10 Yangian for Cotangent Bundle of Grassmannian (Shivang)

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Goal:

- $Y(gl_2)$ and it's Evaluation representation
- Nakajima Quiver Variety for a κ, (Fixed points).
- Stable Envelopes
- R-matrices $\rightsquigarrow$ Yangian $\rightsquigarrow$ $Y(gl_2)$
- quantum determinant for $Y(gl_2)$ $\rightsquigarrow$ $Z(Y(gl_2))$
- Core-Yangian Conductor.

1.) $Y(gl_2)$

$T_{ij}^{(k)}$ for $k \geq 1$, $i,j \in \{1,2\}$.

$$\left[T_{ij}^{(r)}, T_{k,l}^{(s)}\right] = \sum_{a=1}^{\min(r,s)} T_{kj}^{a-1} T_{i,l}^{r+s-a} - T_{k,j}^{r+s-a} T_{i,l}^{a-1} \qquad \text{———} \quad (1)$$

$$T_{ij}(u) = \delta_{ij} + \sum_{k \geq 1} T_{ij}^{(k)} u^{-k} \quad \in \quad Y(gl_2)[u^{-1}]$$

$$(u-v)\left[T_{ij}(u), T_{k,l}(v)\right] = T_{kj}(u)\,T_{i,l}(v) - T_{k,j}(v)\,T_{i,l}(u) \qquad (1)'$$

$\mathbb{Q}^2$

$$T(u) = \begin{bmatrix} T_{11}(u) & T_{12}(u) \\ T_{21}(u) & T_{22}(u) \end{bmatrix} \in \underline{\mathrm{End}(\mathbb{Q}^2) \otimes Y(\mathfrak{gl}_2)[[u^{-1}]]}$$

$\mathbb{Q}^2$

$T(u) \otimes Id \otimes -$

$$T_1(u) \in \mathrm{End}(\mathbb{Q}^2) \otimes \mathrm{End}(\mathbb{Q}^2) \otimes Y(\mathfrak{gl}_2)[[u^{-1}]]$$

$$T_2(u) \in$$

$Id \otimes T(u) \otimes -$

$$R(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u}{u-1} & \frac{-1}{u-1} & 0 \\ 0 & \frac{-1}{u-1} & \frac{u}{u-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v)$$ — 'RTT' relation $\in \mathrm{End}(\mathbb{Q}^2) \otimes \mathrm{End}(\mathbb{Q}^2) \otimes Y(\mathfrak{gl}_2)[[u^{-1}, v^{-1}]]$

$$Y(\mathfrak{gl}_2) \xrightarrow{\sim} Y(\mathfrak{gl}_2)$$
$$T(u) \longmapsto f(u)\,T(u)$$

where $f(u) = 1 + \sum_{k>0} a_k u^{-k} \in \mathbb{Q}[[u^{-1}]]$.

$\frac{1}{1-1/u} \rightsquigarrow$

Evaluation Representation

$Y(\mathfrak{gl}_2)$ $\qquad \deg(T_{ij}^{k}) = k-1$

$$\rightsquigarrow \operatorname{gr} Y(\mathfrak{gl}_2) = \mathcal{U}(\mathfrak{gl}_2[z]) .$$

$$\mathfrak{gl}_2 \hookrightarrow \mathbb{Q}^2$$
$$\text{End}(\mathbb{Q}^2).$$

$$Y(\mathfrak{gl}_2) \xrightarrow{\rho} \text{End}(\mathbb{Q}^2) \qquad \rightsquigarrow \qquad \text{End}(\mathbb{Q}^2)\otimes\text{End}(\mathbb{Q}^2) \xrightarrow{R_{12}} \text{End}(\mathbb{Q}^2)\otimes\text{End}(\mathbb{Q}^2)$$

$$T_{ij}(u) \longmapsto \frac{\delta_{ij} - e_{ji}u^{-1}}{1-\frac{1}{u}}$$

$$T(u) \longmapsto R(u) \in \underline{\text{End}(\mathbb{Q}^2)\otimes\text{End}(\mathbb{Q}^2)} \otimes^{-1}$$

$$\begin{bmatrix} T_{11}(u) & T_{12}(u) \\ T_{21}(u) & T_{22}(u) \end{bmatrix} \longmapsto \begin{bmatrix} \begin{bmatrix} \frac{1-1/u}{1-1/u} & 0 \\ 0 & \frac{1}{1-1/u} \end{bmatrix} & \\ & \end{bmatrix} = R(u)$$

Since $R(u-v)\,T_1(u)\,T_2(v) \longmapsto R(u-v)\,R(u)\,R(v) = R(v)\,R(u)\,R(u-v)$

$$Y(gl_2) \xrightarrow{-a} Y(gl_2) \xrightarrow{\rho} End(\mathbb{Q}^2). \quad \text{for any } a \in \mathbb{Q}$$

$$T(u) \longmapsto T(u-a)$$

u

$$\rho_a : Y(gl_2) \longrightarrow End(\mathbb{Q}^2)$$

$$T_{ij}(u) \longmapsto \delta_{ij} - e_{ji}(u-a)^{-1}$$

$$\underbrace{1 - \frac{1}{u-a}}$$

$$\rho_a' : Y(gl_2) \longrightarrow End(\mathbb{Q}^2)$$

$$T_{ij}(u) \longmapsto \delta_{ij} - e_{ji}(u-a)^{-1}$$

$$T_{ij}^{(k)} \longmapsto -a^{k-1} e_{ji}$$

$U(gl_2[u])$

$$U(gl_2[u]) \xrightarrow{\sim} gr.\, Y(gl_2)$$

$$u^{r-1} E_{ij} \longmapsto \bar{T}_{ij}^{(r)}.$$

$$\bar{\rho}_a' : U(gl_2[u]) \longrightarrow End(\mathbb{Q}^2)$$

$a = a_1, a_2 \ldots a_n$

$$\tilde{P}'_{a_1 a_2 \cdots a_n} : U(gl_2[u]) \longrightarrow \underbrace{End(\mathbb{Q}^2) \otimes \cdots \otimes End(\mathbb{Q}^2)}_{n\text{-tim.}}$$

$$\underline{P_{a_1, a_2 \cdots a_n}} : Y(gl_2) \longrightarrow End(\mathbb{Q}^2) \otimes \cdots \otimes End(\mathbb{Q}^2)$$

$$T(u) \longmapsto R_{1,2}(u-a_1)\, R_{1,3}(u-a_2) \cdots R_{1,n+1}(u-a_n)$$

$\underline{\text{Claim:}}$ $n, a_1, a_2 \cdots a_n$, $\bigcap \ker(P_{a_1 a_2 \cdots a_n}) = 0$

$g \in Y(gl_2)$ $\exists\ P_{a_1 a_2 \cdots a_n}$ $P_{a_1 \cdots a_n}(g) \neq 0$

$P_{a_i} = \mathbb{Q}^2(a_i)$

$\tilde{P}_{a_1 a_2 \cdots a_n}$

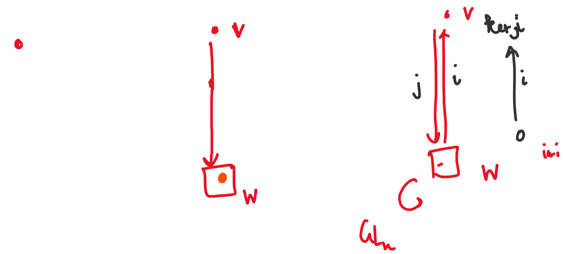
$\underline{U(gl_2[u])}$ $\bar{g} = \sum (\cancel{g \otimes \cdots g_k})(u-a_1)^{k_1}(u-a_2)^{k_2} \cdots (u-a_n)^{k_n}$

$\cancel{\otimes}$ $\underline{P_{\lambda_1 \lambda_2 \cdots \lambda_n}}$

$$Y(gl_2) \subset \prod_{a_1 \cdots a_n} End\left(\mathbb{Q}^2(a_1) \otimes \cdots \mathbb{Q}^2(a_n)\right)$$

2.) Nakajima Variety of a Point:

$$Q \rightsquigarrow Q_f \rightsquigarrow \overline{Q}_f$$



$$G_V \hookrightarrow \mathrm{Rep}(\overline{Q}_f, \overset{k}{V}, \overset{n}{W})$$

$$g\cdot(i,j) = (gi, jg^{-1})$$

$$\mu: \mathrm{Rep}(\overline{Q}_f, V, W) \longrightarrow \mathfrak{gl}_V^* \underset{tr}{\cong} \mathfrak{gl}_V$$

$$(i,j) \longmapsto -ij$$

$$\mu^{-1}(0) /\!/_{\chi_\theta} G_V = M_\theta(k,n) \xrightarrow{\pi} Gr(k,n)$$

$$i \longmapsto \ker i$$

$$\theta>0 \rightsquigarrow i|_V = 0 \rightsquigarrow \underline{\mathrm{Hom}(W/V, V)}$$

$$M_\theta(k,n) = T^*Gr(k,n)$$

Fixed Points:

$$GL(n)\times \mathbb{C}_\hbar \hookrightarrow T^*Gr(k,n) \qquad -\hbar$$

$\cup$

$A =$

$$(\mathbb{C}^*)^n \hookrightarrow Gr(k,n)$$

$(z_1 \cdots z_n)$ $\mathbb{C}^n$ $e_i \mapsto z_i e_i$

$$\mathbb{C}^* \curvearrowright T^* Gr(k,n) \qquad \mathbb{C}^n = \underbrace{\mathbb{C}^{n_1}}\oplus\underbrace{\mathbb{C}^{n_2}} \quad \rightsquigarrow \quad T^* Gr(k,n)^{\mathbb{C}^*} = \coprod_{k_1+k_2=k} T^* Gr(k_1,n_1) \times T^* Gr(k_2,n_2)$$

$$(z_1 \cdots z_r) = (\mathbb{C}^*)^r \curvearrowright T^* Gr(k,n) \qquad \mathbb{C}^n = \underbrace{\mathbb{C}^{n_1}}\oplus\mathbb{C}^{n_2} \cdots \oplus \mathbb{C}^{n_r} \quad \rightsquigarrow \quad T^* Gr(k,n)^{(\mathbb{C}^*)^r} = \coprod_{\substack{k_1+k_2+\cdots+k_r \\ = k}} T^* Gr(k_1,n_1) \times T^* Gr(k_2,n_2) \cdots \times T^* Gr(k_r,n_r)$$

$r=n$

$$T^* Gr(k,n)^A = \coprod_{k_1+k_2+\cdots+k_n=k} \underbrace{T^* Gr(k_1,1)} \times \underbrace{T^* Gr(k_2,1)} \cdots \times T^* Gr(k_n,1)$$

$e_1 \quad e_2 \cdots e_k \rightsquigarrow$

$k_1, k_2 \cdots k_k = 1$

$$= \coprod_{\substack{S \subset \{1 \cdots n\} \\ |S| = k}} L_S \qquad L_S = \bigoplus_{i \in S} \mathbb{C} e_i$$

$$M(n) := \coprod_k T^* Gr(k,n)$$

$$M(1) = T^* Gr(0,1) \sqcup T^* Gr(1,1) = |0\rangle \sqcup |1\rangle .$$

$$M(n)^A = M(1) \times M(1) \cdots \times M(1)$$

$H_T(X^A)$; $T = A \times \mathbb{C}^*_\hbar$

$\hookrightarrow$ $H_T(\mathcal{M}(n)^A) = H^\bullet_T(\mathcal{M}(1)) \underset{H_T(pt)}{\otimes} \cdots \underset{H_T(pt)}{\otimes} H^\bullet_T(\mathcal{M}(1))$

Stable Envelopes:-

$$X = T^*Gr(k,n) \xrightarrow{\text{Proper}} X_0 = \mathcal{M}_0(k,n)/\mathrm{GL}_n$$

$A \hookrightarrow \underbrace{T^*Gr(k,n)}$

$\downarrow$

$\mathfrak{a}_{\mathbb{R}} = \mathrm{Cochar}(\mathbb{C}^*, A) \otimes \mathbb{R}$ $\qquad \mathcal{L}_s$

$$A \hookrightarrow T_{\underset{\sim Z}{\mathcal{L}_s}} T^*Gr(k,n) \sim T_{\mathcal{L}_s} Gr(k,n) \oplus T^*_{\mathcal{L}_s} Gr(k,n)$$

$\qquad \| $

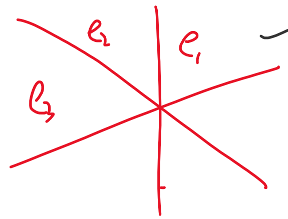
$\mathrm{Hom}(\mathcal{L}_s, \mathbb{C}^n/\mathcal{L}_s)$

$(z_1 \dots z_r) \rightsquigarrow z_j/z_i$

$\underline{(a_i - a_j)} = 0$

$\overline{\mathrm{Attr}_\mathfrak{C}(Z)} \cap Z' \neq 0$

$\mathfrak{a}_{\mathbb{R}} \setminus \bigcup (a_i - a_j = 0) =$



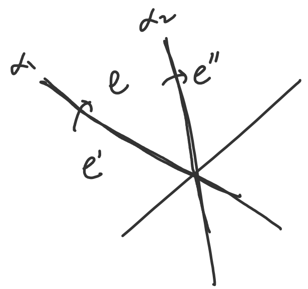
$\rightsquigarrow \mathrm{Att}_e(\mathcal{L}_s) \rightsquigarrow$ $\quad z' \underset{e}{\leq} z$

$$\mathrm{Att}^f_e(\mathcal{L}_s) = \bigsqcup_{z' \leq z} \mathrm{Att}_e([z'])$$

Theorem: $\mathrm{Stab}_{e,\varepsilon} : \underbrace{H_T^\bullet(T^*Gr(k,n)^A)}_{\text{Exists unique.}} \longrightarrow H_T^\bullet(T^*Gr(k,n)) \xrightarrow{res} H_T^\bullet(T^*Gr(k,n)^A)$

R-matrix: e', e''

$$\mathrm{Stab}_{e'}^{-1} \circ \mathrm{Stab}_{e''} \in \mathrm{End}\left(H_T^\bullet(X^A) \otimes \mathbb{Q}(t)\right)$$



$$R_{e',e''} = \mathrm{Stab}_{e'}^{-1} \circ \mathrm{Stab}_{e''}$$
$$= \left(\mathrm{Stab}_{e'}^{-1} \circ \mathrm{Stab}_{e}\right) \circ \left(\mathrm{Stab}_{e}^{-1} \circ \mathrm{Stab}_{e''}\right)$$

$R_\alpha \to$ Root R-matrices

$$\begin{bmatrix} \frac{u}{u-\hbar} & \frac{-\hbar}{u-\hbar} \\ \end{bmatrix}$$

$A = \mathbb{C}^*$

$K_{<,>}(u) =$
$$\begin{bmatrix} \frac{-\hbar}{u-\hbar} & \frac{u}{u-\hbar} \end{bmatrix}$$

$A' \subset A \rightsquigarrow X^A \subset X^{A'}$

$\mathrm{Cochar}(A') \subset \mathrm{Cochar}(A)$

$\mathfrak{C}' \subset \mathfrak{C}$

$A/A' \hookrightarrow X^{A'}$

$$\begin{tikzcd} H_T^\cdot(X^A) \arrow[r, "\mathrm{Stab}_{\mathfrak{C}/\mathfrak{C}'}"] \arrow[dr, "\mathrm{Stab}_{\mathfrak{C}}"'] & H_T^\cdot(X^{A'}) \arrow[d, "\mathrm{Stab}_{\mathfrak{C}'}"] \\ & H_T^\cdot(X) \end{tikzcd}$$

$R_\alpha \rightsquigarrow A > \ker(\alpha) = A^\alpha$

H

$\mathbb{C}^* = A/A^\alpha \hookrightarrow X^{A^\alpha}$

$$M(n) = \bigsqcup_k T^* Gr(k,n)$$

$\alpha = a_i - a_j$

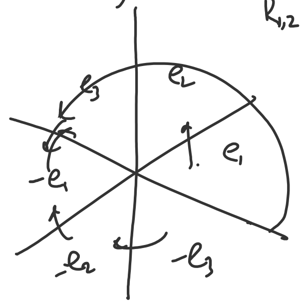
$A^\alpha \subset A = (\mathbb{C}^*)^n$

$$\mathcal{M}(n)^A = \underbrace{\mathcal{M}(2)}_{\|} \times \prod_{k \neq i,j} \mathcal{M}(1)$$

$\mathbb{B}t \sqcup T^*\mathbb{P}^1 \sqcup \mathbb{B}t$

$$\forall\, i,j \qquad R_{i,j}(a_i - a_j) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a_i - a_j}{a_i - a_j - t} & \frac{-t}{\cdots} & 0 \\ 0 & \cdots & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to Y(\mathfrak{gl}_2)$$

$R_{1,2} \circ R_{1,3} \circ R_3$



Yangian for

$$Y_\cdot \subset \prod_{i_1, i_2 \cdots i_n} \mathrm{End}_{H^\cdot_{G_\hbar}(\mathbb{B}t)[u_1, \dots u_n]} \left(H^\cdot_{G_\hbar}(\mathcal{M}(1))[u_1] \otimes H^\cdot_{G_\hbar}(\mathcal{M}(1))[u_2] \cdots \otimes H^\cdot_{G_\hbar}(\mathcal{M}(1))[u_n] \right)$$

$$\| $$

$$H^\cdot_T(\mathcal{M}(n)^A) = H^\cdot_{T/A}(\mathcal{M}(n)^A) \otimes \mathbb{Q}[a]$$

$H^\cdot_{G_\hbar}(\mathbb{B}t) = \mathbb{Q}_\hbar$

$H^\cdot_A(\mathbb{B}t)$

$\mathbb{Q}$

$\mathbb{Q}^2_\hbar[u_1]$

$$W = \mathbb{Q}^2_\hbar \otimes \mathbb{Q}^2_\hbar[u_1] \otimes \mathbb{Q}^2_\hbar[u_2] \cdots \otimes \mathbb{Q}^2_\hbar[u_n]$$

$$m(u) = m_{11}(u)\, e_1 \otimes e_1^* + m_{12}(u)\, e_1 \otimes e_2^* + m_{21} + m_{22} \in \mathbb{Q}^2_\hbar \otimes \mathbb{Q}^{2\vee}_\hbar[u].$$

$$\underset{0}{Q^2_\hbar[u]} \otimes (Q^2_\hbar)^\vee \quad \otimes Q^2_\hbar[u_1] \otimes Q^2_\hbar[u_2] \cdots \otimes Q^2_\hbar[u_n]$$

$$R_{0,n} \cdots R_{0,2} R_{0,1}$$

$$\underbrace{Q^2_\hbar \otimes (Q^2_\hbar)^\vee}_{tr} \otimes Q^2_\hbar[u_1] \otimes \cdots \otimes Q^2_\hbar[u_n]$$

$$\downarrow$$

$$Q^2[u] \otimes W \xrightarrow{\mathrm{Res}_{u \to \infty}} W$$

$$E(m(u)) \in \mathrm{End}(W) \rightsquigarrow Y_\hbar \xleftarrow{\sim} Y(\mathfrak{gl}_2)$$

Quantum Determinant for $Y(\mathfrak{gl}_2)$

$$q\det(u) = \underbrace{T_{11}(u)\, T_{22}(u-1) - T_{21}(u)\, T_{12}(u-1)}_{} \in Y(\mathfrak{gl}_2)[u^{-1}].$$

$$\|$$

$$1 + \sum_{k>0} q\det_k u^{-k} \qquad \rightsquigarrow \qquad q\det_k \qquad\qquad t_{ij}^{(k)}$$

$$q\det_k = \underline{t_{11}^{(k)} + t_{22}^{(k)}} + \underbrace{\qquad}_{<k-1}$$

$\leadsto$ algebraically indept

$gr\, Y(\mathfrak{gl}_2) = \mathcal{U}(\mathfrak{gl}_2)[u]$

$T_{ij}^{(k)} \leftarrow E_{ij} u^{k-1}$

$q\det$

$\mathcal{U}(Z(\mathfrak{g})[u])$

$$\underline{(E_{11}+E_{22})} u^{k-1} \in \mathcal{U}(\mathfrak{gl}_2)[u].$$

$\mathfrak{g}[u]$

$[g_1 u^k, g_2 u^s]$

$= u^{k+s}[g_1, g_2]$

$\mathfrak{g} \to$ reductive Lie algebra

Lemma:

$$Z(\mathcal{U}(\mathfrak{g}[u])) = \underline{\mathcal{U}(Z(\mathfrak{g})[u])}$$

$- \; Z(\mathfrak{gl}_2)$

$$\leadsto Z(\mathcal{U}(\mathfrak{gl}_2(z))) = \mathcal{U}((E_{11}+E_{22})u^{k-1})$$

$$\Rightarrow \quad Z(Y(\mathfrak{gl}_2)) = \langle q\det_k \; ; \; k \geq 1 \rangle$$

$$Z(\mathfrak{g}) = \{ x \in \mathfrak{g} \mid \forall y \in \mathfrak{g}, \; [x,y]=0 \}$$

Lemma: $\mathfrak{a} \subseteq \mathfrak{g}$ such that $\mathfrak{a}$ is reductive in $\mathfrak{g}$

$\hookrightarrow$ finite dimension

Let $\mathfrak{b}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.

Then the centralizer of $\mathcal{U}(\mathfrak{a}[z])$ in $\mathcal{U}(\mathfrak{g}[z])$ is equal to $\mathcal{U}(\mathfrak{b}[z])$

- Moreover, $Z(\mathfrak{g}) = 0$, then center of $\mathcal{U}(\mathfrak{g}[z])$ is also trivial.

$Z(\mathcal{U}(\mathfrak{g}[z])) = \mathcal{U}(z$

- $Z(\mathcal{U}(\mathfrak{sl}_2[z])) = 0$

$Y(\mathfrak{sl}_2)$:
- RTT
- J-presentation
- Drinfeld

$f(u) = 1 + \sum_{i>0} a_i u^{-i}$

$\tau_f : Y(\mathfrak{gl}_2) \longrightarrow Y(\mathfrak{gl}_2)$

$Y(\mathfrak{sl}_2) \subset Y(\mathfrak{sl}_2)$

$Y(\mathfrak{sl}_2) \subset Y(\mathfrak{gl}_2)$

$T(u) \longrightarrow f(u)\,T(u)$.

$Y(\mathfrak{gl}_2) \simeq Y(\mathfrak{sl}_2) \otimes Z(Y(\mathfrak{gl}_2))$

$$Y(\mathfrak{sl}_2) = \frac{Y(\mathfrak{gl}_2)}{\langle Z(Y(\mathfrak{gl}_2))\rangle} = \frac{Y(\mathfrak{gl}_2)}{\langle q\det_k = 0\rangle}$$

$$\begin{array}{c} W_i \\ \downarrow \\ \mathcal{M}_\theta(r, w) \end{array}$$

Mo: $\mathbb{Y}_a \subset Y_{\mathfrak{a}}$

$$\mathrm{Ch}_k(W_i) = \langle q\det_k \rangle \qquad \mathbb{Y}_\cdot = Y(\mathfrak{sl}_2)$$