Schur-Weyl duality for quantum wreath products

① Introduction / motivation

Recall: $\mathcal{H}_n(q) = \Bbbk\langle H_i : 1 \le i \le n-1\rangle \Big/ \begin{matrix} H_i^2 = (q-1)H_i + q \\ H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1};\quad H_iH_j = H_jH_i \quad |i-j|>1. \end{matrix}$ ← deforms $\Bbbk[S_n]$

Schur-Weyl duality: $U_q(\mathfrak{gl}_d) \curvearrowright V = \mathbb{C}^d \rightsquigarrow U_q^{Lus}(\mathfrak{gl}_d) \underset{\text{iso for } n\to\infty}{\twoheadrightarrow} S_{d,n}(q) \curvearrowright V^{\otimes n} \curvearrowleft \mathcal{H}_n(q)/I_d \underset{\text{iso for } d\ge n.}{\twoheadleftarrow} \mathcal{H}_n(q)$ double centralizes

$$\bigoplus_{\lambda\vdash n} x_\lambda \mathcal{H}_n \simeq \bigoplus_{\lambda\vdash n} S_\lambda V \otimes x_\lambda \mathcal{H}_n.$$

$\uparrow$ TL_n for $d=2$

Why are $S_{n,d}(q)$ important? Quasi-her. cover of $\mathcal{H}_n(q)$, so helps to study rep. theory of roots of unity.

Affine Hecke algebra (Bernstein presentation): $\mathcal{H}_n^{aff}(q) = \Bbbk\langle H_i, x_j^{\pm}\rangle \Big/ \begin{matrix} H_ix_i = x_{i+1}H_i + (q-1)x_i \\ H\text{-relations}\end{matrix}$ ← deforms $\Bbbk[S_n] \otimes \Bbbk[x^{\pm1}]^{\otimes n}$

In some situations, similar deformations occur, but with $\Bbbk[x^{\pm}] \rightsquigarrow B$

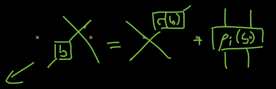
1) Categorification of PBW of quantum affine algebras $\rightsquigarrow$ imaginary root part is controlled by def. of $\Bbbk[S_n] \otimes (Z[x])^{\otimes n}$, Z-zigzag algebra [Kleshchev-Muth]
2) "pro-p" invariants of Gelfand-Graev module W over $GL_n(\mathbb{Q}_p) \curvearrowleft$ def. of $\Bbbk[S_n] \otimes \Bbbk[\mathbb{F}_p^\times][x]^{\otimes n}$ [Goo-Gurevich-Kerosienich]; $\widetilde{GL_n}(\mathbb{Q}_p)$ metaplectic $\rightsquigarrow$ $\Bbbk[\mathbb{Z}/p\text{-}\mathbb{Z}][x]$ (B-L).
3) CoHAs of fin. length sheaves on smooth surfaces

Moreover, "small char" seems to be controlled by Schur versions

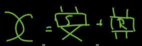
1) [MM] for $\mathfrak{gl}_2$, conjecturally [LM, Davidson-Kujawa-Muth] for other $\hat{\mathfrak{g}}$
2) [Buciumas-Lai] $\mathrm{End}_{\mathcal{H}}(W) \rightsquigarrow \mathrm{Sch}(\mathcal{H})$

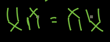
Goal → Deformations of $\Bbbk[S_n] \otimes B^{\otimes n}$
→ Schur versions.

② Quantum wreath products



Def B (unital) assoc. alg. $\underset{\substack{\uparrow \\ \text{quantum wreath product}}}{B \wr \mathcal{H}_n(q)} = B^{\otimes n}\langle H_i : 1 \le i \le n-1\rangle \Big/ \begin{matrix} H_ib = \sigma_i(b)H_i + \rho_i(b) \qquad b \in B^{\otimes n} \\ H_i^2 = S_iH + R_i \\ H_iH_j = H_jH_i, \quad H_{i+1}H_iH_{i+1} = H_iH_{i+1}H_i + D_iH_i + \bar{D}_iH_{i+1} + E_i \end{matrix}$





where: $\sigma : B^{\otimes 2} \to B^{\otimes 2}$ alg. homomorphism

$\rho : B^{\otimes 2} \to B^{\otimes 2}$ left σ-twisted derivation: $\rho(bb') = \sigma(b)\rho(b') + \rho(b)b'$

$S, R \in B^{\otimes 2}$, $D, \bar{D}, E \in B^{\otimes 3}$, $S_i = 1^{\otimes i-1} \otimes S \otimes 1^{\otimes n-i-1}$ etc.

$\mathfrak{q} = (\sigma, \rho, R, S, D, \bar{D}, E)$

First sanity check: is the size of $B \wr \mathcal{H}_n(q)$ the same as $\Bbbk[S_n] \otimes B^{\otimes n}$? Def $B \wr \mathcal{H}_n(q)$ has PBW basis if the spanning set $\{H_w b\}$ is lin. indep.

Lai-Nakano-Xiang: gave a complicated list of (suff. and necessary) conditions when $D = \bar{D} = E = 0$

Elias: a more complicated (non-explicit) list in general

Thm (M., 2026; particular case) $\sigma =$ flip, $\rho(b) = rb - \sigma(b)r$, $r \in B^{\otimes 2}$, $Z(F) \supset \Bbbk[x]$ s.t. F is a free $\Bbbk[x]$-module.

$\exists$ Basis $\Rightarrow R = \gamma - \sigma(r)r$; γ is symmetric & central

For each such $R \in B^{\otimes 2}$, $\exists!$ choice of $S, D, \bar{D}, E$ s.t. $\exists$ basis: $S = r_{12} + r_{21}$,
$D = r_{12}r_{13} + r_{13}r_{21} - r_{23}r_{13}$
$\bar{D} = r_{23}r_{12} + r_{13}r_{32} - r_{12}r_{13}$
$E = r_{12}r_{13}r_{23} - r_{23}r_{13}r_{12} + r_{13}(R_1 - R_2)$

Example $B = F[x]$ or $F[x^{\pm1}]$, where F is (unital) assoc. algebra. $\Delta_{0,1} \in F^{\otimes 2}$ s.t. $\Delta f = \sigma(f)\Delta$ (e.g. F sym. Frobenius, $\Delta = \Delta(1)$)

$r = \dfrac{\Delta_0 + \Delta_1 x_1}{x_1 - x_2}$, $R \in Z(F \otimes F) \cap (F \otimes F)^{S_2}$ $\rightsquigarrow$ $S = \Delta_1$
$D = \bar{D} = E = 0.$

$\Delta_1 = 0$: Savage ; $F =$ Zigzag $\rightsquigarrow$ Kleshchev-Muth
$\Delta_0 = 0$: Rosso-Savage ; $F = \Bbbk[C_{p-1}]$ $\rightsquigarrow$ GGK
$\Delta_0 = c\Delta_1$: Mathas-Stoppel (RAFH, WIP)

For simplicity, we will always work with this example from now on.

③ Schur-Weyl duality

First check: want $B \wr \mathcal{H}_n$ to act on $B^{\otimes n}$. ($U(\mathfrak{gl}_1)$-case: $U(\mathfrak{gl}_n(B)) \curvearrowright B^{\otimes n} \curvearrowleft B \wr \mathcal{H}_n$)

$H_i(1) = -\alpha_i \rightsquigarrow H^2(1) = -H\alpha = -\sigma(\alpha)H(1) - \rho(\alpha) = \sigma(\alpha)\alpha \Rightarrow R = (\sigma(\alpha) + S)\alpha.$
For simplicity $\rho(\alpha) = 0$ $\qquad = SH(1) + R = -S\alpha + R$
$H_1H_2H_1 = H_2H_1H_2 \Rightarrow \alpha_{23}\alpha_{13}\alpha_{12} = \alpha_{12}\alpha_{13}\alpha_{23}$

Rmk Minuses b/c of some left vs right

Assume $\exists\, \alpha \in F \otimes F$ s.t. the above equations hold (non-trivial condition! Seems to hold in the examples we care about)

Def $e_n = \sum_{w \in S_n} \alpha_w H_{w^{-1}w_0}$; $\alpha_w = \alpha_{i_k} \cdot s_{i_k}(\alpha_{i_{k-1}}) \cdots s_{i_k} \cdots _{i_2}(\alpha_{i_1})$

Easy to see that $e_n^2 = m_n e_n$, $m_n = \sum \alpha_w \pi(w)(\bar{\alpha}_{\pi(w^{-1}w_0)})$, π-involution of S_n, $\bar{\alpha} = \sigma(\alpha) + S$ $\Rightarrow e_n$ is quasi-idempotent

Ex $e_2 = H + \alpha$; $e_2^2 = H^2 + \alpha H + H\alpha + \alpha^2 = (S + \alpha + \sigma(\alpha))(H + \alpha) = (\alpha + \bar{\alpha})e_2 = m_2e_2.$ Ex For affine Hecke, $m_n = [n]!_q$

There is a certain action of $B \wr \mathcal{H}_n \curvearrowright (\Bbbk^d \otimes B)^{\otimes n}$ Def $B \wr S_{d,n} := \mathrm{End}_{B \wr \mathcal{H}_n}(\Bbbk^d \otimes B)^{\otimes n}$ — wreath Schur algebra.

Thm (Lai-M. '25) For RAFH algebras, $B \wr S_{d,n} \curvearrowright (\Bbbk^d \otimes B)^{\otimes n} \simeq \bigoplus_{\lambda \vdash n} e_\lambda B^{\otimes d} \curvearrowleft B \wr \mathcal{H}_n$ satisfies DCP

Rmk In the paper, we only treat the case when Δ is central (i.e. F comm. Frob).

We give 2 proofs; one inspired by equiv. loc (only works under some inv. assumptions), and another more algebraic (works w/o invertibility)

This was done b/c of relation to Kleshchev-Muth.

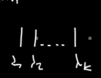
Anyway, the second pf actually goes through w/o Δ being central. The only thing one needs to do is to be very careful about the order in the product $\prod \alpha_{ij}$.

④ Generators & relations

Important property: ∀ $\lambda \models n$, denote $e_\lambda = e_{\lambda_1}\otimes\dots\otimes e_{\lambda_k}$. Then $\exists\, e^\lambda, \tilde{e}^\lambda$ $e_\lambda \tilde{e}^\lambda = e_n = e^\lambda e_\lambda$.

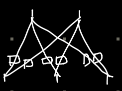
In particular, we get the obvious maps of $\mathbb{H}_n$-modules $\mathbb{H}_n e_\lambda \xrightarrow{\cdot\tilde{e}^\lambda} \mathbb{H}_n e_n$, $\mathbb{H}_n e_n = \mathbb{H}_n e^\lambda e_\lambda \hookrightarrow \mathbb{H}_n e_\lambda$. Can also multiply by elts

$$\begin{matrix} b & \in e_\lambda\mathbb{H}\cap\mathbb{H}e_\lambda \supset e_\lambda\mathbb{H}e_\lambda \\ \| & \| \\ b_1\otimes\dots\otimes b_n & \in \bigotimes e_{\lambda_i}\mathbb{H}\cap\mathbb{H}_{\lambda_i}e_{\lambda}. \end{matrix}$$

Introduce the following notation: $\mathbb{H}_n e_\lambda =$ 

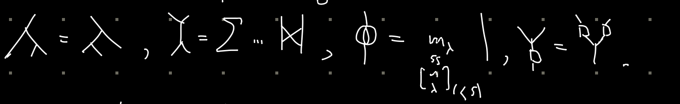
,  $: \mathbb{H}_n e_\lambda \to \mathbb{H}_n e_n$ (merge)
 $: \mathbb{H}_n e_n \to \mathbb{H}_n e_\lambda$ (split)
, $b :$ 

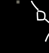
– "renormalization of H"

Thm (LM '25) The wreath Schur algebra has a "chicken-feet basis" 

Rmk Again, proved in comm. case, but once e_n is properly defined this goes through.

In order to get a gen/rel description, enough to massage product of two basis elts into another (as we know the basis already); similar relations appear in Song-Wang.

Prop 

⚠ Not yet: precise form of  $= \sum \dots$  !

Recall that the classical q-Schur algebra is given by the first three equations. The map $U_q(\mathfrak{gl}_d) \longrightarrow S_{d,n}(q)$ is given by $E_i \longmapsto$ , $F_i \longmapsto$ ,

Similarly one can obtain map from quantum loop $U_q(L\mathfrak{gl}_n) \to S^{aff}_{d,n}(q)$ by sending $E_i^k \mapsto$ .

This suggests that in our situation we should have some kind of quantization of $\mathcal{U}(\mathfrak{gl}_n(B))$ with generators $e_i(b), f_i(b)$, $b \in B = F[x]$.

WIP (Lai-M.) Construct this quantization; prove that $U_q(\mathfrak{gl}_n(B)) = \varprojlim_{d\to\infty} B \wr S_{n,d}$.

Rmk Hard part: understanding relations $[e_i, f_j] = 4_{i+j}$; boils down to the unknown rel. above.

Upshot New "affine quantum gps" in the situations described in the intro