that fixes the orbit.

$a \in A$

$a \cdot \hat{x} = g \cdot \hat{x}$ $\qquad g \in G_V$

There is an exact sequence

$1 \to G_V \to G^x \to A \to 1$

And this splits into a semi direct product

and we get a homomorphism

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[rr, bend right, "id"'] & G^x \arrow[r] & A \end{tikzcd}$$

$a \mapsto (g^{-1}, a)$

$\underbrace{\qquad\qquad}_{id}$ $\quad$ '' $(0, \sim)$

$\phi(A)$ to fix $\tilde{X}$

$\tilde{X}$ becomes an A-module

$\tilde{X}$ splits into a direct sum

So can view $\tilde{X}$ as a repr of

$\overline{Q^{\heartsuit}} \times A^*$ $\quad$ A^* characters of A $\quad$ $A^* \cong \mathbb{Z}$

Example:

$Q = \bullet$ $\qquad$ $\overline{Q^{\heartsuit}} = \begin{smallmatrix}\bullet \\ \downarrow\uparrow \\ \square\end{smallmatrix}$ $\quad$ v, w

$V \rightleftarrows W$ $\qquad$ $w = w' + w''$

$\bigoplus V_{z^n} \rightleftarrows \bigoplus W_{z^n} = W_z \oplus W_1$

$\bullet \rightleftarrows \square$ $\quad z^2$

$\bullet \rightleftarrows \square$ $\quad z$

$\bullet \rightleftarrows \square$ $\quad 1$

$\bullet \rightleftarrows \square$ $\quad z^{-1}$

$\bullet \rightleftarrows \square$ $\quad z^{-2}$

Arbitrary splitting will be

$V_{z^2} \rightleftarrows 0$

$V_z \rightleftarrows W_z = \mathbb{C}^{w'}$ $\qquad$ this is empty

$V_1 \rightleftarrows W_1 = \mathbb{C}^{w''}$

$$V_1 \rightleftarrows W_1 = \mathbb{C}^{w''} \qquad \text{this is empty}$$

$$\begin{array}{ccc} V_{z^1} & \rightleftarrows & 0 \\ & & 0 \end{array}$$

$$V_{z^n} = 0 \qquad n \neq 0 \text{ or } 1$$

The relevant splittings are

$$\begin{array}{ccc} 0 & \rightleftarrows & 0 \\ 0 & \rightleftarrows & 0 \\ V_{z'} & \rightleftarrows & W_{z'} \\ V_1 & \rightleftarrows & W_1 \\ 0 & \rightleftarrows & 0 \\ 0 & \rightleftarrows & 0 \end{array} \qquad \coprod_{v'+v''=v} \mathcal{M}_\theta(v',w') \times \mathcal{M}_\theta(v'',w'')$$

$$\theta = 1$$

$$\mathcal{M}_1^A(v,w) = \coprod_{\substack{v'+v''=v \\ v' \le w' \\ v'' \le w''}} T^*Gr(v',w') \times T^*Gr(v'',w'')$$