KLR/Schur algebras of curves & semicuspidal representations

Plan
1) KLR algebras for quivers, PBW basis questions
2) KLR & Schur for curves
3) Relation between the two.

1) KLR algebras & categorification

KLR = Khovanov, Lauda, Rouquier

KLR algebras were introduces to categorify quantum groups.

$Q = (I, E)$ a quiver. (vertices, edges) $\quad \alpha \in \mathbb{Z}^I_{\geq 0}$

$\mathrm{Rep}_\alpha Q$ = stack of reps of Q of dim. α.

$[\mathbb{C}^N / GL_{n_1} \times \dots GL_{n_k}]$.

$\tilde{\alpha} = (\epsilon_{i_1}, \dots, \epsilon_{i_{|\alpha|}})$ - composition of α $\qquad \epsilon_i = (0, \dots, \underset{\substack{\uparrow \\ i}}{1}, \dots, 0)$.

$$\sum_j \epsilon_{i_j} = \alpha$$

$Fl_{\tilde{\alpha}} Q$ = stack of flags of reps of Q of type $\tilde{\alpha}$

$$0 \subset V_1 \subset V_2 \subset \dots \subset V_{|\alpha|}$$
s.t. $\dim V_{i+1}/V_i = \epsilon_i$

$\pi_{\tilde{\alpha}} : Fl_{\tilde{\alpha}} Q \to \mathrm{Rep}_\alpha Q$ - forgets the flag

$$\pi_\alpha = \coprod_{\tilde{\alpha} \text{ comp. of } \alpha} \pi_{\tilde{\alpha}} : \coprod_{\tilde{\alpha}} Fl_{\tilde{\alpha}} Q \to \mathrm{Rep}_\alpha Q.$$

Notation $\underline{k}$ = a field or $\mathbb{Z}$

Def $R(\alpha) := \mathrm{Ext}^*((\pi_\alpha)_* \underline{k}, (\pi_\alpha)_* \underline{k})$

Rmk Alternatively: $R(\alpha)$ = diagrams with $|\alpha|$ strands, dots, crossings, I colours, + relations (S_n-rel + $\times - \times = ||$)

$R(\alpha)\otimes R(\beta) \longrightarrow R(\alpha+\beta)$ $\rightsquigarrow$ induction and restriction functors on module categories.

<u>Thm</u> (KLR) $\left(\bigoplus_\alpha R(\alpha)\text{-gr.pr.mod}, \text{ind}, \text{res}\right)$

Categorifies the bialgebra $U_q(n_Q^-)$

$U_q(n^-)$ has PBW basis. (for fixed convex order on the positive roots).

<u>Q</u> Can we categorify it?

[Brundan, Kleshchev, McNamara].

Let Q be of Dynkin type.

- $R(\alpha)$-mod is properly stratified by subcategories of

roughly, every object has a filtration, s.t. graded pieces live in $Cusp_{\alpha'}$

Cuspidal modules $Cusp_{\alpha'}$

$Res_{\beta,\gamma}=0$ $\quad \beta=\sum$ roots $\geq \alpha'$, $\gamma=\sum$ roots $\leq \alpha'$

- each $Cusp_\alpha$ (positive root) contains a unique irreducible s.t. the proj. cover generate PBW basis.

<u>Rmk</u> In positive char., have to be careful, consider $Cusp_{n\alpha}$

What if Q is of affine type?

We also have a proper stratification by $\frac{1}{2}Cusp_{n\alpha}$ $(\geq \rightsquigarrow >, \leq \rightsquigarrow <)$

For real roots - same

For imaginary δ - much more complicated.

$\frac{1}{2}\mathrm{Cusp}_{n\delta}$ has a projective generator $P \Rightarrow \simeq \mathrm{End}\,P$-mod

char $k=0$: [Kleshchev-Muth]

P is simple enough $\Rightarrow$ $\mathrm{End}\,P$ is computed

$\mathrm{End}\,P \cong Z_n$ "affine zigzag algebra"

char $k=p$: $\mathrm{End}\,P$ is not computed.

Expected: "Schur version" of Z_n.

In effect: this is true for $Q = \bullet \rightrightarrows \bullet$

2| KLR/Schur algs of curves

jt w/ Ruslan Maksimau

Let C be a smooth curve $/\mathbb{C}$,

$\mathrm{Tor}\,C$ = stack of torsion sheaves on C.

sheaves of length n

$\mathrm{Tor}\,C = \coprod_n \mathrm{Tor}_n C$

Ex. $\to$ $C = \mathbb{A}^1$

Sheaf of lg n $\Leftrightarrow$ $\mathbb{C}[x] \curvearrowright \mathbb{C}^n$ $\Leftrightarrow$ el-t of $\mathfrak{gl}_n$

$\mathrm{Tor}_n \mathbb{A}^1 = [\mathfrak{gl}_n /_{\mathrm{ad}} GL_n] = \mathrm{Rep}_n \circlearrowright$

$\to$ $C = \mathbb{C}^*$

— " — $\Leftrightarrow$ $\mathbb{C}[x,x^{-1}] \curvearrowright \mathbb{C}^n$ $\Leftrightarrow$ el-t of GL_n

$\mathrm{Tor}_n \mathbb{C}^* = [GL_n /_{\mathrm{Ad}} GL_n]$.

some smooth variety

$\to$ In general, $\mathrm{Tor}_n C = [Q_n / GL_n]$

$\lambda = (\lambda_1, \dots, \lambda_k) \quad \sum \lambda_i = n$

$\rightsquigarrow \mathcal{F}_\lambda$ = moduli of flags $\mathcal{E}_1 \subset \dots \subset \mathcal{E}_k$ of tor. sheaves
$lg(\mathcal{E}_i / \mathcal{E}_{i-1}) = \lambda_i$.

$\pi : \coprod_\lambda \mathcal{F}_\lambda \longrightarrow \mathrm{Tor}_n$.

<u>Def</u> [MM] The <u>Schur algebra</u> of C $\qquad \pi_{1^n} : \mathcal{F}_{1,\dots,1} \to \mathrm{Tor}_n$

$$S_n^C = \mathrm{Ext}^*(\pi_* \underline{k}, \pi_* \underline{k})$$

The <u>KLR algebra</u> of C

$$R_n^C = \mathrm{Ext}^*((\pi_{1^n})_* \underline{k}, (\pi_{1^n})_* \underline{k}) \subset S_n.$$

<u>Properties</u>

1) $S_n = \bigoplus_{\lambda,\mu} S_{\lambda\mu} = \bigoplus \mathrm{Ext}^*((\pi_\lambda)_* \underline{k}, (\pi_\mu)_* \underline{k})$

2) $\lambda = \mu$, by adjunction between π^* & π_* $\quad H_*(\mathcal{F}_\lambda) \hookrightarrow S_{\lambda\lambda}$

<u>Fact</u> $H_*(\mathcal{F}_\lambda) = \left(H_*(C^n)[x_1, \dots, x_n] \right)^{S_{\lambda_1} \times \dots \times S_{\lambda_k}}$

3) λ is subpartition of μ $\qquad$ e.g. $\mu = (2,3) \supset \lambda = (\underbrace{1,1}, \underbrace{2,1})$

$$\begin{tikzcd} \mathcal{F}_\lambda \arrow[r, "\text{forget}"] \arrow[dr, "\pi_\lambda"'] & \mathcal{F}_\mu \arrow[d, "\pi_\mu"] \\ & \mathrm{Tor}_n \end{tikzcd} \rightsquigarrow \quad \begin{aligned} S_\mu^\lambda &= [\mathcal{F}_\lambda] \in H_*(\mathcal{F}_\lambda) \subset S_{\lambda\mu} \text{ - \underline{split}} \\ M_\lambda^\mu &= [\mathcal{F}_\lambda] \in H_*(\mathcal{F}_\lambda) \subset S_{\mu\lambda} \text{ - \underline{merge}} \end{aligned}$$

4) $S_n = \mathrm{Ext}^*(\pi_* \underline{k}, \pi_* \underline{k}) \curvearrowright \mathrm{Ext}^*(\pi_* \underline{k}, \underline{k}) = \bigoplus_\lambda H_*(\mathcal{F}_\lambda)$

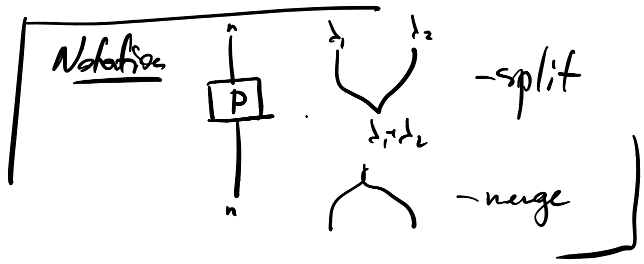
$$= \bigoplus_\lambda \left(H_*(C^n)[x_1, \dots, x_n] \right)^{S_{\lambda_1} \times \dots \times S_{\lambda_n}}.$$

–polynomial representation.

<u>Thm</u> [M-M.] char $k=0$

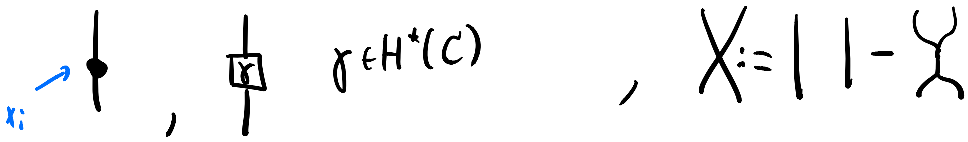
- Polynomial representation is faithful.
- split, merge (weird symmetrization), polynomial operators act by explicit formulas
- S_n is generated by ops above.

<u>Rmk</u> have explicit basis.

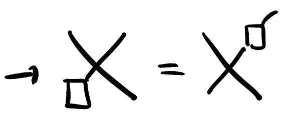


<u>Thm</u> [M.M.]

R_n it is generated

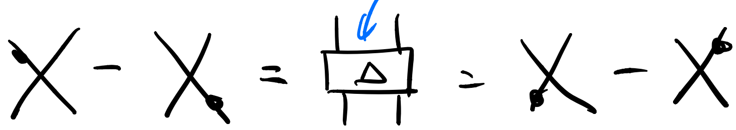


<u>relations</u>: → S_n-relations for crossings



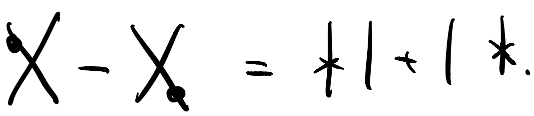
class of Δ-diagonal in $H^*(C\times C)$.

(∗)



<u>E.g.</u> $C=\mathbb{P}^1$ $\quad H^*(C)=k[c]/c^2 \quad c =:$

(∗)



<u>Observation</u> $R_n^{\mathbb{P}^1}$ is precisely the affine zigzag alg Z_n of Kleshchev-Muth for $Q=\bullet \rightrightarrows \bullet$

Heuristically, expect $\frac{1}{2}\mathrm{Cusp}_{n\delta} \simeq S_n^{\mathbb{P}^1}$-mod. in any characteristic.

Not quite true.

Rmk For $C = \mathbb{P}^1$, any coefficient ring works.

$\frac{1}{2}\mathrm{Cusp}_{n\delta} \simeq \mathrm{End}\,P$ - mod

Thm [M-M]

We have a homomorphism $\mathrm{End}\,P \xrightarrow{\Phi} S_n^{\mathbb{P}^1}$ in any characteristic.

If char $k = 0$, Φ is an iso.

If $k = \mathbb{Z}$, Φ is injective (Δ we know the image).

If char $k > 0$, neither inj nor surj.

Cor $\tilde{S}_n^{\mathbb{F}_p} := \mathrm{Im}\,\Phi \otimes_{\mathbb{Z}} \mathbb{F}_p$. Then $\mathrm{End}\,P \simeq \tilde{S}_n^{\mathbb{F}_p}$.

Rmk 1) Why? Basically, thanks to der. eq. $D^b\mathrm{Coh}\,\mathbb{P}^1 \xrightarrow{\sim} D^b\,\mathrm{Rep}\,\bullet\rightrightarrows\bullet$

2) $\tilde{S}_n^{\mathbb{F}_p}$ has a polynomial representation. Don't know if it is faithful.

3) Φ is not surjective. - related to:

$H^*(\mathrm{Tor}_n\,\mathbb{P}^1, \mathbb{Z})$ - not gen. by

Künneth-Chern classes of

the universal sheaf $\mathcal{E}_n \in \mathrm{Coh}((\mathrm{Tor}_n\,\mathbb{P}^1) \times \mathbb{P}^1)$

4) the proof of injectivity is algebraic.

In this situation, no good theory of parity sheaves!

Rmk For affine quivers of type A, E have other der. equivalences

$D^b(\mathrm{Rep}\,Q) \simeq D^b\,\mathrm{Coh}\,(\text{weighted } \mathbb{P}^1)$

WIP extend our result to other types.