Maulik-Okounkov Yangians

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Goal: introduce MO Yangians.

① Setup

dim. vectors

$Q^{=(I,E)}$ - quiver $\rightsquigarrow$ $\mathcal{M}(w) = \coprod_{v} \mathcal{M}_{\cancel{\theta}}(v,w)$, $\theta = (1,\dots,1)$

$$\begin{array}{ccc} \circ \rightleftarrows \circ & \leftarrow v \\ \Updownarrow \quad \Updownarrow & \\ \square \quad \square & \leftarrow w \end{array}$$

$\mathcal{M}(w) \curvearrowleft A = \mathbb{G}_m^{w} = \prod_{i\in I} \mathbb{G}_m^{w_i}$

Properties: (i) $\mathcal{M}(w)^A = \prod_i \mathcal{M}(w_i) = \prod_{i\in I} \mathcal{M}(\delta_i)^{w_i}$

$\mathcal{M}(\delta_i) \times \mathcal{M}(\delta_i) \times \dots$ w_i times

framing $(0,\dots,\underset{\text{i-th place}}{1},\dots,0)$

(ii) All roots of $\mathcal{M}(w)$ are roots of $PGL_w = P(\prod_i GL_{w_i})$

i.e. $\alpha_{ij} = a_i - a_j$

corr. to one of the 1-dim'l subtori.

Pf $N = N_+ \oplus N_-$, $N_- = \sum \mathrm{Hom}(\underbrace{W_i'}_{\Sigma a_j}, \underbrace{V_i''}_{\Sigma a_i}) + \sum \mathrm{Hom}(V_i', W_i'') - \sum_{\dots} \mathrm{Hom}(V_i', V_j'')$

dep. on Q.

$\rightsquigarrow$ wts are $a_j - a_i$ (in K-theory)

$\mathcal{M}(w)^{\mathbb{C}^*\times\mathbb{C}^*} = \coprod \underbrace{\mathcal{M}(w_1)}_{V_i', W_i'} \times \underbrace{\mathcal{M}(w_2)}_{V_i''}$

$$\begin{array}{cc} V_i \to & \circ \rightleftarrows \circ \\ & \Updownarrow \quad \Updownarrow \\ w_i \to & \square \quad \square \end{array}$$

(iii) $\mathcal{M}(w)^{A_{\alpha_{ij}}} = \mathcal{M}(\delta_i + \delta_j) \times \prod_{k\neq i,j} \mathcal{M}(\delta_k)$

bigger torus

Recall: we constructed the stable envelope map : $\mathrm{Stab}_{\mathfrak{C}} : H^{\cdot}_{G_A}(X^A) \to H^{\cdot}_{G_A}(X) \quad \forall\, \mathfrak{C} \subset \mathfrak{a}_{\mathbb{R}}$ chamber.

Def $R_{\mathfrak{C}',\mathfrak{C}} := \mathrm{Stab}_{\mathfrak{C}'}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}} \in \mathrm{End}\left(H^{\cdot}_{G_A}(X^A)\right) \otimes \mathbb{Q}(\mathfrak{g}_A)$ — R-matrix

(2) Properties of R-matrices

$R_{\mathfrak{C}'',\mathfrak{C}} = R_{\mathfrak{C}'',\mathfrak{C}'} \circ R_{\mathfrak{C}',\mathfrak{C}}$ tautologically.

$\Rightarrow$ all R-matrices are defined by root R-matrices, i.e. s.t. $\mathfrak{C}, \mathfrak{C}'$ are separated by wall $\alpha = 0$.

$A_\alpha = \mathrm{Ker}\,\alpha \quad , \quad X^\alpha := X^{A_\alpha} \curvearrowleft A/A^\alpha$

$\rightsquigarrow R_\alpha := R_{<0,>0} \in \mathrm{End}\left(H^{\cdot}_{G_A}(X^A)\right) \otimes \mathbb{Q}\left(\mathfrak{g}_A/\mathfrak{g}_{A_\alpha}\right)$

Lm $R_{\mathfrak{C}',\mathfrak{C}} = R_\alpha$

Follows from:

Prop Let $\mathfrak{C}$ chamber, $\bar{\mathfrak{C}}$ - face of $\mathfrak{C}$, $\bar{\mathfrak{a}} = \mathrm{Span}\,\bar{\mathfrak{C}}$ $\rightsquigarrow \mathfrak{C} \to \mathfrak{C}/\bar{\mathfrak{C}} \subset \mathfrak{a}/\bar{\mathfrak{a}}$

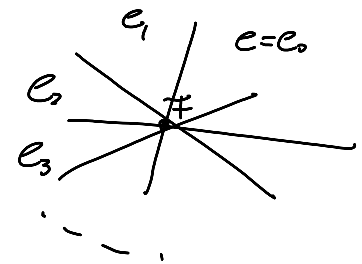
$$\begin{tikzcd}
H^{\cdot}_{G_A}(X^A) \arrow[rr, "\mathrm{Stab}_{\mathfrak{C}}"] \arrow[dr, "\mathrm{Stab}_{\mathfrak{C}/\bar{\mathfrak{C}}}"'] & & H^{\cdot}_{G_A}(X) \\
& H^{\cdot}(X^{\bar{A}}) \arrow[ur, "\mathrm{Stab}_{\bar{\mathfrak{C}}}"'] &
\end{tikzcd}$$

This diagram commutes.

Pf $\mathrm{Stab}_{\mathfrak{C}}$ is unique! $\square$

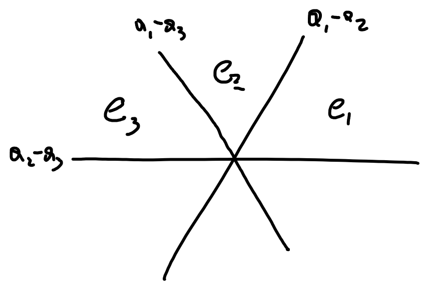
Another corollary of "associativity":

<u>Prop</u> $\mathcal{F} \in \mathcal{C}$ a codimension 2 face; $\mathcal{C}_i$ - chambers containing $\mathcal{F}$ in cyclic order



$$\Rightarrow \quad R_{e_0,e_1} \circ R_{e_1,e_2} \circ \dots \circ R_{e_n,e_0} = 1.$$

<u>Ex</u> $\mathbb{G}_m^3 \curvearrowright \mathcal{M}(w)$ $\qquad \mathcal{F} = \{a_1 = a_2 = a_3\}$



$$\rightsquigarrow \quad R_{12}(a_1-a_2)\, R_{13}(a_1-a_3)\, R_{23}(a_2-a_3) = R_{23}(a_2-a_3)\, R_{13}(a_1-a_3)\, R_{12}(a_1-a_2)$$

satisfies <u>YBE</u>

$R \in End(V \otimes V) \quad \rightsquigarrow R_{12}, R_{13}, R_{23}$

$R_{12} \in End(V_1 \otimes V_2 \otimes V_3)$

$\uparrow$
R_{23}

<u>Prop</u> $R_\alpha = 1 + O(\alpha^{-1})$ $\qquad$ i.e. power series in α^{-1}

<u>Pf</u> Exercise $\square$

homothety of the base of the univ. deformation

For sympl. resolutions (e.g. $\mathcal{M}(w)$) $\quad \rightsquigarrow \quad R_\alpha = 1 + O(\hbar)$

In our case $\hbar$ - character of torus scaling half of the arrows in $Q^\heartsuit$

$$\rightsquigarrow R_\alpha = 1 + \frac{\hbar}{\alpha} r_\alpha + O(\alpha^{-2}) \qquad , \quad r_\alpha \in \mathrm{End}(H^\cdot_{G_A}(X^A))$$ — classical R-matrix

Expand YBE in powers of α^{-1} + compare 2nd degree terms:

$$\begin{cases} [r_{12}, r_{13}+r_{23}]=0 \\ [r_{23}, r_{12}+r_{13}]=0 \end{cases} \quad \underset{\bar r_{ij} = \frac{r_{ij}}{a_i-a_j}}{\rightsquigarrow} \quad [\bar r_{12},\bar r_{13}]+[\bar r_{12},\bar r_{23}]+[\bar r_{13},\bar r_{23}]=0$$ — classical YBE

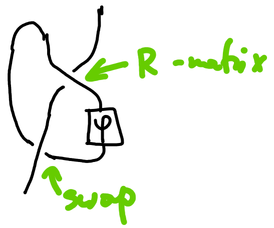
Rmk $r\in\mathfrak g\otimes\mathfrak g \rightsquigarrow \delta:\mathfrak g\to\mathfrak g\otimes\mathfrak g$, $X\mapsto (\mathrm{ad}X\otimes id)\, r$ cYBE for r assures that $(\mathfrak g,\delta)$ is a Lie bialgebra.

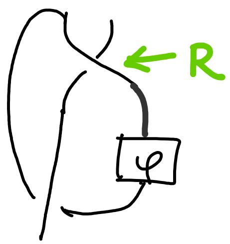
③ Maulik-Okounkov Yangian

A^1 – 1 dim torus $\qquad \mathcal M(w)^{A^1} = \mathcal M(w_1)\times \mathcal M(w_2)$

$\rightsquigarrow$ operators on $H^\cdot(\mathcal M(w_2)) \to H^\cdot(\mathcal M(w_1))$

$$\mathrm{tr}_{H^\cdot(\mathcal M(w_2))}\Big[(\underbrace{\varphi}_{\in \mathrm{End}(H^\cdot(\mathcal M(w_1)))}\otimes id)\circ R\Big] =$$





For us, $w_i = \delta_i$.

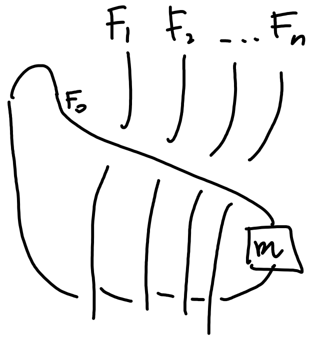
Define $F_i(u_i) := H^{\cdot}_{\mathbb{C}^*_{a_i}}(\mathcal{M}(\delta_i))$

$F_0 \in \{F_i\}$

$\boxed{\text{Def}}$ For each $m(u) \in \mathrm{End}(F_0)[u]$, consider

$$E(m(u)) = -\frac{1}{\hbar}\,\mathrm{Res}_{u=\infty}\left(\mathrm{tr}_{F_0}\, m(u)\, R_{F_0,F_n}(u-u_n)\circ\cdots\circ R_{F_0,F_1}(u-u_1)\right)$$

pick a coefficient

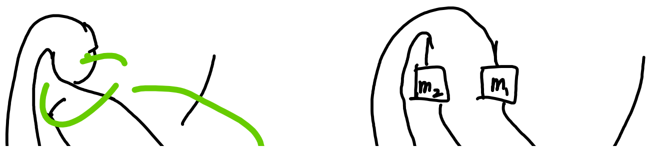


$\rightsquigarrow$ get an element of $\prod_{(i_j)} \mathrm{End}_{\mathbb{C}[u_1,\dots,u_k]} F_{i_1}(u_1)\otimes\cdots\otimes F_{i_k}(u_k)$

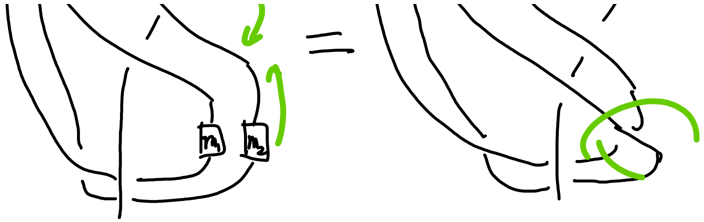
The <u>Yangian</u> Y_Q is the subalgebra of $\prod \mathrm{End}$ generated by all $E(m(u))$'s.

This gives gen's of Y_Q, but not rel's.

YBE $\rightsquigarrow$ $(m_1(u_1)\otimes m_2(u_2))\, R_{F_1F_2}(u_1-u_2) = R_{F_1F_2}(u_1-u_2)\,(m_2(u_2)\otimes m_1(u_1))$



RTT = TTR.



④ Some properties

1) If you consider only $E(m_0)$, where $m_0 \in \mathrm{End}(F_0)$ $\cancel{[u]}$,
they form a Lie subalgebra $\mathfrak{g}_Q \subset Y_Q$. ("BPS Lie algebra").

$\mathfrak{g}_Q$ is "similar" to Kac-Moody Lie algs.

2) Y_Q contains all $c_k(V_i) \cap -$, $c_k(W_i) \cap -$

3) Y_Q is filtered: $\deg E(m_0 \cdot u^n) \leq n$.

Thm $gr(Y_Q) \simeq \mathcal{U}(\mathfrak{g}_Q[u])$ + Y_Q is gen. by $\mathfrak{g}_Q$ + cup product with taut. Chern classes.

Example $Q = \bullet$, $w = 2$

$$\mathcal{M}(2) = \underset{\substack{\| \\ pt_0}}{\mathcal{M}(0,2)} \sqcup \underset{\substack{\| \\ T^*\mathbb{P}^1}}{\mathcal{M}(1,2)} \sqcup \underset{\substack{\| \\ pt_2}}{\mathcal{M}(2,2)}$$

$A = \mathbb{G}_m^2$, $\mathcal{M}(2)^A = \mathcal{M}(1) \times \mathcal{M}(1)$

$$\mathcal{M}(1) = \underset{\substack{\| \\ pt}}{\mathcal{M}(0,1)} \sqcup \underset{\substack{\| \\ pt}}{\mathcal{M}(1,1)}$$

$\mathcal{M}(0,1) \times \mathcal{M}(0,1) \rightsquigarrow pt_0$

$\mathcal{M}(0,1) \times \mathcal{M}(1,0) \rightsquigarrow 0$

$\mathcal{M}(1,0) \times \mathcal{M}(0,1) \rightsquigarrow \infty$

$$\mathcal{M}(1,0) \times \mathcal{M}(0,1) \to \sim$$

$$\mathcal{M}(1,1) \times \mathcal{M}(1,1) \longrightarrow pt_2$$

We work inside $\mathbb{C}^2 \otimes \mathbb{C}^2$ basis: $v_0\otimes v_0$, $v_0\otimes v_1$, $v_1\otimes v_0$, $v_1 \otimes v_1$

$$v_i \otimes v_j \leftrightarrow \mathcal{M}(i,1) \times \mathcal{M}(j,1).$$

$$R(u) = \begin{array}{c} \begin{array}{cccc} v_0\otimes v_0 & v_0\otimes v_1 & v_1\otimes v_0 & v_1\otimes v_1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u}{u+\hbar} & \frac{\hbar}{u+\hbar} & 0 \\ 0 & \frac{\hbar}{u+\hbar} & \frac{u}{u+\hbar} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} v_0\otimes v_0 \\ v_0\otimes v_1 \\ v_1\otimes v_0 \\ v_1\otimes v_1 \end{array}$$

$$m(u) \in End(\mathbb{C}^2)[u] \qquad m(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & d(u) \end{pmatrix}$$

$$m\otimes id = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

$$m\otimes id \circ R(u-u_1) = \begin{pmatrix} a & b\frac{\hbar}{u+\hbar-u_1} & b\frac{u-u_1}{u+\hbar-u_1} & 0 \\ 0 & a\frac{u-u_1}{u+\hbar-u_1} & a\frac{\hbar}{u+\hbar-u_1} & b \\ c & d\frac{\hbar}{u+\hbar-u_1} & d\frac{u-u_1}{u+\hbar-u_1} & 0 \\ 0 & c\frac{u-u_1}{u+\hbar-u_1} & c\frac{\hbar}{u+\hbar-u_1} & d \end{pmatrix}$$

$$tr_{F_0}(\,---\,) = \left(\begin{array}{c} v_i\otimes v_j \to v_k \otimes v_l \\ \leadsto \\ \delta_{ik}\,(v_j \to v_l) \end{array} \right)$$

$$\| \qquad \begin{pmatrix} a + d\frac{u-u_1}{u+\hbar-u_1} & b\frac{\hbar}{u+\hbar-u_1} \\ c\frac{\hbar}{u+\hbar-u_1} & a\frac{u-u_1}{u+\hbar-u_1} + d \end{pmatrix}$$

$$\frac{u-u_1}{u+\hbar-u_1} = \frac{1-u_1u^{-1}}{1-(u_1-\hbar)u^{-1}} = (1-u_1u^{-1})\Big(1+\sum_k (u_1-\hbar)^k u^{-k}\Big)$$

$$\frac{\hbar}{u+\hbar-u_1} = \hbar u^{-1}\Big(1+\sum_k (u_1-\hbar)^k u^{-k}\Big)$$

For simplicity $m = m_0 \cdot u^n$ $\qquad$ Res $=$ take coefficient at $u^{-(n+1)}$

$$\mathrm{Res} = \begin{pmatrix} d\big((u_1-\hbar)^{n+1} - u_1(u_1-\hbar)^n\big) & b\hbar(u_1-\hbar)^n \\ c\hbar(u_1-\hbar)^n & a\big((u_1-\hbar)^{n+1} - u_1(u_1-\hbar)^n\big) \end{pmatrix} = \cancel{-\hbar}\,\underbrace{(u_1-\hbar)}_{\text{parameter}}{}^n \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

basically evaluation representation of $Y(\mathfrak{gl}_2)$

$\mathfrak{J}Q$: $n=0$ $\rightsquigarrow$ $\mathfrak{gl}_2$.

Cupping with $c_k(V)$ $\quad\rightsquigarrow\quad$ $(u_1-\hbar)^k \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Cupping with $c_1(W)$ $\quad\rightsquigarrow\quad$ qdet $\in Y(\mathfrak{gl}_2)$

$g \in \prod \mathrm{End}(F_i)$ s.t. $[g\otimes g, R(u)] = 0$ $\quad\rightsquigarrow\quad$ $E(gu^k)$ – commutative subalgebra (bethe)

$\Gamma_g =$ projection $\quad H^\cdot(\mathcal{M}(w)) \twoheadrightarrow H^\cdot(\mathcal{M}(0,w))$ $\quad\rightsquigarrow\quad$ subalg. of cupping with taut classes.