COHAs of 0-dimensional sheaves on surfaces

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jt w/ A. Mellit
O. Schiffmann
E. Vasserot

Alexandre Minets
MPI Bonn

Motivation

S quasi-proj surface

$\mathrm{Hilb}_n S := \{ I \subset \mathcal{O}_S \text{ ideal} : \lg \mathcal{O}_S/I = n \}$ – Hilbert scheme of points

smooth variety (Fogarty)

<u>Question</u>: understand $H^*(\mathrm{Hilb}_n S)$.

Helpful perspective: action of algebras.

<u>Thm</u> (Nakajima) Let S be projective.

$$\mathfrak{h}_S = \left\langle q_i(\lambda)\ {\scriptstyle i \in \mathbb{Z}_{\neq 0},\ \lambda \in H^*(S)},\ C \right\rangle \Big/ \begin{array}{c} C \text{ central} \\ [q_i(\lambda), q_j(\mu)] = i\,\delta_{i,-j}\, C \int_S \lambda\mu \end{array}$$ – Heisenberg Lie algebra

$$\mathfrak{h}_S \curvearrowright \bigoplus_{n \geq 0} H^*(\mathrm{Hilb}_n S) \quad \text{via} \quad \begin{tikzcd} & \{I \subset J \subset \mathcal{O}_S\} \arrow[dl] \arrow[dr] & \\ S \times \mathrm{Hilb}_n S & & \mathrm{Hilb}_{n \pm i} S \end{tikzcd}$$

<u>Rmk</u> If S qproj, $q_{-i}(\mu)$ labeled by $\mu \in H^*_c(S)$.

Motivation

$\bigoplus_n H^*(\mathrm{Hilb}_n S)$ is a h.w $\mathcal{U}(\mathfrak{h}_S)$-module. $\Rightarrow$ $\boxed{\sum P_t(\mathrm{Hilb}_n S) q^n = \mathrm{Exp}\left(P_t(S) q\right)}$

What about ring structure on $H^*(\mathrm{Hilb}_n S)$?

Let $S = \mathbb{A}^2$ $\qquad D_{n,0} := q_n \qquad D_{0,m} := \mathrm{ch}_m\left(\Gamma(\mathcal{O}_S/I)\right) \cup -$

Thm (Lehn)

- $W^{\geq} = \langle D_{m,n} : m,n \geq 0 \rangle / [D_{mn}, D_{m'n'}] = (m'n - mn') D_{m+m', n+n'-1} \curvearrowright \bigoplus H^*(\mathrm{Hilb}_n \mathbb{A}^2)$
- $H^*(\mathrm{Hilb}_n \mathbb{A}^2)$ is gen. by $D_{0,m}$ as a module over itself

  Action: $D_{0,n} = \dfrac{(-1)^n}{(n+1)!} \displaystyle\sum_{k_0,\dots,k_n > 0} q_{k_0+\dots+k_n} \partial_{k_0} \cdots \partial_{k_n}$, $\quad \partial_m := m \dfrac{\partial}{\partial q_m}$

Lehn-Sorger
Li-Qin-Wang : similar result for S with $K_S = 0$ $\Rightarrow$ descr. of $H^*(\mathrm{Hilb}_n S)$

Question How does this generalize to other moduli of sheaves on S?

Cohomological Hall algebras

Idea: far-reaching generalization of Nakajima operators

$\mathcal{C}$ - abelian category $\rightsquigarrow$ $\mathcal{M}_\mathcal{C}$ moduli stack of objects

$\mathcal{E}_\mathcal{C}$ —"— of short exact sequences

$$\begin{tikzcd}[column sep=small] & \mathcal{E}_\mathcal{C} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{M}_\mathcal{C} \times \mathcal{M}_\mathcal{C} & & \mathcal{M}_\mathcal{C} \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 0 \to A \to B \to C \to 0 \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (A, C) & & B \end{tikzcd}$$

$$\overset{?}{\rightsquigarrow} \quad A(\mathcal{M}_\mathcal{C}) \otimes A(\mathcal{M}_\mathcal{C}) \longrightarrow A(\mathcal{M}_\mathcal{C})$$

some invariant

$A = \mathbb{F}_q$-points $\rightsquigarrow$ Ringel-Hall algebras

If we want A to be "cohomological", need restrictions on $\mathcal{C}$ to make sense of q^*

Most general setting: CoHAs for 3CY categories (Kontsevich-Soibelman)

$\mathrm{Ext}^i(A,B) \simeq \mathrm{Ext}^{3-i}(B,A)^\vee$

Taxonomy of COHAs

hdim(e)	Coh. theory	local (quiver reps)	$\leftarrow$ toric	global (coherent sheaves)
3d	$\mathcal{M}_e$ (-1)-shifted sympl. $H^*(\varphi_{\mathcal{M}_e}) = H^*_{crit}(\mathcal{M}_e)$	$\mathrm{Rep}(J_{Q,W})$ [Kontsevich-Soibelman]	$\rightsquigarrow$ wall crossing	Coh M, M 3CY [Khan-Kinjo]: $M = \mathrm{Tot}_S K_S$ General M - modulo Joyce's conj.
$\uparrow$ dim. red. [Davison, Kinjo] $\downarrow$				
2d	q quasi-smooth $H^*(\mathbb{D}_{\mathcal{M}_e}) = H^{BM}_*(\mathcal{M}_e)$	$\mathrm{Rep}\,\Pi_Q$ [Schiffmann-Vasserot, Yang-Zhao]	$\rightsquigarrow Y_\hbar(\mathfrak{g}_Q)$	Coh S, S surface [Kapranov-Vasserot]
1d	q smooth $H^*(\mathcal{M}_e)$	$\mathrm{Rep}\,Q$ [Rimanyi Przezdziecki]	$\rightsquigarrow$ categorifications of $U_q(\mathfrak{g}_Q)$	Coh C, C curve

Today: global 2d.

0-dimensional (2d) COHAs

For general S, COHA is too big.

<u>Def</u> Let $\mathrm{Coh}_{0,n} S :=$ moduli of coh. sheaves of length n on S.

<u>Ex</u>
- $\mathrm{Coh}_{0,1} S = S \times B\mathbb{G}_m$
- $S = \mathbb{A}^2 \rightsquigarrow \mathrm{Coh}_{0,n} \mathbb{A}^2 \simeq \left[\{ (X,Y) \in \mathfrak{gl}_n^2 \mid [X,Y] = 0 \} / GL_n \right]$
- $\mathrm{supp} : \mathrm{Coh}_{0,n} S \longrightarrow \mathrm{Sym}_n S$ coarse moduli.

<u>Def</u> $H_0(S) := \bigoplus_{n \geq 0} H_*^{BM}(\mathrm{Coh}_{0,n} S)$ – COHA of 0-dim sheaves on S

<u>Rmk</u> [SV] for $\mathbb{A}^2$, [M] for T^*C, [Yu Zhao] for all S.

<u>Variant</u> $H_0^c(S) := \bigoplus_{n \geq 0} H_c^*(\mathrm{supp}_* \mathbb{D}_{\mathrm{Coh}_{0,n} S})$

compactly supported COHA

$$
\begin{tikzcd}
 & \mathcal{E}_{n,m} \arrow[dl] \arrow[dr] & \\
\mathrm{Coh}_{0,n} S \times \mathrm{Coh}_{0,m} S \arrow[d] & & \mathrm{Coh}_{0,m+n} S \arrow[d] \\
\mathrm{Sym}_n S \times \mathrm{Sym}_m S \arrow[rr] & & \mathrm{Sym}_{m+n} S
\end{tikzcd}
$$

<u>Rmk</u> For $S \subset \bar{S}$ compactification,

$$H_0^c(S) \to H_0(\bar{S}) \to H_0(S) \quad \text{alg. morphisms.}$$

Hecke patterns

In a sense, $H_0(S)$ is "universal algebra of Nakajima-like operators".

Assume S is proper

Let $Coh^{\geq 1}(S)$ substack of sheaves with no fin. length subsheaves

$X \subset Coh^{\geq 1}(S)$ locally closed.

Def X is a Hecke pattern if

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \qquad \mathcal{G} \in Coh_0(S)\,,\ \mathcal{F} \in X \implies \mathcal{E} \in X.$$

Example $Hilb_n S$, Gieseker stable sheaves

For simplicity: all sheaves in X have fixed rk, ch_1 ; $X = \coprod_{i \in \mathbb{Z}} X_i \qquad i = ch_2.$

$$\begin{tikzcd}
 & \{\mathcal{F}' \subset \mathcal{F} : \lg \mathcal{F}/\mathcal{F}' = n\} =: Fl_{k,n} \arrow[dl] \arrow[dr] & \\
Coh_{0,n} \times X_k & & X_{k+n}
\end{tikzcd}
\qquad \rightsquigarrow \quad H_0(S) \curvearrowright \bigoplus_{k \in \mathbb{Z}} H^{BM}_*(X_k)$$

Rmk When S is not proper, we have trichotomy:

weak			$H_0^c(S)$	
normal	Hecke pattern	$\implies$ action of	$H_0(\bar{S})$	on $H^{BM}_*(X)$
strong			$H_0(S)$	

Explicit formulas

X moduli $\Longrightarrow$ $\mathcal{F}_u \to X \times S$ universal sheaf.

$$\rightsquigarrow \quad ch_k(\gamma) := \int_S ch_k(\mathcal{F}_u) \cup \gamma \in H^*(X) \qquad \forall k \geq 0,\ \gamma \in H^*(S)$$

power sums

More generally, $\forall$ symmetric function $f = f(p_1, p_2, \dots)$

$$f_k(\gamma) := \int_S f(1!ch_1, 2!ch_2, \dots) \cup \gamma.$$

$p_k \longleftrightarrow k!\,ch_k$
$e_k \longleftrightarrow c_k$ (Chern)
$h_k \longleftrightarrow s_k$ (Segre)

Consider $Fl_{k,1} = \{ \mathcal{F}' \subset \mathcal{F} : \lg \mathcal{F}/\mathcal{F}' = 1 \}$

"$\mathcal{F}' \subset \mathcal{F} \iff x \in S$ + line in $\mathcal{F}|_x$" $\Rightarrow$ $Fl_{k,1} \simeq \mathbb{P}_{X_{k+1} \times S}(\mathcal{F}_u)$

Rmk Need to interpret RHS in derived sense (Q. Jiang).
This requires $X \subset Coh^{\geq 1}$.

Corollary Let $\gamma u^i \in H^*(Coh_{0,1} S) \simeq H^*(S)[u]$. $\rightsquigarrow$ $\gamma u^i \cdot [X_k] = h_{i-r+1}(\gamma) \cdot [X_{k+1}]$

In $K_0(Fl_{k,1})$, $[\mathcal{F}_{X_k}] + [\mathcal{F}_{Coh_{0,1}}] = [\mathcal{F}_{X_{k+1}}]$ $\rightsquigarrow$ $[p_k, \gamma u^i] = \dots\dots$

These two relations open the door to computing $H_\bullet(S)$.

W-algebras

Denote $\Delta_S \in H^4(S\times S)$ diagonal, $c_1 = c_1(TS) \in H^2(S)$, $s_2 = c_1^2 - c_2(TS) \in H^4(S)$

Def $W^?(S) = \langle \psi_n(\lambda), T_n(\lambda) : \begin{smallmatrix} n\geq 0 \\ \lambda \in H^*(S)\end{smallmatrix} \rangle / \text{relations}:$

- $[\psi_n, \psi_m] = 0$ • $[\psi_n(\lambda), T_m(\mu)] = n\, T_{m+n-1}(\lambda\mu)$ • $[T_m(\lambda\mu), T_n(\nu)] = [T_m(\lambda), T_n(\mu\nu)]$

- $$[T_m(\lambda), T_{n+3}(\mu)] - 3[T_{m+1}(\lambda), T_{n+2}(\mu)] + 3[T_{m+2}(\lambda), T_{n+1}(\mu)] - [T_{m+3}(\lambda), T_n(\mu)]$$
$$+ [T_m(\lambda), T_{n+1}(s_2\mu)] - [T_{m+1}(\lambda), T_n(s_2\mu)] + \{T_m, T_n\}(c_1\Delta_S\lambda\mu) = 0$$

- $$\sum_{\sigma\in\mathfrak{S}_3} [T_{n_{\sigma(1)}}(\lambda_{\sigma(1)}), [T_{n_{\sigma(2)}}(\lambda_{\sigma(2)}), T_{n_{\sigma(3)}+1}(\lambda_{\sigma(3)})]] = 0$$

$W^+(S) := \langle T_n(\lambda)\rangle \subset W^?(S)$

Thm (MMSV) Let S be cohomologically pure, i.e. $H^*(\bar{S}) \twoheadrightarrow H^*(S)$.

Then $\exists$ iso of algebras
$$H_0(S) \longrightarrow W^+(S)$$
$$\lambda u^n \longmapsto T_n(\lambda)$$
In particular, $H_0(S)$ gen. in degree 1.

+ cpt. supp. version

Rmk "Coh. purity" condition is technical, should be able to drop it.

Properties of $W^{\mathfrak{z}}(S)$

1) If $c_1 \Delta = 0$, $W^{\mathfrak{z}}(S) \simeq \mathcal{U}(w^{\mathfrak{z}}(S))$ for explicit Lie algebra $w^{\mathfrak{z}}(S)$

   If furthermore $s_2 = 0$,

$$w^{\mathfrak{z}}(S) = \left\langle D_{mn}(\lambda) : \begin{matrix} m,n \geq 0 \\ \lambda \in H^*(S) \end{matrix} \right\rangle \Big/ [D_{mn}(\lambda), D_{m'n'}(\mu)] = (m'n - mn') D_{m+m', n+n'-1}(\lambda\mu)$$

   E.g. for $S = \mathbb{A}^2$ $\rightsquigarrow$ Lehn's algebra.

2) $W^{\mathfrak{z}}(S)$ is bigraded:

$$\deg \psi_n(\lambda) = (0, 2n-2+\deg\lambda)$$
$$\deg T_n(\lambda) = (1, 2n-2+\deg\lambda)$$

3) $\dim W^{\mathfrak{z}}(S) = \operatorname{Exp}\left(\dfrac{P_z(S)\, z^{-2}}{(1-z^2)(1-w)} \right)$

4) $\langle D_{m0} \rangle$ – half of Heisenberg

   $\langle D_{m1} \rangle$ – half of Virasoro

5) $W^{\mathfrak{z}}(S)$ acts on the tautological subspace of $H^{BM}(X)$, X Hecke pattern.

Sketch of proof

A priori, no map $W^+(S) \longrightarrow H_0(S)$. However, given a Hecke pattern $X = \coprod X_k$

$$
\begin{tikzcd}
T_n(\lambda) \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \lambda \cdot u^n \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
W^+(S) \arrow[d] & H_0'(S) \arrow[r, hook] & H_0(S) \arrow[d] \\
\mathrm{End}(V^{taut}) \arrow[r] & \mathrm{Hom}(V^{taut}, V) & \mathrm{End}(V) \arrow[l]
\end{tikzcd}
$$

$V = H_\bullet^{BM}(X)$

V^{taut} - tautological part

If (i) $V = V^{taut}$

(ii) $W^+(S) \curvearrowright V$ faithful

(iii) $\dim W^+(S) = \dim H_0(S)$

then we're done.

(iii) : [KV] via factorization methods, [DHSM] via BPS methods.

(i) : holds for $X = \coprod \mathrm{Hilb}_n S$ if S is pure

(ii) : holds for $\mathcal{U}(\mathfrak{h}_S^+) \subset W^+(S)$ by Nakajima, Lehn

Lm $I \subset W^+(S)$ two-sided ideal
$I \cap \mathcal{U}(\mathfrak{h}_S^+) = 0 \Rightarrow I = 0.$

$\square$

Cor $W^{?'}(S)$ acts on V, not just V^{taut}.

Double of $W^{\geq}(S)$

This explicit description of $H_0(S)$ allows us to write down its "double"

Def Let $W^{\leq} = W^{\geq, op}$, $W^- = W^{+, op}$; $r \in \mathbb{Z}$

Define $W^{(r)}(S) = W^-(S) \otimes W^0(S) \otimes W^+(S)$ as a vector space, with relations:

- $W^0(S) \otimes W^+(S) \simeq W^{\geq}(S)$, $W^-(S) \otimes W^0(S) \simeq W^{\leq}(S)$ subalgebras
- $[T_n^-, T_m^+] = (-1)^{r+1} \sum_{0 \leq i \leq j \leq n+m} (-1)^j \binom{r-j+1}{i+1} (h_{n+m-j}\, e_{j-i})(c_1^i \lambda_\mu)$

Thm (MMSV) If X is a two-sided Hecke pattern (i.e. for $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ $\mathcal{F} \in X \Leftrightarrow \mathcal{G} \in X$) of rank r

then $W^{(r)}(S) \curvearrowright H_*^{BM}(X)^{taut}$

Rmk
- we should be able to remove "taut"
- weak/strong versions.

Properties
- $W^{(r)}(S)$ contains Heisenberg with central charge $r \int \lambda_\mu$
  Virasoro $r\left(\int c_2 \lambda_\mu - (1-r^2) \int c_1^2 \lambda_\mu + 2\psi_0(c_1 \lambda_\mu)\right)$
- has a "Fock-like" module $\Lambda^{(r)}(S) \simeq \mathbb{Q}[s^{\pm 1}][\, p_i(\lambda) : 2i + \deg \lambda \geq 4] / _{p_0(pt) - r}$

  Action similar to Lehn's differential operators

Rank 0 case

Let X be a Hecke pattern of rank 0 $\Rightarrow$ X parameterizes sheaves with 1-dim'l support

Examples
- $\mathrm{Higgs}_n = \{(\mathcal{E},\theta) : \mathcal{E} \in \mathrm{Bun}_n C,\ \theta : \mathcal{E} \to \mathcal{E} \otimes L\}$ on $S = \mathrm{Tot}_C L$
- $\mathrm{Mukai}_n = \{\mathcal{F}$ with purely 1-dim supp., $c_1(\mathcal{E}) = nC\}$ on $S = K3$ $\quad C \subset S$ smooth curve
- $PT_n = \left\{ \mathcal{O}_{\mathbb{A}^2} \xrightarrow{\varphi} \mathcal{F} : \begin{array}{l} \mathrm{supp}\,\mathcal{F} = \{y=0\} \\ \mathrm{coker}\,\varphi \text{ has fin length} \end{array} \right\}$ on $S = \mathbb{A}^2$

In this case central charge of $\mathfrak{h}_S \subset W(S)$ vanishes $\Rightarrow$ $W(S) \curvearrowright H^{BM}(X)$ might have non-trivial kernel

For Higgs_n, [HMMS] : factors through "rational version". $\rightsquigarrow$ there is a geometric $\mathfrak{sl}_2 \curvearrowright H^*(\mathrm{Higgs})$
This was crucial in our proof of $P=W$ conjecture.

We have a similar degeneration for K3's $\rightsquigarrow$ $P=C$ for Mukai_n (work in progress)

In fact, this is reflective of a more general phenomenon.

Thm (M., in progress) Suppose $c_1(\mathcal{E}) = nC$ for $\mathcal{E} \in X$.
Then $W^{(0)}(S) \curvearrowright H^{BM}(X)$ factors through "Costello-Grojnowski's" spherical trigonometric DAHA of GL_n, associated to $H^*(S)$

Cor $\mathrm{sDAHA}_{GL_n} \curvearrowright H^{BM}_*(PT_n)$.