A PRIMER ON COHA

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Abstract. We recall definition of classical and cohomological Hall algebras and illustrate this notion with a couple of examples.

1. Motivation

Definition 1.1. The *Hilbert scheme* Hilb_n *of n points on* $\mathbb{A}^2$ is the moduli space of ideals $I \subset \mathbb{C}[X,Y]$ such that $\dim \mathbb{C}[X,Y]/I = n$.

Theorem 1.2. (Fogarty) Hilb_n *is a smooth quasi-projective variety.*

One of the first questions one might ask about a smooth variety is what its cohomology groups are. For Hilb_n, the answer is given by the celebrated Göttsche's formula:

Theorem 1.3 (Ellingsrud-Strømme, Göttsche)**.** *We have*

$$\sum_{n\geq 0} q^n P_t(\mathrm{Hilb}_n) = \prod_{m\geq 1} \frac{1}{1-t^{2m-2}q^m}, \tag{1.1}$$

where P_t denotes the Poincaré polynomial.

The right hand side of (1.1) looks like the graded dimension of a bigraded vector space $\mathbb{C}[X]$, where $\deg X^m = (2m-2, m)$. One can think of several possible interpretations of this formula:

(1) by Grothendieck's trace formula, knowing cohomology groups of a variety amounts more or less to counting the number of its points over finite fields. Thus, one can infer that there exists some kind of Jordan-Hölder decomposition of objets in Hilb_n, and indecomposables are parametrized by positive integers. This is not true verbatim; however, a related fact is that the fixed points under the natural $(\mathbb{C}^\times)^2$-action on Hilb_n are parameterized by partitions of n.
(2) for a simple Lie algebra $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{t} \oplus \mathfrak{n}^-$, its Verma modules are isomorphic to $Sym(\mathfrak{n}^-)$ as $\mathfrak{t}$-graded vector spaces. This suggests $\oplus_n H^*(\mathrm{Hilb}_n)$ might admit an action of some Lie algebra with $\mathfrak{n}^- \cong \mathbb{C}[X]$ as a vector space.

Recall that the *Heisenberg Lie algebra* $\mathcal{H}$ is defined as $\bigoplus_{i\in\mathbb{Z}_{>0}}(\mathbb{C}\partial_i \oplus \mathbb{C}x_i) \oplus \mathbb{C}c$, where the Lie bracket is given by

$$[\partial_i, \partial_j] = [x_i, x_j] = 0, \qquad [\partial_i, x_j] = \delta_{ij} c, \qquad c \text{ is central.}$$

Nakajima proved that there exists a geometric action of $\mathcal{H}$ on $\oplus_n H^*(\mathrm{Hilb}_n)$, which makes the latter into the Fock space representation of $\mathcal{H}$.

The Hilbert scheme of n points can be also seen as the moduli of torsion-free sheaves $\mathcal{F}$ on $\mathbb{P}^2$, equipped with trivialization $\mathcal{F}|_{\mathbb{P}^1_\infty} \simeq \mathcal{O}_{\mathbb{P}^1}$ at infinity, and numerical invariants given by

$$(\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), c_2(\mathcal{F})) = (1, 0, n).$$

What if the rank of $\mathcal{F}$ is greater than 1? One has similar formulas for Poincaré polynomials of such moduli spaces, but their dimensions are bigger. This suggests that there exists a larger algebra acting on them. This algebra naturally appears in the context of cohomological Hall algebras (CoHAs).

2. Ringel-Hall algebra

Before getting to the actual definition of cohomological Hall algebra, let us study a simpler object. In this section we work over a finite field $\mathbf{k} = \mathbb{F}_q$.

Let $\mathcal{C}$ be an abelian $\mathbf{k}$-linear category. We denote by $\operatorname{Ob}\mathcal{C}$ the set of equivalence classes of objects in $\mathcal{C}$, and demand that $\mathbb{C}$ satisfies the following conditions:

- $\operatorname{Ext}^i(\mathcal{A}, \mathcal{B}) = 0$ for any $i \geqslant 2$ and $\mathcal{A}, \mathcal{B} \in \operatorname{Ob}(\mathcal{C})$,
- $\#\operatorname{Ext}^1(\mathcal{A}, \mathcal{B}), \#\operatorname{Hom}(\mathcal{A}, \mathcal{B}) < \infty$ for any $\mathcal{A}, \mathcal{B} \in \operatorname{Ob}(\mathcal{C})$.

Examples of such categories include the category $\operatorname{Rep} Q$ of finite dimensional representations of a quiver Q, and the category $\operatorname{Coh} X$ of coherent sheaves over a smooth projective curve X.

Definition 2.1. The *Ringel-Hall algebra* $H(\mathcal{C})$ of $\mathcal{C}$ is the space $\operatorname{Fun}_c(\operatorname{Ob}\mathcal{C}, \mathbb{C})$ of complex-valued functions $\operatorname{Ob}\mathcal{C}$ with finite support, with product given by

$$f * g = q_* p^* (f \boxtimes g).$$

Here the maps are given by the following diagram

$$\operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C} \xleftarrow{\;p\;} \mathcal{F}l_2 \xrightarrow{\;q\;} \operatorname{Ob}\mathcal{C} \qquad (E_1, E_2/E_1) \xleftarrow{\;p\;} E_1 \subset E_2 \xrightarrow{\;q\;} E_2 \tag{2.1}$$

where $\mathcal{F}l_2$ is the set of two step flags of objects in $\mathcal{C}$.

More explicitly, we have

$$\mathbf{1}_A * \mathbf{1}_B = \sum_{C \in \operatorname{Ob}\mathcal{C}} \# \{A' \subset C : A' \simeq A, C/A' \simeq B\}\, \mathbf{1}_C,$$

where $\mathbf{1}_A$ is the characteristic function of the object A.

Proposition 2.2. *$H(\mathcal{C})$ is an associative algebra.*

Let us illustrate this definition with a couple of examples.

Example 2.3. Let $\mathcal{C}$ be the category of finite-dimensional vector spaces. Then $\operatorname{Ob}\mathcal{C} = \{\mathbf{k}^m, m \geq 0\}$, and the product is given by

$$\mathbf{1}_{\mathbf{k}^m} * \mathbf{1}_{\mathbf{k}^n} = \#\operatorname{Gr}(m, m+n)\mathbf{1}_{\mathbf{k}^{m+n}}.$$

In particular, $H(\mathcal{C}) \simeq \mathbb{C}[\mathbf{1}_{\mathbf{k}}]$ is a commutative algebra.

Remark 2.4. In order to correctly state the results below, we need to twist the product in $H(\mathcal{C})$ by a correction factor:

$$\mathbf{1}_A * \mathbf{1}_B \rightsquigarrow \left(\frac{\#\operatorname{Hom}(B, A)}{\#\operatorname{Ext}^1(B, A)}\right)^{1/2} \mathbf{1}_A * \mathbf{1}_B.$$

Example 2.5. Let $\mathcal{C} = \operatorname{Rep} Q$ be the category of finite-dimensional representations of the quiver $Q = \bullet \longrightarrow \bullet$. It has 3 indecomposable representations:

$$S_1 = (\mathbf{k} \longrightarrow 0), \quad S_2 = (0 \longrightarrow \mathbf{k}), \quad T_{12} = (\mathbf{k} \xrightarrow{\sim} \mathbf{k}).$$

So, the only two representations with dimension vector $(2, 1)$ are $T_{12} \oplus S_1$ and $S_1^{\oplus 2} \oplus S_2$. In particular, the elements

$$\mathbf{1}_{S_1} * \mathbf{1}_{S_1} * \mathbf{1}_{S_2}, \quad \mathbf{1}_{S_1} * \mathbf{1}_{S_2} * \mathbf{1}_{S_1}, \quad \mathbf{1}_{S_2} * \mathbf{1}_{S_1} * \mathbf{1}_{S_1}$$

have a linear relation in $H(\mathcal{C})$:

$$S_1^2 S_2 - (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) S_1 S_2 S_1 + S_2 S_1^2 = 0.$$

The relation above is the Serre relation in $U_\nu(\mathfrak{sl}_3)$, where $\nu = q^{\frac{1}{2}}$.

This example is an instance of the following theorem:

Theorem 2.6. (Ringel, Green) *Let Q be a quiver without loops, and $\mathfrak{g} = \mathfrak{g}_Q$ is the corresponding Kac-Moody Lie algebra. Then we have an inclusion*

$$U_\nu(\mathfrak{n}^+) \hookrightarrow H(\operatorname{Rep} Q),$$

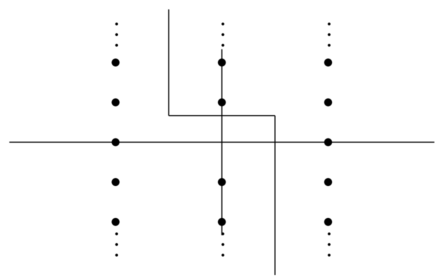
where $\nu = q^{\frac{1}{2}}$. This map is an isomorphism for quivers of type ADE.

Example 2.7. Let $Q = \bullet\circlearrowleft$ be the Jordan quiver, and let $\mathcal{C} = \operatorname{Rep}^{nil} Q$ be the category of its nilpotent representations. Then $H(\operatorname{Rep} Q) = \mathbb{C}[E_i]_{i\in\mathbb{N}}$. This computation largely follows from the existence of Jordan normal form.

Example 2.8. Finally, let $\mathcal{C} = \operatorname{Coh}(\mathbb{P}^1)$.

Theorem 2.9 (Kapranov)**.** *There exists an inclusion $U_q(\mathcal{L}\mathfrak{sl}_2)^+ \subset H(\operatorname{Coh}(\mathbb{P}^1))$.*

Here, by positive half we mean the nonstandard Borel subalgebra, spanned by the roots lying to the right of the line in the diagram below:



Here fat dots represent the root system of $\mathcal{L}\mathfrak{sl}_2$.

3. Cohomological version

From now on our ground field is $\mathbf{k} = \mathbb{C}$. We want to devise a cohomological version of Ringel-Hall algebra, such that we get a version of quantum groups as an output. Let us first restrict attention to the case $\mathcal{C} = \operatorname{Rep} Q$. The easiest thing we can do is to replace functions by homology groups.

Definition 3.1. $\operatorname{CoHA}(Q) = (H_*^{\mathrm{BM}}(\operatorname{Rep} Q), q_*p^*)$, were the maps p, q are the same as in (2.1).

Let us discuss this definition. First, $\operatorname{Rep} Q$ is considered as an Artin stack. More specifically, it is a stack of the form V/G, so that its Borel-Moore homology groups are computed by equivariant homology. Furthermore, the map q is proper, and p is a vector bundle stack with fibre $\operatorname{Ext}^1(B, A)/\operatorname{Hom}(B, A)$ over $(A, B) \in \operatorname{Rep} Q \times \operatorname{Rep} Q$. Thus the product map q_*p^* is well defined.

As for Ringel-Hall algebras, $\operatorname{CoHA}(Q)$ is an associative algebra.

Example 3.2. Let us consider the simplest example $Q = \bullet$. Then $\operatorname{Rep} Q = \operatorname{Vect} = \coprod_n BGL_n$ as an algebraic stack. Recall that

$$H_*^{BM}(\operatorname{Vect}) = \bigoplus_n H_*^{BM}(BGL_n) = \bigoplus_n \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}.$$

Observe that this space is larger than its analogue from Example 2.3. Still, it has the same size as the positive half of $U(\mathcal{L}\mathfrak{sl}_2)^+$. In particular, one could expect that the product in $\operatorname{CoHA}(\bullet)$ is commutative. Let us show that this is not true.

Take $x_1^m, x_1^n \in H_*(B\mathbb{G}_m)$, and let us compute $x_1^m * x_1^n \in H_*(BGL_2)$. For brevity, we will write $G = GL_2$, and B, $T \simeq \mathbb{G}_m \times \mathbb{G}_m$ for standard Borel subgroup and maximal torus in G respectively.

The diagram (2.1) then becomes

$$\begin{array}{ccc} & BB & \\ {}^{p}\swarrow & & \searrow^{q} \\ BT & & BG \end{array} \tag{3.1}$$

The map $p^* : H_*(BT) \to H_*(BB)$ is an isomorphism. Let us compute q_*. Passing to equivariant homology, we are reduced to computing the pushforward $H_*^G(G/B) = H_*^G(\mathbb{P}^1) \to H_*^G(\mathrm{pt})$. By equivariant localization theorem, the following square commutes, up to multiplication by the Euler class of inclusion $\iota : \{0, \infty\} \hookrightarrow \mathbb{P}^1$:

$$\begin{array}{ccc} H_*^G(\mathbb{P}^1) & \xrightarrow{q_*} & H_*^G(\mathrm{pt}) \\ {\scriptstyle \iota^*}\downarrow & & \downarrow \\ H_*^T(0 \sqcup \infty) & \longrightarrow & H_*^T(\mathrm{pt}) \end{array}$$

Since $T_0\mathbb{P}^1 = \mathfrak{n}_- = x_1 - x_2$ as a T-character, we get

$$x_1^m * x_1^n = \frac{x_1^m x_2^n}{x_1 - x_2} + \frac{x_1^n x_2^m}{x_2 - x_1} = \frac{x_1^m x_2^n - x_1^n x_2^m}{x_1 - x_2} \neq x_1^n * x_1^m.$$

Note that if the denominators in the product above were squared, we would get an equality between two sides. Because of this consideration, let us replace Vect by $T^*\mathrm{Vect}$, and the correspondences by their conormal bundles. The diagram 3.1 turns into

$$\begin{array}{ccc} & N^*\left(BB/(BT \times BG)\right) & \\ {}^{p}\swarrow & & \searrow^{q} \\ T^*BT & & T^*BG \end{array}$$

Thanks to Pavel's talk, we have $T^*BG = \mathfrak{g}^*/G$ and $N^*(BB/(BT \times BG)) = \widetilde{\mathfrak{g}}^*/G$ (up to derived shifts), and the map q becomes the Grothendieck-Springer resolution. One can check that if we localize in this situation, the difference of Euler classes becomes trivial, so that the product becomes commutative:

$$x_1^m * x_1^n = x_1^m x_2^n + x_1^n x_2^m = x_1^n * x_1^m.$$

The upshot is that in order to get something resembling Lie algebra $\mathfrak{g}$, one has to consider cotangent stack $T^* \operatorname{Rep} Q$. It is well known that the latter (at least after truncating the derived part) is isomorphic to the category $\operatorname{Rep} \Pi_Q$ of representation of the *preprojective algebra* Π_Q. Note that the category $\operatorname{Rep} \Pi_Q$ has cohomological dimension 2, that is Ext^2 doesn't vanish. Therefore the map p in (2.1) is no longer smooth, so that the pullback is not defined on the nose.

In order to get a well-defined pullback p^*, we will upgrade $\mathcal{F}l_2$ to a derived stack $\widetilde{\mathcal{F}l}_2$. Namely, note that for any pair of representations $V_1, V_2 \in \operatorname{Rep} \Pi_Q$, there exists a quasi-isomorphism

$$\mathrm{RHom}(V_1, V_2)[1] \quad \simeq \quad \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{E}_1,$$

where $\mathcal{E}_i$ are locally-free sheaves on $\operatorname{Rep} \Pi_Q \times \operatorname{Rep} \Pi_Q$. The stack $\mathcal{F}l_2$ is then precisely the total space of $\mathrm{Ker}(\mathcal{E}_0 \to \mathcal{E}_1)$. We have:

$$\begin{array}{ccc} \mathcal{F}l_2 & \longrightarrow & \mathrm{Tot}\, \mathcal{E}_0 \\ {\scriptstyle p}\downarrow & & \downarrow \\ \operatorname{Rep} \Pi_Q \times \operatorname{Rep} \Pi_Q & \longrightarrow & \mathrm{Tot}\, \mathcal{E}_1 \end{array}$$

where the lower horizontal map is the zero section. Note that this diagram is a cartesian square set-theoretically, but not scheme-theoretically. Let us define $\widetilde{\mathcal{F}l}_2$ as the homotopy fiber product:

$$\widetilde{\mathcal{F}l}_2 := (\operatorname{Rep} \Pi_Q \times \operatorname{Rep} \Pi_Q) \times_{\mathrm{Tot}\, \mathcal{E}_1} \mathrm{Tot}\, \mathcal{E}_0.$$

Then the pullback along

$$\widetilde{\mathcal{F}l}_2 \to \operatorname{Rep}\Pi_Q \times \operatorname{Rep}\Pi_Q$$

is defined by construction; let us denote it by $p^!$.

One can check that $q_* p^!$ produces an associative product on $H_*(\operatorname{Rep}\Pi_Q)$. The same trick works for any other reasonable category of cohomological dimension 2.

Example 3.3. When Q is of type ADE, $\mathrm{CoHA}(\Pi_Q)$ is isomorphic to the positive half of Yangian $Y_\hbar(\mathfrak{g}_Q)$ (up to a certain localization).

From now on, we will be consistently sketchy with taking or not taking Drinfeld double of algebras.

Example 3.4. Let Q be the Jordan quiver. Then

$$T^*(\operatorname{Rep} Q) = \bigsqcup_n \{(X,Y) \in \mathfrak{gl}_n : [X,Y] = 0\} / GL_n = \operatorname{Coh}(\mathbb{C}^2)^{\text{fin.length}}.$$

The resulting $\mathrm{CoHA}(\mathbb{C}^2)$ acts by correspondences on $\oplus_n H^*(\mathrm{Hilb}_n)$, as well as higher rank moduli spaces.

Let us replace Borel-Moore homology by K-theory, and consider it equivariantly with respect to the natural action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$. The K-theoretic Hall algebra $\mathrm{KHA}^{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2)$ is called the *elliptic Hall algebra*. It is also known under the following names:

- Ringel-Hall algebra of an elliptic curve;
- quantum toroidal $\mathfrak{gl}_1$;
- limit of spherical DAHAs of type A_n, $n \to \infty$;
- HOMFLY skein algebra of a torus (for $q = t$);
- Feigin-Tsymbaliuk shuffle algebra.

Example 3.5 (M., work in progress). Another example of a category of cohomological dimension 2 is $\operatorname{Coh} S$, where S is a smooth surface. In keeping with our affinization intuition from the example 3.2, it is natural to expect that $\mathrm{KHA}^{\mathbb{C}^*\times\mathbb{C}^*}(T^*\mathbb{P}^1)$ contains a half of quantum toroidal algebra $U_{q,t}(\widehat{\widehat{\mathfrak{gl}}}_2)$.

More generally, let Γ be a finite subgroup of SU_2, and S is the resolution of singularity $\mathbb{C}^2/\Gamma$. Then the following inclusion should take place:

$$U_q(\widehat{\widehat{\mathfrak{g}}}_\Gamma) \hookrightarrow \mathrm{KHa}^{\mathbb{C}^*}(S_\Gamma),$$

where the right-hand side has two quantization parameters if Γ is of type A.