COHERENT SHEAVES ON SURFACES, COHAS AND DEFORMED $W_{1+\infty}$-ALGEBRAS

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Abstract. We compute the COHA of zero-dimensional sheaves on an arbitrary smooth quasi-projective surface S with pure cohomology, deriving an explicit presentation by generators and relations. When S has trivial canonical bundle, this COHA is isomorphic to the enveloping algebra of deformed trigonometric $W_{1+\infty}$-algebra associated to the ring $H^*(S,\mathbb{Q})$. We also define a double of this COHA, show that it acts on the homology of various moduli stacks of sheaves on S and explicitly describe this action on the products of tautological classes. Examples include Hilbert schemes of points on surfaces, the moduli stack of Higgs bundles on a smooth projective curve and the moduli stack of 1-dimensional sheaves on a $K3$ surface in an ample class. The double COHA is shown to contain Nakajima's Heisenberg algebra, as well as a copy of the Virasoro algebra.

Contents

0. Introduction

Let S be a smooth quasi-projective complex surface. In the pioneering work [31], Nakajima constructed an action of a Heisenberg algebra $\mathfrak{h}_S$ on the direct sum $\mathbf{V}(S) := \bigoplus_{n\geqslant 0} H^*(\mathrm{Hilb}_n(S),\mathbb{Q})$ of cohomology groups of the Hilbert schemes of points on S. Here, $\mathfrak{h}_S$ is modeled on the cohomology ring $H^*(S,\mathbb{Q})$. What's more, Nakajima identified $\mathbf{V}(S)$ with the Fock space representation of $\mathfrak{h}_S$, thereby providing a very fruitful bridge between the enumerative geometry of $\mathrm{Hilb}(S)$ and the representation theory of Heisenberg algebras. This has led to a flurry of remarkable results on the topology of Hilbert schemes of points on surfaces or of instanton spaces (see, among many others, [23], [24], [43], [41], ...) and has served as model for the theory of quiver varieties. Similar constructions exist also in the K-theoretic context, see e.g. [33], and may be upgraded to the T-equivariant setting in the presence of a torus action on S.

Nakajima operators arise from the correspondences

$$\mathrm{Hilb}_n(S) \times S \xleftarrow{q} \mathrm{Hilb}_{n,n+k}(S) \xrightarrow{p} \mathrm{Hilb}_{n+k}(S)$$

(and their transposes), where $\mathrm{Hilb}_{n,n+k}(S)$ is the nested Hilbert scheme parametrizing pairs of subschemes $Z \subset Z'$ of respective lengths n, $n+k$. The scheme $\mathrm{Hilb}_{n,n+k}(S)$ carries the tautological vector bundle $H^0(S, Z'/Z)$; taking cup product with the characteristic classes of these bundles yields additional operators, generating a much larger algebra than $U(\mathfrak{h}_S)$. For $S = \mathbb{C}^2$ equipped with the natural $(\mathbb{C}^*)^2$-action, such an algebra was studied in [39], where it was identified with the so-called *affine Yangian of* $\mathfrak{gl}_1$ (see also [28] for a different approach)[1]. In *loc.cit.* the same algebra was shown to act on the cohomology of any of the *instanton spaces*, which are moduli spaces of higher rank (framed, torsion-free) sheaves on $\mathbb{C}^2$. The affine Yangian of $\mathfrak{gl}_1$ is in turn a two-parameter deformation of the algebra $W_{1+\infty}$ of differential operators on the circle, and its representation theory is strongly related to that of affine W-algebras of $\mathfrak{gl}_r$.

The aim of this paper is to provide a generalization of the results (except for the link to affine W-algebras) to the case of an *arbitrary* smooth quasi-projective surface S which is cohomologically pure (for instance, projective). This provides actions of explicit infinite-dimensional algebras that we call *deformed $W_{1+\infty}$-algebras* on the Borel-Moore homology of many interesting moduli stacks of coherent sheaves on S. Our approach is based on the theory of (2-dimensional) cohomological Hall algebras, which we now succinctly recall.

0.1. **Cohomological Hall algebras.** Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category (satisfying suitable finiteness conditions, such as in e.g. [8, § 5.1]) and let $\mathfrak{M}_\mathcal{C}$ denote the derived stack of objects in $\mathcal{C}$. The prime example of interest for us is the category of coherent sheaves on an algebraic surface S. Extensions in $\mathcal{C}$ are controled by the (Hecke) correspondence

$$\mathfrak{M}_\mathcal{C} \times \mathfrak{M}_\mathcal{C} \xleftarrow{q} \widetilde{\mathfrak{M}}_\mathcal{C} \xrightarrow{p} \mathfrak{M}_\mathcal{C} \tag{0.1}$$

where $\widetilde{\mathfrak{M}}_\mathcal{C}$ is the stack of short exact sequences in $\mathcal{C}$. Here p, resp. q associate to a sequence its middle, resp. extreme terms. The properties of the maps p, q depend heavily on the global dimension of $\mathcal{C}$; crucially, q is quasi-smooth when $\mathcal{C}$ is of global dimension at most 2. When in addition p is proper, the composition $p_*q^! : H_*(\mathfrak{M}_\mathcal{C}, \mathbb{Q})^{\otimes 2} \to H_*(\mathfrak{M}_\mathcal{C}, \mathbb{Q})$ yields a structure of an associative algebra on the Borel-Moore homology $H_*(\mathfrak{M}_\mathcal{C}, \mathbb{Q})$; this is the cohomological Hall algebra (COHA) of $\mathcal{C}$. Furthermore, any locally closed susbtack $\mathfrak{M}^\circ_\mathcal{C} \subset \mathfrak{M}_\mathcal{C}$ for which (0.1) restricts to a correspondence

$$\mathfrak{M}_\mathcal{C} \times \mathfrak{M}^\circ_\mathcal{C} \xleftarrow{q} \widetilde{\mathfrak{M}}^\circ_\mathcal{C} \xrightarrow{p} \mathfrak{M}^\circ_\mathcal{C}$$

gives rise to a $H_*(\mathfrak{M}_\mathcal{C}, \mathbb{Q})$-module structure on $H_*(\mathfrak{M}^\circ_\mathcal{C}, \mathbb{Q})$ (such substacks are called *Hecke patterns* in [22]). In other words, the same algebra $H_*(\mathfrak{M}_\mathcal{C}, \mathbb{Q})$ acts simultaneously on the homology of all Hecke patterns. Hecke patterns may for instance be constructed using stability conditions and/or framings. This construction appears in [38] (in the K-theoretic setting) and in [39] in the context of quiver varieties and instanton spaces, where it gives rise to *Yangians* of Kac-Moody algebras, a family of infinite-dimensional quantum groups. The construction of the COHA was later extended to much more general contexts, see e.g. [44],[29], [37], [4], [22], [45], [34],. . .

The main object of study of this paper is the COHA of the category of zero-dimensional coherent sheaves on a smooth surface S. Such COHAs were previously considered in [29], [22] and, in the K-theoretical context, in [45] and [33], where quadratic relations (of Ding-Iohara type) between degree one generators were found and actions on smooth moduli spaces were constructed. Here, we

[1]the K-theoretic version was considered in [38] where it was identified with the elliptic Hall algebra (see also [13])

focus on the Borel-Moore homology COHA and fully determine this COHA under the assumption that S has a pure cohomology. As far as we are aware, the only case in which this COHA was fully determined before was $S = \mathbb{A}^2$ with a torus action, see [6], [39].

0.2. **Deformed $W_{1+\infty}$-algebras associated to a surface.** In § 3 and § 5, to which we refer for details, we introduce and begin the study of a family of associative algebras $W(S)$ attached to smooth, pure surfaces S. Let us begin by assuming that S is proper. Let c_1, c_2 be the Chern classes of S and $s_2 = c_1^2 - c_2$. The algebra $W^{(\mathbf{c})}(S)$ is generated by collections of elements $\{T_n^{\pm}(\lambda), \psi_n(\lambda) \,|\, n \geqslant 0, \lambda \in H^*(S, \mathbb{Q})\}$ and a central element $\mathbf{c}$ modulo relations among which the most important ones are

$$[\psi_m(\lambda), T_n^{\pm}(\mu)] = \pm m T_{m+n-1}^{\pm}(\lambda\mu),$$

$$\begin{aligned}[][T_m^{\pm}(\lambda), T_{n+3}^{\pm}(\mu)] - 3[T_{m+1}^{\pm}(\lambda), T_{n+2}^{\pm}(\mu)] + 3[T_{m+2}^{\pm}(\lambda), T_{n+1}^{\pm}(\mu)] - [T_{m+3}^{\pm}(\lambda), T_n^{\pm}(\mu)]&\\ - [T_m^{\pm}(\lambda), T_{n+1}^{\pm}(s_2\mu)] + [T_{m+1}^{\pm}(\lambda), T_n^{\pm}(s_2\mu)] \pm \{T_m^{\pm}, T_n^{\pm}\}(c_1\Delta_S\lambda\mu) = 0&,\end{aligned}$$

$$\sum_{w \in S_3} w \cdot [T_{m_3}^{\pm}(\lambda_3), [T_{m_2}^{\pm}(\lambda_2), T_{m_1+1}^{\pm}(\lambda_1)]] = 0$$

as well as the double relation (3.17), which expresses the commutators $[T_m^+(\lambda), T_{m'}^-(\mu)]$ as polynomials in ψ_n's. We denote by $W^{\pm}(S)$, resp. $W^0(S)$ the subalgebras generated by $\{T_n^{\pm}(\lambda)\}$ and $\{\psi_n(\lambda), \mathbf{c}\}$ respectively. The algebra $W^{(\mathbf{c})}(S)$ is $\mathbb{Z} \times \mathbb{N}$-graded, where $T_n^{\pm}(\lambda), \psi_n(\lambda)$ are put in degrees $(\pm 1, 2n - 2 + \deg(\lambda))$ and $(0, 2n - 2 + \deg(\lambda))$ respectively.

0.3. **Main results.** Let us now describe our main results, referring to the body of the text for details:

Theorem A (Theorem 3.2, Propositions 3.12, 3.15, 3.20)**.** *Let S be a smooth and proper surface. The following hold:*

(a) *There is a triangular decomposition $W^{(\mathbf{c})}(S) \simeq W^-(S) \otimes W^0(S) \otimes W^+(S)$,*
(b) *The graded character of $W^+(S)$ is given by*

$$P_{W^+(S)}(z, w) = \mathrm{Exp}\left(\frac{P_S(z) z^{-2} w}{(1 - z^2)(1 - w)}\right)$$

where $P_S(z)$ is the Poincaré polynomial of S,

(c) *There are embeddings $U(\mathfrak{h}_S) \hookrightarrow W^{(\mathbf{c})}(S)$ and $U(\mathrm{Vir}_S) \hookrightarrow W^{(\mathbf{c})}(S)$, where $\mathfrak{h}_S$ and Vir_S are the Heisenberg and Virasoro algebras modeled on $H^*(S, \mathbb{Q})$; the respective central charges of $\mathfrak{h}_S$ and Vir_S as functions of $\lambda, \mu \in H^*(S)$ are given by*

$$C_{\mathfrak{h}} = \mathbf{c} \int_S \lambda\mu, \qquad \eta_{\mathrm{Vir}} = \mathbf{c} \left(\int_S c_2 \lambda\mu - (1 - \mathbf{c}^2) \int_S c_1^2 \lambda\mu + 2\psi_0(c_1\lambda\mu) \right).$$

When the surface S has trivial canonical bundle, the W-algebra turns out to be the enveloping algebra of a Lie algebra. More precisely:

Theorem A′ (Theorem 3.5)**.** *Let S be a proper surface, such that $c_1 = 0$ and $s_2 = q^2$ for some $q \in H^2(S)$. Then*

$$W^+(S) \simeq U(\mathfrak{w}^+(S)),$$

where $\mathfrak{w}^+(S)$ is the Lie algebra spanned by elements $z^m D^n \lambda$ with $m \geqslant 1, n \geqslant 0, \lambda \in H^*(S, \mathbb{Q})$, subject to the relations

$$[z^m D^n \lambda\,,\, z^{m'} D^{n'} \mu] = z^{m+m'} \frac{(D + m'q)^n D^{n'} - D^n (D + mq)^{n'}}{q} \lambda\mu.$$

We also construct a natural representation $\mathbf{F}^{(r)}(S)$ of $W^{(r)}(S) := W^{(\mathbf{c})}(S)_{|\mathbf{c}=r}$ for any integer $r\geqslant 0$, which we call the *level r Fock space*. As a vector space,

$$\mathbf{F}^{(r)}(S) := \Lambda(S)[s, s^{-1}]_{|\mathbf{r}=r}$$

where $\Lambda(S) = \mathbb{Q}[p_n(\lambda), \mathbf{r} \mid n\geqslant 0, \lambda \in H^*(S, \mathbb{Q})]$. The space $\Lambda(S)$ may be understood as the ring of universal tautological classes on the stack $\mathfrak{Coh}(S)$ of coherent sheaves on S, see § 1.6. We prove in § 4 the following

Theorem A$''$ (Corollary 4.7, Remark 2.10)**.** *For any $r \in \mathbb{N}$, there is an action of $W^{(r)}(S)$ on $\mathbf{F}^{(r)}(S)$ via Fourier modes of the vertex operators*

$$\Theta^+(z) = \exp\Big(\sum_{\gamma;k\geqslant 1} \frac{p_k}{k}(\gamma) \otimes \gamma^* s^{-k}\Big)_{[s<r]} \exp\Big(-\sum_{\gamma;n\geqslant 0} \frac{\partial}{\partial \kappa_n(\gamma)} \otimes \gamma s^n\Big)$$

$$\Theta^-(z) = \exp\Big(-\sum_{\gamma;k\geqslant 1} \frac{\tau_{c_1} p_k}{k}(\gamma) \otimes \gamma^* s^{-k}\Big)_{[s<-r]} \exp\Big(\sum_{\gamma;n\geqslant 0} \frac{\partial}{\partial \kappa_n(\gamma)} \otimes \gamma s^n\Big)$$

where $\{\gamma\}$, $\{\gamma^*\}$ are dual bases of $H^*(S, \mathbb{Q})$ and where the elements $\{\kappa_n(\lambda)\}$ are related to the $\{p_k(\lambda)\}$ through relation (2.24) *involving the Todd class of S, and where τ_{c_1} is a certain shift automorphism of $\Lambda(S)$, see § 0.4. This representation is faithful for $r > 0$, but is neither irreducible nor highest weight.*

In the case of open surfaces S, we may replace $H^*(S, \mathbb{Q})$ by either $H^*(S, \mathbb{Q})$ or $H^*_c(S, \mathbb{Q})$, the cohomology with compact supports. It turns out that both are important for applications. This leads us to define not one, but *four* versions $W^{(\mathbf{c})}_{\uparrow\uparrow}(S), W^{(\mathbf{c})}_{\uparrow\downarrow}(S), W^{(\mathbf{c})}_{\downarrow\downarrow}(S)$ and $W^{(\mathbf{c})}_{\downarrow\uparrow}(S)$ of deformed $W_{1+\infty}$-algebra, depending on a choice of H^* or H^*_c for each half $W^+(S), W^-(S)$. Assuming that S is pure, we extend Theorems A, A$'$ and A$''$ to the setup of open surfaces, see § 5. All the above results continue to hold in the presence of a torus T acting on S, where we now consider all spaces as (free) modules over $H^*(BT)$. Finally, from the construction, it is immediate that the assignment $S \mapsto W^{(\mathbf{c})}_{\uparrow\uparrow}(S)$ is functorial with respect to open immersions; similar result holds for the other types of W-algebras, see § 5.

Let us now return to COHAs. Let S be smooth and cohomologically pure, and let $\mathfrak{Coh}_{n\delta}$ denote the derived stack of length n coherent sheaves on S. The COHAs which we are interested in are

$$\mathbf{H}_0(S) := \bigoplus_{n\geqslant 0} H_*(\mathfrak{Coh}_{n\delta}, \mathbb{Q})$$

and its 'compactly supported' version $\mathbf{H}^c_0(S) := \bigoplus_{n\geqslant 0} H_*(\mathfrak{Coh}_{n\delta}/\operatorname{Sym}^n(S), \mathbb{Q})$, which is defined using hyperbolic Borel-Moore homology with respect to the support map $\mathfrak{Coh}_{n\delta} \to \operatorname{Sym}^n(S)$, see Appendix A for definitions. Both $\mathbf{H}_0(S)$ and $\mathbf{H}^c_0(S)$ are functorial with respect to open immersions.

Theorem B (§ 7.6)**.** *There are canonical algebra isomorphisms $\Theta_S : \mathbf{H}_0(S) \simeq W^+_{\uparrow}(S)$ and $\Theta^c_S : \mathbf{H}^c_0(S) \simeq W^+_{\downarrow}(S)$. In particular, $\mathbf{H}_0(S)$ is spherically generated.*

These isomorphisms are compatible with open immersions. We may include the ring of universal tautological classes $\Lambda(S)$ by forming semi-direct products

$$\widetilde{\mathbf{H}}^{(c)}_0(S) = \Lambda(S) \ltimes \mathbf{H}^{(c)}_0(S), \qquad \widetilde{\mathbf{H}}^c_0(S) = \Lambda(S) \ltimes \mathbf{H}^c_0(S).$$

The isomorphisms above extend to $\widetilde{\mathbf{H}}_0(S) \simeq W^{\geqslant}_{\uparrow}(S)$ and $\widetilde{\mathbf{H}}^c_0(S) \simeq W^{\geqslant}_{\downarrow}(S)$. The above results hold mutatis mutandis in the presence of a torus T acting on S.

Corollary. *If $c_1\Delta_S = 0$ and there exists $q \in H^2(S, \mathbb{Q})$ such that $q^2 = s_2$, then $\mathbf{H}_0(S) \simeq U(\mathfrak{w}^+(S))$.*

This corollary is in accordance with the general philosophy of Donaldson-Thomas theory for 2 Calabi-Yau categories. In particular, $\mathfrak{w}^+(S)$ is the Lie algebra constructed by Davison-Kinjo [12]; as a vector space, it is isomorphic to $\mathfrak{g}^{\mathrm{BPS}}[u]$, where $\mathfrak{g}^{\mathrm{BPS}}$ is the BPS Lie algebra of $\mathcal{C}oh_0(S)$ which was determined in [8].

Our proof of Theorem B involves the construction and comparison of suitable faithful representations of both $W^{\geqslant}(S)$ and $\mathbf{H}_0^c(S)$. Following [22] (see also [11]) we introduce several notions of Hecke patterns in § 6. We deduce from the general formalism of COHAs that a left/right S-strong, resp. S-weak Hecke pattern gives rise to a left/right action of $\mathbf{H}_0(S)$, resp. $\mathbf{H}_0^c(S)$ on $\mathbf{V}(X) := \bigoplus_\alpha H_*(X_\alpha, \mathbb{Q})$.

We have evaluation map $\mathrm{ev} : \Lambda(S) \to H^*(X, \mathbb{Q})$. For any class $\alpha \in \bigoplus_i H^{2i}(\overline{S}, \mathbb{Q})$, we let $[X_\alpha]$, resp. $[X_\alpha^{cl}]$ be the virtual, resp. classical fundamental class of X_α, and we denote by

$$\mathbf{V}^{\mathrm{vtaut}}(X) := \bigoplus_\alpha \mathrm{ev}(\Lambda(S)) \cap [X_\alpha], \qquad \mathbf{V}^{\mathrm{taut}}(X) := \bigoplus_\alpha \mathrm{ev}(\Lambda(S)) \cap [X_\alpha^{cl}],$$

the subspace of virtual resp. classical tautological classes in $\mathbf{V}(X)$. Abusing notation, we denote the induced maps from the Fock space $\mathbf{F}^{(r)}$ to $\mathbf{V}^{\mathrm{vtaut}}(X)$, $\mathbf{V}^{\mathrm{taut}}(X)$ by ev as well. We collect properties of Hecke patterns in the following theorem:

Theorem C (Proposition 6.8, Corollary 7.13)**.** *The following hold:*

(a) *Let X be a left S-strong Hecke pattern of rank r. The action Ψ_X^+ preserves $\mathbf{V}^{\mathrm{vtaut}}(X)$ and there is a commutative diagram*

$$\begin{tikzcd} W_\uparrow^+(S) \otimes \mathbf{F}^{(r)} \arrow[r, "\Phi^+"] \arrow[d, "\Theta_S \otimes \mathrm{ev}"] & \mathbf{F}^{(r)}(S) \arrow[d, "\mathrm{ev}"] \\ \mathbf{H}_0^+(S) \otimes \mathbf{V}^{\mathrm{vtaut}}(X) \arrow[r, "\Psi_X^+"] & \mathbf{V}^{\mathrm{vtaut}}(X) \end{tikzcd} \tag{0.2}$$

Similar results hold for S-weak Hecke patterns, and for right Hecke patterns.

(b) *Let X be a two-sided Hecke pattern, so that we have both an action of $W_\uparrow^+(S)$ (or $W_\downarrow^+(S)$) and of $W_\uparrow^-(S)$ (or $W_\downarrow^-(S)$) on $\mathbf{V}_X$. Then* (0.2) *extends to an action of $W^{(r)}(S)$ on $\mathbf{V}_X$, fitting in a commutative diagram*

$$\begin{tikzcd} W^{(r)}(S) \otimes \mathbf{F}^{(r)} \arrow[r, "\Phi"] \arrow[dr, "\Phi_X"'] & \mathbf{F}^{(r)}(S) \arrow[d, "\mathrm{ev}"] \\ & \mathbf{V}^{\mathrm{vtaut}}(X) \end{tikzcd}$$

Here the appropriate version of $W^{(r)}(S)$ depends on whether X is (left or right) S-strong or S-weak.

(c) *Assuming that X satisfies the regularity condition* (2.10)*, the results of (b-c) remain valid if we replace $\mathbf{V}^{\mathrm{vtaut}}(X)$ with $\mathbf{V}^{\mathrm{taut}}(X)$.*

We conjecture that in the case of two-sided Hecke patterns of rank r, the action of $W^{(r)}(S)$ on $\mathbf{V}^{\mathrm{vtaut}}(X)$ extends to an action on the whole of $\mathbf{V}(X)$. The approach which we take here is, however, restricted to tautological classes. One may hope to apply the machinery of [11] to this problem.

We provide in § 7 and § 8 some examples of regular two-sided Hecke patterns such as Hilbert schemes of points on S (in which case our results complement those of Lehn [23]) and stacks of Higgs bundles on a smooth projective curve. The action of the W-algebra on the homology of the stack of Higgs bundles is an essential ingredient in the proof of the $P = W$ conjecture which is given in [18]. Other examples include moduli of instantons and moduli stacks of one-dimensional sheaves on $K3$ surfaces, with possible applications to χ-independence problems.

The paper is organized as follows. In § 1 we define various forms of COHAs of sheaves on a smooth surface S. The formulas for action of length one Hecke correspondences on tautological classes are established in § 2. We introduce and study deformed W-algebras in §§ 3-5. Theorem C is proven in § 6. We zoom in on the action of W-algebras on Hilbert schemes of points in § 7, which results in a proof of Theorem B. Further examples of Hecke patterns, such as moduli of Higgs bundles, are considered in § 8. Finally, § 9 contains some natural conjectures, concerning in particular a possible extension of our results to threefolds. Although we use the language of derived algebraic geometry, our approach throughout is 'low-tech' as we work with absolute (rather than relative) Borel-Moore homology. We believe that it should be possible to lift our results to the setting of local COHAs in [8] (i.e. to adequate sheaves on the space $\mathrm{Sym}^\bullet(S)$).

0.4. **Notations.** Throughout the paper, all geometric objects are defined over the base field $\mathbb{C}$.

Stacks. In this paper, a (derived) stack will mean a 1-Artin (derived) stack which is locally a quotient stack of finite type. Let $cl : X^{cl} \to X$ be the classical truncation of a derived stack X. Restriction to X^{cl} will be often indicated by a superscript $(-)^{cl}$. For instance, for any object $\mathcal{E} \in D(Coh(X))$ we set $\mathcal{E}^{cl} = cl^*\mathcal{E}$. If $\mathcal{E}$ is a perfect complex of finite amplitude on a derived stack X then we define the total space of $\mathcal{E}$ to be $\mathbb{V}(\mathcal{E}) = \mathrm{Spec}\,\mathrm{Sym}(\mathcal{E}^\vee)$. We will make use of a similar notion of projectivization $\mathbb{P}(\mathcal{E})$, studied by Q. Jiang [19]. Note that if V is a finite-dimensional vector space then $\mathbb{P}(V)$ parametrizes hyperplanes of V. Unless specifically mentioned, all fiber products and tensor products are derived.

Borel-Moore homology. For a stack X, there is a well-defined notion of cohomology or Borel-Moore homology with $\mathbb{Q}$-coefficients which we will denote by $H^*(X,\mathbb{Q})$ and $H_*(X,\mathbb{Q})$ respectively, see e.g. [22]. When the stack X is pure dimensional we usually write d_X for its dimension and $[X] \in H_{2d_X}(X,\mathbb{Q})$ for its fundamental class. If X is smooth then there is an isomorphism of vector spaces $H^i(X,\mathbb{Q}) = H_{2d_X-i}(X,\mathbb{Q})$ such that $c \mapsto c \cap [X]$. For a derived stack X there is also a well-behaved notion of cohomology $H^*(X,\mathbb{Q})$ and of Borel-Moore homology $H_*(X,\mathbb{Q})$, see [20], [35]. The push-forward map cl_* yields isomorphisms $H^*(X,\mathbb{Q}) = H^*(X^{cl},\mathbb{Q})$ and $H_*(X,\mathbb{Q}) = H_*(X^{cl},\mathbb{Q})$; we will often identify the two spaces without mention. Note, however that some operators on cohomology or Borel-Moore homology do depend on the derived structure. We collect some results of that theory in Appendix A.

Algebras. The degree of an homogeneous element a of a graded vector space will be denoted by $|a|$. When considering superalgebras, we apply the rule of sign for the multiplication of tensor products. In particular, we denote by $[-,-]$ the super-commutator $[a,b] = ab - (-1)^{|a|\cdot|b|}ba$, and by $\{-,-\}$ the anti-super-commutator $[a,b] = ab + (-1)^{|a|\cdot|b|}ba$.

Symmetric functions. Let Λ be the Macdonald ring of symmetric functions, which is given by

$$\Lambda = \mathrm{Sym}(t\mathbb{Q}[t]) = \mathbb{Q}[e_1, e_2, \ldots].$$

We will use standard notations for the elements in Λ, as in [26], and will sometimes denote the unit of Λ by e_0 or h_0. It is convenient to add a formal element p_0 of degree 0; we will denote by $\Lambda' = \Lambda \otimes \mathbb{C}[p_0]$ the resulting algebra. The specialization maps $\pi_N : \Lambda \to \mathbb{C}[x_1, x_2, \ldots, x_N]^{\mathfrak{S}_N}$ extend to Λ' by setting $\pi_N(p_0) = N$. We will occasionally use the following shift operation: for c a formal (even) variable, there is an algebra map

$$\tau_c : \Lambda' \to \Lambda'[c], \qquad \tau_c F(x_1, x_2, \ldots) = F(x_1 + c, x_2 + c, \ldots). \tag{0.3}$$

For instance,

$$\tau_c(p_k) = \sum_{i=0}^{k} \binom{k}{i} p_i c^{k-i}, \qquad \tau_c(e_k) = \sum_{i=0}^{k} \binom{p_0 - i}{k-i} e_i c^{k-i}$$

for any $k \geqslant 0$. For $\mathcal{E}$ a coherent sheaf on a stack X and $f = F(e_1, e_2, \ldots) \in \Lambda$ we define

$$f(\mathcal{E}) = F(c_1(\mathcal{E}), c_2(\mathcal{E}), \ldots) \in H^*(X, \mathbb{Q}). \tag{0.4}$$

We extend this to elements $f \in \Lambda'$ by setting $p_0(\mathcal{E}) = \mathrm{rk}(\mathcal{E})$.

1. Cohomological Hall algebra of zero-dimensional sheaves on a surface

1.1. **The stack of coherent sheaves on a surface S.** Let S be a smooth connected quasi-projective surface. Unless mentioned otherwise, we will make the following assumption:

$$\boxed{\text{The surface } S \text{ has pure cohomology.}}$$

We denote by t_1, t_2 the Chern roots of S, so that the Chern classes of S are $c_1 = t_1 + t_2$, $c_2 = t_1 t_2$ and the Todd class is

$$\mathrm{Td}_S = t_1 t_2 / (1 - e^{-t_1})(1 - e^{-t_2}). \tag{1.1}$$

We will sometimes use its graded version

$$\mathrm{Td}_S(x) = x^2 t_1 t_2 / (1 - e^{-t_1 x})(1 - e^{-t_2 x}) = \sum_{k \geqslant 0} \mathrm{Td}_S^{(k)} x^k.$$

Set $s_2 = t_1^2 + t_1 t_2 + t_2^2$. We will also consider the cohomology with compact support $H_c^*(S, \mathbb{Q})$. Recall that there is an algebra morphism $H_c^*(S, \mathbb{Q}) \to H^*(S, \mathbb{Q})$ and cup product maps $H_c^i(S, \mathbb{Q}) \otimes H^j(S, \mathbb{Q}) \to H_c^{i+j}(S, \mathbb{Q})$. We set $K_0^c(S)_{\mathbb{Q}} = \bigoplus_i H_c^{2i}(S, \mathbb{Q})$ and denote by

$$\langle \alpha, \beta \rangle = \int_S \alpha^\vee \cup \beta \cup \mathrm{Td}_S$$

the Mukai pairing on $K_0^c(S)_{\mathbb{Q}}$, where if $\alpha = \sum_k \alpha_k$ with $\alpha_k \in H_c^{2k}(S, \mathbb{Q})$ then $\alpha^\vee = \sum_k (-1)^k \alpha_k$.

Taking the generic rank of a coherent sheaf yields a linear map $\mathrm{rk} : K_0^c(S)_{\mathbb{Q}} \to \mathbb{Q}$. The class of the structure sheaf of a point will be denoted by δ. Note that for any α,

$$\langle \alpha, \delta \rangle = \langle \delta, \alpha \rangle = \mathrm{rk}(\alpha). \tag{1.2}$$

Let us pick some $\alpha \in K_0^c(S)_{\mathbb{Q}}$. Consider the derived stack $\mathfrak{Coh}_\alpha(S)$ parametrizing coherent sheaves $\mathcal{E} \in \mathcal{C}oh(S)$ with proper support and Chern character $\mathrm{ch}(\mathcal{E}) = \alpha$, see e.g. [34]. Its underlying classical stack will be denoted $\mathcal{C}oh_\alpha(S)$. When the surface S is understood, we may abbreviate $\mathfrak{Coh}_\alpha = \mathfrak{Coh}_\alpha(S)$ and likewise for $\mathcal{C}oh_\alpha$. In addition, when $\alpha = r[\mathcal{O}_S] + \alpha'$, where $\alpha' \in \bigoplus_{i>0} H_c^{2i}(S, \mathbb{Q})$, we may write $\mathfrak{Coh}_{r,\alpha'}$ instead of $\mathfrak{Coh}_\alpha$. Note that $\mathfrak{Coh}_\alpha$ is empty for $\mathrm{rk}(\alpha) \neq 0$ if S is not complete. The stack $\mathcal{C}oh_\alpha$ is singular in general, but $\mathfrak{Coh}_\alpha$ is quasi-smooth and of virtual dimension $d_\alpha = -\langle \alpha, \alpha \rangle$. Unless $\alpha \in \mathbb{N}\delta$, the stack $\mathfrak{Coh}_\alpha$ is of infinite type. However, it may always be covered by open global quotient stacks which are of finite type. We say that a coherent sheaf $\mathcal{E}$ on S is of dimension $\geqslant d$, for $d = 1, 2$ if it contains no subsheaf with support of dimension strictly less than d. Let $\mathfrak{Coh}_\alpha^{\geqslant d}$ be the stack parametrizing dimension $\geqslant d$ sheaves in $\mathfrak{Coh}_\alpha$; it is open in $\mathfrak{Coh}_\alpha$. We denote by $\mathcal{E}_\alpha \in \mathcal{C}oh(\mathfrak{Coh}_\alpha \times S)$ the tautological sheaf. Its restriction to $\mathfrak{Coh}^{\geqslant d} \times S$ will be denoted $\mathcal{E}_\alpha^{\geqslant d}$.

1.2. **The stack of zero-dimensional sheaves.** For $d \in \mathbb{N}$, the stack $\mathfrak{Coh}_{d\delta}$ is the derived moduli stack of (zero-dimensional) sheaves on S of length d. Its underlying classical stack $\mathcal{C}oh_{d\delta}$ is irreducible of dimension d while $\mathfrak{Coh}_{d\delta}$ is of virtual dimension $-\langle d\delta, d\delta\rangle = 0$. Let us set $\mathfrak{Coh}^0 = \bigsqcup_d \mathfrak{Coh}_{d\delta}$ and $\mathcal{C}oh^0 = \mathfrak{Coh}^{0,cl}$.

Example 1.1.

(a) If $S = \mathbb{A}^2$, then $\mathcal{C}oh_{d\delta}$ is the commuting stack $\mathcal{C}_{\mathfrak{gl}_d} = \{(x, y) \in \mathfrak{gl}_d^2\ ;\ [x, y] = 0\}/GL_d$.

(b) When $S = \mathrm{Tot}(\mathcal{L})$ is the total space of a line bundle $\mathcal{L}$ over a smooth curve C, $\mathcal{C}oh_{d\delta}$ is the classical stack of $\mathcal{L}$-twisted Higgs sheaves of length d, i.e. it parametrizes pairs $(\mathcal{F}, \theta)$ with $\mathcal{F}$ a length d torsion sheaf on C and $\theta \in \mathrm{Hom}(\mathcal{F}, \mathcal{F} \otimes \mathcal{L})$.

Let $\Delta_S : S \to S \times \overline{S}$ be the diagonal map and

$$\Delta = \Delta_{S*}(1) \in H^*(S \times \overline{S}, \mathbb{Q})$$

be the class of the diagonal. Let $\rho \in Coh(B\mathbb{G}_m)$ be the linear character and $u = c_1(\rho)$. We have

$$\mathcal{C}oh_\delta \simeq S \times B\mathbb{G}_m, \qquad \mathcal{E}_\delta = \Delta_{S*}(\mathcal{O}_S) \boxtimes \rho \in Coh(S \times \overline{S} \times B\mathbb{G}_m), \tag{1.3}$$

$$H^*(\mathcal{C}oh_\delta, \mathbb{Q}) = H^*(S, \mathbb{Q})[u] \quad , \quad \mathrm{ch}(\mathcal{E}_\delta) = \Delta \cup e^u \cup \mathrm{Td}_S^{-1}. \tag{1.4}$$

Since $\mathcal{C}oh_\delta$ is smooth, there is an isomorphism

$$H^i(\mathcal{C}oh_\delta, \mathbb{Q}) = H_{2-i}(\mathcal{C}oh_\delta, \mathbb{Q}) \quad , \quad c \mapsto c \cap [\mathcal{C}oh_\delta]. \tag{1.5}$$

We will sometimes assume that the surface S is acted upon by a torus T. In this case there is an induced action of T on the stacks $\mathfrak{Coh}_{d\delta}$, and all homology groups acquire module structure over $\mathbf{R}_T = H^*(BT, \mathbb{Q})$.

1.3. **The COHA of zero-dimensional sheaves.** We now introduce, following [22, §4], the cohomological Hall algebra of zero-dimensional sheaves on S. See also [45] for another construction of this COHA, [29] for the case of the cotangent bundle of a curve, and [39] for the case of $S = \mathbb{A}^2$. We consider the $\mathbb{Z}^2$-graded vector space

$$\mathbf{H}_0(S) = H_*(\mathcal{C}oh^0, \mathbb{Q}), \qquad \mathbf{H}_0(S)[l, n] = H_n(\mathcal{C}oh_{l\delta}, \mathbb{Q}).$$

Let us briefly recall the definition of the COHA product. Fix $\alpha = a\delta, \beta = b\delta$ and $\gamma = \alpha + \beta$. Let $\widetilde{\mathfrak{Coh}}_{\alpha;\beta}$ be the derived stack parametrizing short exact sequences $0 \to \mathcal{T}' \to \mathcal{T} \to \mathcal{T}'' \to 0$ with $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ respectively in $\mathfrak{Coh}_\gamma, \mathfrak{Coh}_\beta$ and $\mathfrak{Coh}_\alpha$. There is a convolution diagram

$$\mathfrak{Coh}_\alpha \times \mathfrak{Coh}_\beta \xleftarrow{q_{\alpha,\beta}} \widetilde{\mathfrak{Coh}}_{\alpha;\beta} \xrightarrow{p_{\alpha,\beta}} \mathfrak{Coh}_\gamma \tag{1.6}$$

in which the maps $p_{\alpha,\beta}$ and $q_{\alpha,\beta}$ assign to the sequence $0 \to \mathcal{T}' \to \mathcal{T} \to \mathcal{T}'' \to 0$ the object $\mathcal{T}$ and the pair of objects $(\mathcal{T}'', \mathcal{T}')$ respectively. The classical truncation of that diagram reads

$$\mathcal{C}oh_\alpha \times \mathcal{C}oh_\beta \xleftarrow{q^{cl}_{\alpha,\beta}} \widetilde{\mathcal{C}oh}_{\alpha;\beta} \xrightarrow{p^{cl}_{\alpha,\beta}} \mathcal{C}oh_\gamma. \tag{1.7}$$

The map $p_{\alpha,\beta}$ is proper and representable. The map $q_{\alpha,\beta}$ is neither representable nor smooth, but it is quasi-smooth. More precisely, consider the complex

$$\mathcal{C}_{\alpha,\beta} = \mathrm{RHom}_S(\mathcal{E}_\alpha, \mathcal{E}_\beta)[1] = \mathbf{R}p_{12*}\mathbf{R}\mathcal{H}om(p_{13}^*\mathcal{E}_\alpha, p_{23}^*\mathcal{E}_\beta)[1]$$

of perfect amplitude $[-1, 1]$. Here p_{ij} stands for the projection from $\mathfrak{Coh}_\alpha \times \mathfrak{Coh}_\beta \times S$ to the i-th and j-th components. There is a canonical isomorphism of derived stacks over $\mathfrak{Coh}_\alpha \times \mathfrak{Coh}_\beta$

$$\mathbb{V}(\mathcal{C}_{\alpha,\beta}) \simeq \widetilde{\mathfrak{Coh}}_{\alpha,\beta}$$

(recall that $\mathbb{V}(\mathcal{C}_\bullet)$ stands for the total space of a complex $\mathcal{C}_\bullet$). We may thus define a virtual pullback morphism

$$q^!_{\alpha,\beta} : H_i(\mathcal{C}oh_\alpha \times \mathcal{C}oh_\beta, \mathbb{Q}) \to H_{i-2\langle\alpha,\beta\rangle}(\widetilde{\mathcal{C}oh}_{\alpha;\beta}, \mathbb{Q}).$$

It is useful to rephrase this construction in classical terms. Let us fix an explicit representative of the complex $\mathcal{C}_{\alpha,\beta}$

$$0 \to \mathcal{V}_{-1} \to \mathcal{V}_0 \to \mathcal{V}_1 \to 0.$$

Let $\mathcal{C}^{cl}_{\alpha,\beta}$ be the restriction of $\mathcal{C}_{\alpha,\beta}$ to $\mathcal{C}oh_\alpha \times \mathcal{C}oh_\beta$. Let $\tau_{\leqslant 0}$ and $(-)^{\leqslant 0}$ be the standard and stupid truncations. By [22], there is an isomorphism

$$\mathbb{V}(\tau_{\leqslant 0}(\mathcal{C}^{cl}_{\alpha,\beta})) = \widetilde{\mathcal{C}oh}_{\alpha,\beta}.$$

This yields a factorization

$$\begin{tikzcd} \mathbb{V}(\mathcal{C}^{cl,\leqslant 0}_{\alpha,\beta}) \arrow[d, "\pi"'] & \mathbb{V}(\tau_{\leqslant 0}\mathcal{C}^{cl}_{\alpha,\beta}) \simeq \widetilde{\mathcal{C}oh}_{\alpha;\beta} \arrow[l, "\iota"'] \arrow[dl, "q_{\alpha,\beta}"] \\ \mathcal{C}oh_\alpha \times \mathcal{C}oh_\beta & \end{tikzcd}$$

The map π is a linear stack, in particular it is smooth. Hence it yields an isomorphism

$$\pi^* : H_i(\mathcal{C}oh_\alpha \times \mathcal{C}oh_\beta, \mathbb{Q}) \to H_{i+2d_0}(\mathbb{V}(\mathcal{C}^{cl,\leqslant 0}_{\alpha,\beta}), \mathbb{Q}).$$

Further, there is a refined Gysin pullback

$$\iota^! : H_i(\mathbb{V}(\mathcal{C}^{cl,\leqslant 0}_{\alpha,\beta}), \mathbb{Q}) \to H_{i-2d_1}(\widetilde{\mathcal{C}oh}_{\alpha,\beta}, \mathbb{Q}).$$

Here d_0 and d_1 are the ranks of $\mathcal{C}^{\leqslant 0}_{\alpha,\beta}$ and $\mathcal{V}_1$, so $d_0 - d_1 = -\langle\alpha,\beta\rangle$ is the virtual rank of $\mathcal{C}^{cl}_{\alpha,\beta}$. We have $q^!_{\alpha,\beta} = \iota^! \circ \pi^*$, see [20]. In particular, the morphism $\iota^! \circ \pi^*$ thus defined is independent of the presentation of the complex $\mathcal{C}$ (see [22, § 3] for a direct proof). We set

$$\star = (p_{\alpha,\beta})_* \circ q^!_{\alpha,\beta} : H_*(\mathcal{C}oh_\alpha, \mathbb{Q}) \otimes H_*(\mathcal{C}oh_\beta, \mathbb{Q}) = H_*(\mathcal{C}oh_\alpha \times \mathcal{C}oh_\beta, \mathbb{Q}) \to H_{*-2\langle\alpha,\beta\rangle}(\mathcal{C}oh_\gamma, \mathbb{Q}).$$

Theorem 1.2 ([22, thm. 4.4.2], [45])**.** *The convolution product $\star$ defines on $\mathbf{H}_0(S)$ a structure of a graded associative algebra.*

Remark 1.3. Note that $\langle\delta,\delta\rangle = 0$, hence the product in $\mathbf{H}_0(S)$ is degree-preserving.

In the presence of a torus T, we can likewise consider the T-equivariant COHA which is an algebra with underlying vector space given by

$$\mathbf{H}^T_0(S) = H^T_*(\mathcal{C}oh^0, \mathbb{Q}).$$

An open inclusion $i : S \to S'$ of smooth surfaces gives an open inclusion of derived stacks $\underline{i} : \mathfrak{Coh}^0(S) \to \mathfrak{Coh}^0(S')$ and thus to a map $\underline{i}^* : \mathbf{H}_0(S') \to \mathbf{H}_0(S)$.

Lemma 1.4. *The map $\underline{i}^*$ is an algebra homomorphism.*

Proof. The stack $\mathfrak{Coh}_{d\delta}(S)$ is an open substack of $\mathfrak{Coh}_{d\delta}(S')$ for any d, and the convolution diagram used to define the product is compatible with open base change. □

We will need the following result on the Hilbert series of $\mathbf{H}_0(S)$. We define

$$h_{\mathbf{H}_0(S)}(z,w) = \sum_{l,n} \dim(\mathbf{H}_0(S)[l,n])(-z)^n w^l.$$

Let $h_S(z) = \sum_n \dim(H_n(S,\mathbb{Q}))(-z)^n$ be the Borel-Moore homology Poincaré polynomial of S.

Theorem 1.5 ([22, thm. 7.1.6])**.** *Let q, t be formal variables of respective degrees $[0,-2]$ and $[1,0]$. There is a canonical isomorphism of graded vector spaces*

$$\mathbf{H}_0(S) = \mathrm{Sym}\left(H_*(S \times B\mathbb{G}_m, \mathbb{Q}) \otimes qt\mathbb{Q}[t]\right) = \mathrm{Sym}\left(H_*(S, \mathbb{Q}) \otimes qt\mathbb{Q}[q,t]\right).$$

In particular, the Hilbert series of $\mathbf{H}_0(S)$ is given by

$$h_{\mathbf{H}_0(S)}(z,w) = \mathrm{Exp}\left(\frac{h_S(z)z^{-2}w}{(1-z^{-2})(1-w)}\right)$$

where Exp *is the plethystic exponential.*

Theorem 1.5 is proved for an arbitrary smooth surface S using factorization homology techniques, which do not extend to the T-equivariant setting. However, in [9] the question of equivariant formality is treated in the much greater generality of relative COHAs, which includes our COHAs by [8, §11.1]. In particular, as soon as S is T-equivariantly formal, $\mathbf{H}_0(S)$ is a free $\mathbf{R}_T$-module of (graded) rank given by $h_{\mathbf{H}_0(S)}(z,w)$ by [9, Theorems 11.5, 11.6]. By [15, Theorem 14.1] this assumption is satisfied for S cohomologically pure.

1.4. **The compactly supported COHA of zero dimensional sheaves.** When S is not proper, we will also consider a variant of $\mathbf{H}_0(S)$ defined using *hyperbolic* Borel-Moore homology, see Appendix A.1 its definition and properties. More precisely, set $\mathrm{Sym}(S) = \bigsqcup_n \mathrm{Sym}^n(S)$ and let $\mathrm{supp} : \mathfrak{Coh}^0 \to \mathrm{Sym}(S)$ be the support map, $\pi : \mathrm{Sym}(S) \to \mathrm{pt}$ projection to a point. We set

$$\mathbf{H}_0^c(S) = \bigoplus_d H_*^c(\mathcal{C}oh_{d\delta}, \mathbb{Q}),$$

$$H_*^c(\mathcal{C}oh_{d\delta}, \mathbb{Q}) = H_*(\mathfrak{Coh}_{d\delta}/\,\mathrm{Sym}^d(S), \mathbb{Q}) := H^{-*}(\pi_! \mathrm{supp}_* \mathbb{D}_{\mathfrak{Coh}_{d\delta}}).$$

The map $H_c^*(S,\mathbb{Q})[u] \to H_*^c(\mathcal{C}oh_\delta, \mathbb{Q})$ given by $x \mapsto x \cap [\mathcal{C}oh_\delta]$ is an isomorphism. Here we use the natural map $H_c^*(S,\mathbb{Q}) \otimes H_*(S,\mathbb{Q}) \to H_*^c(S,\mathbb{Q})$. We complete the induction diagram by introducing the support maps

$$\begin{tikzcd}
\mathfrak{Coh}_{m\delta} \times \mathfrak{Coh}_{n\delta} \arrow[d, "\mathrm{supp}"'] & \widetilde{\mathfrak{Coh}}_{m\delta;n\delta} \arrow[l, "q_{m,n}"'] \arrow[r, "p_{m,n}"] & \mathfrak{Coh}_{(m+n)\delta} \arrow[d, "\mathrm{supp}"] \\
\mathrm{Sym}^m(S) \times \mathrm{Sym}^n(S) \arrow[rr, "\oplus"] & & \mathrm{Sym}^{m+n}(S)
\end{tikzcd} \tag{1.8}$$

where $\oplus$ is the direct sum (a finite map), which allows us to view $\mathfrak{Coh}_{m\delta} \times \mathfrak{Coh}_{n\delta}$ and $\widetilde{\mathfrak{Coh}}_{m\delta;n\delta}$ as derived stacks over $\mathrm{Sym}^{m+n}(S)$. Note that

$$H_*(\mathcal{C}oh_{m\delta} \times \mathcal{C}oh_{n\delta}/\,\mathrm{Sym}^{m+n}(S), \mathbb{Q}) = H_*(\mathcal{C}oh_{m\delta}/\,\mathrm{Sym}^m(S), \mathbb{Q}) \otimes H_*(\mathcal{C}oh_{n\delta}/\,\mathrm{Sym}^n(S), \mathbb{Q}).$$

We may now define the convolution product

$$\star = (p_{m,n})_! \circ q_{m,n}^* : H_*^c(\mathcal{C}oh_{m\delta}, \mathbb{Q}) \otimes H_*^c(\mathcal{C}oh_{n\delta}, \mathbb{Q}) = H_*^c(\mathcal{C}oh_{m\delta} \times \mathcal{C}oh_{n\delta}, \mathbb{Q}) \to H_*^c(\mathcal{C}oh_{(m+n)\delta}, \mathbb{Q}).$$

Proposition 1.6. *The convolution product $\star$ endows $\mathbf{H}_0^c(S)$ with the structure of a graded associative algebra.*

Proof. The proof is in all points analogous to the case of the $\mathbf{H}_0(S)$. Alternatively, one can observe that $\mathbf{H}_0^c(S)$ is obtained by taking compactly supported cohomology of sheaf-theoretical COHA $\mathrm{supp}_* \mathbb{D}_{\mathcal{C}oh^0}$ on $\mathrm{Sym}(S)$, see [8]. □

If the surface S is projective, then we have $\mathbf{H}_0^c(S) = \mathbf{H}_0(S)$. For each open embedding $i : S \to S'$ we have an open embedding of derived stacks $\underline{i} : \mathfrak{Coh}_{n\delta} \to \mathfrak{Coh}_{n\delta}(S')$. Hence, there are pushforward maps $\underline{i}_! : H_*^c(\mathfrak{Coh}_{n\delta}, \mathbb{Q}) \to H_*^c(\mathfrak{Coh}_{n\delta}(S'), \mathbb{Q})$, which combine to $\underline{i}_! : \mathbf{H}_0^c(S) \to \mathbf{H}_0^c(S')$.

Lemma 1.7. *The map $\underline{i}_! : \mathbf{H}_0^c(S) \to \mathbf{H}_0^c(S')$ is an algebra morphism.*

Proof. Quasi-smooth pullback and proper pushforward in hyperbolic homology are compatible with open base change, see Proposition A.6. □

If $\iota : S \to \overline{S}$ is a smooth compactification of S, Lemmas 1.4, 1.7 yield algebra homomorphisms

$$\mathbf{H}_0^c(S) \xrightarrow{\underline{\iota}_!} \mathbf{H}_0^c(\overline{S}) = \mathbf{H}_0(\overline{S}) \xrightarrow{\underline{\iota}^*} \mathbf{H}_0(S). \tag{1.9}$$

The composition $\underline{\iota}^* \underline{\iota}_! : \mathbf{H}_0^c(S) \to \mathbf{H}_0(S)$ is independent of the choice of $\overline{S}$. We will denote it by ϕ_S.

Let $h_S^c(z) = \sum_n \dim(H_n^c(S, \mathbb{Q}))(-z)^n$ be the homology Poincaré polynomial of S. We will need the following variant of Theorem 1.5, which can be found in [9, Corollary 7.11].

Theorem 1.8. *There is a canonical isomorphism of graded vector spaces*

$$\mathbf{H}_0^c(S) = \mathrm{Sym}\left(H_*^c(S \times B\mathbb{G}_m, \mathbb{Q}) \otimes qt\mathbb{Q}[t]\right) = \mathrm{Sym}\left(H_*^c(S, \mathbb{Q}) \otimes qt\mathbb{Q}[q, t]\right).$$

In particular, the Hilbert series of $\mathbf{H}_0^c(S)$ is given by

$$h_{\mathbf{H}_0^c(S)}(z, w) = \mathrm{Exp}\left(\frac{h_S^c(z) z^{-2} w}{(1 - z^{-2})(1 - w)}\right).$$

As explained after Theorem 1.5, a T-equivariant version of Theorem 1.8 follows from [9, §11] provided that S is pure.

1.5. **The COHA of properly supported sheaves.** Following [22, §4], one can extend the construction of COHA product to the stack of properly supported sheaves on S. We set

$$\mathbf{H}(S) = \bigoplus_\alpha H_*(\mathcal{C}oh_\alpha, \mathbb{Q}).$$

As in the case of zero-dimensional sheaves, for any $\alpha, \beta \in K_0^c(S)_\mathbb{Q}$ there is an induction diagram

$$\mathfrak{Coh}_\alpha \times \mathfrak{Coh}_\beta \xleftarrow{q_{\alpha,\beta}} \widetilde{\mathfrak{Coh}}_{\alpha;\beta} \xrightarrow{p_{\alpha,\beta}} \mathfrak{Coh}_\gamma, \tag{1.10}$$

with $q_{\alpha,\beta}$ quasi-smooth and $p_{\alpha,\beta}$ proper.

Theorem 1.9 ([22, thm. 4.4.2])**.** *Convolution with respect to the correspondences* (1.10) *for all $\alpha, \beta \in K_0^c(S)_\mathbb{Q}$ endows the space $\mathbf{H}(S)$ with the structure of a graded associative algebra. The multiplication $H_*(\mathcal{C}oh_\alpha, \mathbb{Q}) \otimes H_*(\mathcal{C}oh_\beta, \mathbb{Q}) \to H_*(\mathcal{C}oh_{\alpha+\beta}, \mathbb{Q})$ is of homological degree $-2\langle \alpha, \beta \rangle$.*

Assume that $i : S \to S'$ is an open immersion into another smooth surface S'. This gives rise to an open immersion $\underline{i} : \mathfrak{Coh}(S) \to \mathfrak{Coh}(S')$ and hence to a restriction morphism $\underline{i}^* : \mathbf{H}(S') \to \mathbf{H}(S)$. The proof of Lemma 1.4 implies that the map $\underline{i}^*$ is an algebra homomorphism.

Remark 1.10. There is no higher rank analog of $\mathbf{H}_0^c(S)$ because in general there is no useful map from $\mathfrak{Coh}$ to a coarse moduli space.

1.6. **Tautological classes.** Let us now study a family of cohomology classes $ch_i(\lambda)$ for $i\geqslant 1$ and $\lambda \in H^*(S,\mathbb{Q})$ on each $\mathcal{C}oh_\alpha$.

Fix a smooth compactification $\iota : S \to \overline{S}$ of S. Since S is pure, the restriction map $\iota^* : H^*(\overline{S},\mathbb{Q}) \to H^*(S,\mathbb{Q})$ is surjective. Let $I(S) \subset H^*(\overline{S},\mathbb{Q})$ be the kernel of this map, which can be identified with the relative cohomology $H^*(\overline{S},S;\mathbb{Q})$. Dually, the map $\iota_! : H^*_c(S,\mathbb{Q}) \to H^*_c(\overline{S},\mathbb{Q}) = H^*(\overline{S},\mathbb{Q})$ is injective. The perfect intersection pairing on $H^*(\overline{S},\mathbb{Q})$ allows us to identify $H^*_c(S,\mathbb{Q})$ with the orthogonal complement $I(S)^\perp$. This is a (typically non-unital) subalgebra under the cup product. Dually, $H^*(S,\mathbb{Q})$ and $H^*(\overline{S},\mathbb{Q})$ are equipped with natural coproducts and ι^* is a surjection of coalgebras.

When S is not proper, it will be convenient to formally add a class [pt] of degree 4 to $H^*(S,\mathbb{Q})$, satisfying $[\mathrm{pt}] \cup H^{>0}(S,\mathbb{Q}) = \{0\}$. Likewise, it will be convenient to formally add a unit 1 to $H^*_c(S,\mathbb{Q})$; we denote the resulting rings by $\overline{H}^*(S,\mathbb{Q})$ and $\overline{H}^*_c(S,\mathbb{Q})$ respectively. We will modify the coproduct accordingly, i.e., if Δ' is the coproduct on $H^*(S,\mathbb{Q})$ we define

$$\Delta : \overline{H}^*(S,\mathbb{Q}) \to \overline{H}^*(S,\mathbb{Q}) \otimes \overline{H}^*(S,\mathbb{Q}),$$
$$\Delta([\mathrm{pt}]) = [\mathrm{pt}]\otimes[\mathrm{pt}], \qquad \Delta(\lambda) = [\mathrm{pt}]\otimes\lambda + \lambda\otimes[\mathrm{pt}] + \Delta'(\lambda) \quad (\lambda \in H^*(S,\mathbb{Q})).$$

Let us now define a variant of Macdonald's ring of symmetric function which is colored by $H^*(S,\mathbb{Q})$. Consider

$$U(S) = \mathrm{Sym}(H^*(S,\mathbb{Q}) \otimes \mathbb{Q}[t]).$$

For each $i\geqslant 1$ and $\lambda \in H^*(S,\mathbb{Q})$, we denote $\underline{\mathrm{ch}}_i(\lambda) = \lambda \otimes t^i$, and set $\deg(\underline{\mathrm{ch}}_i(\lambda)) = 2i + \deg(\lambda) - 4$. In order to keep track of the rank of coherent sheaves, we add an extra element $\mathbf{r}$ of degree 0 and set $U'(S) = U(S) \otimes \mathbb{Q}[\mathbf{r}]$. We view $U'(S)$ as the free graded-commutative algebra generated by the elements $\underline{\mathrm{ch}}_i(\lambda)$ and $\mathbf{r}$ subject to the relations

$$\underline{\mathrm{ch}}_i(\lambda+\mu) = \underline{\mathrm{ch}}_i(\lambda) + \underline{\mathrm{ch}}_i(\mu) \quad , \quad \underline{\mathrm{ch}}_i(a\lambda) = a\,\underline{\mathrm{ch}}_i(\lambda) \quad , \quad \underline{\mathrm{ch}}_i(\lambda)\cdot\underline{\mathrm{ch}}_j(\mu) = (-1)^{|\lambda|\cdot|\mu|}\underline{\mathrm{ch}}_j(\mu)\cdot\underline{\mathrm{ch}}_i(\lambda)$$

for any $a \in \mathbb{Q}$ and $\lambda, \mu \in H^*(S,\mathbb{Q})$.

Definition 1.11. $\Lambda(S)$ is the quotient of $U'(S)$ by the ideal generated by the negative degree elements $\underline{\mathrm{ch}}_1(\lambda)$ for $\deg(\lambda) = 0, 1$. The *universal Chern character* $\underline{\mathrm{ch}}(x)$ is defined as follows:

$$\underline{\mathrm{ch}}(x) = \mathbf{r}\otimes 1 + \sum_{i\geqslant 1} \underline{\mathrm{ch}}_i x^i \in \Lambda(S) \otimes \overline{H}^*_c(S,\mathbb{Q})[[x]], \qquad \underline{\mathrm{ch}}_i := \sum_\lambda \underline{\mathrm{ch}}_i(\lambda)\otimes\lambda^*,$$

where $\sum \lambda\otimes\lambda^* \in H^*(S,\mathbb{Q}) \otimes H^*_c(S,\mathbb{Q})$ is the intersection pairing tensor.

Remark 1.12. The definitions above are compatible with restriction along $\iota : S \to \overline{S}$. Namely, we have a natural quotient $\Lambda(\overline{S}) \twoheadrightarrow \Lambda(S)$, such that the image of $\underline{\mathrm{ch}}_{\overline{S}}(x)$ in $\Lambda(S) \otimes H^*(\overline{S})^*[[x]]$ is precisely $\underline{\mathrm{ch}}_S(x)$.

Remark 1.13. Note that $\underline{\mathrm{ch}}(x)$ belongs to $\mathbb{Q}[\mathbf{r}] \otimes 1 + x\Lambda(S) \otimes H^*_c(S,\mathbb{Q})[[x]]$.

We define a coalgebra structure on $\Lambda(S)$ by requiring the elements $\underline{\mathrm{ch}}_i(\lambda)$ and $\mathbf{r}$ to be primitive. In other words,

$$(\Delta\otimes\mathrm{Id})(\underline{\mathrm{ch}}(x)) = \underline{\mathrm{ch}}(x)_{13} + \underline{\mathrm{ch}}(x)_{23} \in \Lambda(S)\otimes\Lambda(S)\otimes\overline{H}^*_c(S,\mathbb{Q})[[u]].$$

Example 1.14. The primitive elements of degree 0 in $\Lambda(S)$ are linearly spanned by $\mathbf{r}$ and $\underline{\mathrm{ch}}_2(1)$, $\underline{\mathrm{ch}}_1(\lambda)$ for $\lambda \in H^2(S,\mathbb{Q})$.

We also consider an involution

$$\upsilon : \Lambda(S) \to \Lambda(S), \qquad \upsilon(\mathbf{r}) = -\mathbf{r}, \qquad \upsilon(\underline{\mathrm{ch}}_i(\lambda)) = -\underline{\mathrm{ch}}_i(\lambda), \qquad (i \geqslant 1,\ \lambda \in H^*(S,\mathbb{Q})). \tag{1.11}$$

The elements $\underline{\mathrm{ch}}_i$, being even, commute with each other. Hence we may define an algebra morphism

$$p : \Lambda' \to \Lambda(S) \otimes \overline{H}^*_c(S) \quad , \quad p_0 \mapsto \mathbf{r} \otimes 1 \quad , \qquad p_i/i! \mapsto \underline{\mathrm{ch}}_i \quad , \quad i \geqslant 1.$$

We will use the following notation

$$f(\lambda) = \int_S p(f) \cup \lambda \quad , \quad f \in \Lambda \quad , \quad \lambda \in \overline{H}^*(S,\mathbb{Q}), \tag{1.12}$$

and sometimes simply write $\int_S f\lambda$ when there is no risk of confusion. For instance, we have $p_i(\lambda) = i!\underline{\mathrm{ch}}_i(\lambda)$. Observe that

$$(f \cdot g)(\lambda) = (f \otimes g)(\Delta(\lambda)) = \sum f(\lambda^{(1)})g(\lambda^{(2)}) \tag{1.13}$$

for $f, g \in \Lambda'$ and $\lambda \in \overline{H}^*(S,\mathbb{Q})$, where $\Delta(\lambda) = \sum \lambda^{(1)} \otimes \lambda^{(2)}$ in Sweedler's notation. In particular,

$$(p_{i_1} \cdots p_{i_l})(\lambda) = i_1! \cdots i_l! \sum \underline{\mathrm{ch}}_{i_1}(\lambda^{(1)}) \cdots \underline{\mathrm{ch}}_{i_l}(\lambda^{(l)}).$$

We have $\deg(f(\lambda)) = 2\deg(f) + \deg(\lambda) - 4$.

Remark 1.15. Note that with our conventions we have

$$h_0([\mathrm{pt}]) = 1([\mathrm{pt}]) = 1 \in \Lambda(S), \qquad p_0([\mathrm{pt}]) = \mathbf{r} \otimes 1([\mathrm{pt}]) = \mathbf{r} \in \Lambda(S)$$

while $1(\lambda) = p_0(\lambda) = 0$ if $\deg \lambda < 4$, regardless of whether S is proper or not. In a similar vein, we have $f([\mathrm{pt}]) = 0$ for any f in the augmentation ideal of $\mathbb{Q}[p_1, p_2, \ldots]$ when S is not proper.

Now fix $\alpha \in K_0^c(\overline{S})_{\mathbb{Q}}$ a class of rank r and consider a locally closed substack $\mathcal{U}_\alpha \subset \mathcal{C}oh_\alpha(\overline{S})$. Let $\mathcal{E}_\alpha \in Coh(\mathcal{U}_\alpha \times \overline{S})$ denote the restriction of the tautological sheaf to $\mathcal{C}oh_\alpha(\overline{S}) \times \overline{S}$. Consider its Chern character

$$\mathrm{ch}(\mathcal{E}_\alpha, u) = r + \sum_{i \geqslant 1} p_i(\mathcal{E}_\alpha)/i!$$

We have a unique graded ring homomorphism $\mathrm{ev}_\alpha : \Lambda(\overline{S}) \to H^*(\mathcal{C}oh_\alpha, \mathbb{Q})$, defined by

$$(\mathrm{ev}_\alpha \otimes \mathrm{Id})(\underline{\mathrm{ch}}(u)) = \mathrm{ch}(\mathcal{E}_\alpha, u) \in H^*(\mathcal{U}_\alpha, \mathbb{Q}) \otimes H^*(\overline{S}, \mathbb{Q})^*[[u]].$$

Observe that $r = \mathrm{ev}_\alpha(\mathbf{r})$. The following lemma is a straightforward corollary of Remark 1.12.

Lemma 1.16. *Assume that the Chern character of $\mathcal{E}_\alpha$ takes values in $\mathbb{Q} \cdot 1 + H^*(\mathcal{U}_\alpha, \mathbb{Q}) \otimes H^*_c(S, \mathbb{Q})$. Then ev_α factors through $\Lambda(S)$. In particular, the classes $\mathrm{ev}_\alpha(f(\lambda))$ for $f \in \Lambda$ and $\lambda \in H^*(S,\mathbb{Q})$ are independent of the choice of compactification $\overline{S}$.* □

The condition of Lemma 1.16 is verified for instance when $\mathcal{U}_\alpha \subset \mathcal{C}oh_\alpha$. More generally, it holds when the restriction of $\mathcal{E}_\alpha$ to $\mathcal{U}_\alpha \times (\overline{S} \setminus S)$ is a trivial vector bundle; such situations occur when considering moduli stacks of sheaves on $\overline{S}$ which are trivialized along $\overline{S} \setminus S$.

1.7. **Extended COHAs.** Assume now that $\alpha \in \mathbb{N}\delta$. Composing ev_α with the cap product yields an action $\bullet$ of $\Lambda(S)$ on $\mathbf{H}_0(S)$ such that

$$x \bullet c = \mathrm{ev}_\alpha(x) \cap c \quad , \quad c \in H_*(\mathcal{C}oh_\alpha, \mathbb{Q}) \quad , \quad x \in \Lambda(S).$$

This action preserves each $H_*(\mathcal{C}oh_\alpha, \mathbb{Q})$ and is compatible with the (co)homological gradings, i.e. $\deg(x \bullet c) = \deg(c) - \deg(x)$. The same holds for $\mathbf{H}_0^c(S)$, where the action of $H^*(\mathcal{C}oh_\alpha, \mathbb{Q})$ on $H_*^c(\mathcal{C}oh_\alpha, \mathbb{Q})$ is given by Lemma A.3.

Example 1.17. Assuming that S is proper, let us compute $\mathrm{ev}_{n\delta}(p_1([\mathrm{pt}]))$. By definition, we have

$$\sum_\lambda \mathrm{ev}_{n\delta}(p_1(\lambda)) \otimes \lambda^* = c_1(\mathcal{E}_{n\delta}).$$

Hence $\mathrm{ev}_{n\delta}(p_1([\mathrm{pt}])) = c_1(i_x^*(\mathcal{E}_{n\delta}))$ in $H^2(\mathcal{C}oh_{n\delta}, \mathbb{Q})$ for any closed point $i_x : \mathrm{Spec}(\mathbb{C}) \to S$. The support of $i_x^*(\mathcal{E}_{n\delta})$ being of codimension 2, its first Chern class vanishes, hence $\mathrm{ev}_{n\delta}(p_1([\mathrm{pt}])) = 0$ for any $n > 0$.

Proposition 1.18. *The ring $\mathbf{H}_0(S)$ is a $\Lambda(S)$-module algebra, i.e. we have*

$$x \bullet (c_1 \cdot c_2) = \sum (-1)^{|c_1| \cdot |x_i^{(2)}|} (x_i^{(1)} \bullet c_1) \cdot (x_i^{(2)} \bullet c_2) \quad , \quad x \in \Lambda(S) \quad , \quad c_1, c_2 \in \mathbf{H}_0(S), \tag{1.14}$$

where $\Delta(x) = \sum x_i^{(1)} \otimes x_i^{(2)}$. The same holds for $\mathbf{H}_0^c(S)$. The map $\phi_S : \mathbf{H}_0^c(S) \to \mathbf{H}_0(S)$ is a morphism of $\Lambda(S)$-modules.

Proof. We will deal with the case of $\mathbf{H}_0(S)$, the other one is similar. We can assume that $c_i \in H_*(\mathcal{C}oh_{\alpha_i}, \mathbb{Q})$ for $i = 1, 2$. Set $\gamma = \alpha_1 + \alpha_2$. Recall the induction diagram (1.6). We abbreviate $p = p_{\alpha_1;\alpha_2}$ and $q = q_{\alpha_1;\alpha_2}$. By the projection formula

$$x \bullet (c_1 \cdot c_2) = \mathrm{ev}_\gamma(x) \cap p_* q^! (c_1 \otimes c_2) = p_*(p^*(\mathrm{ev}_\gamma(x)) \cap q^!(c_1 \otimes c_2)).$$

There is a short exact sequence of tautological sheaves

$$0 \to q^*(\mathcal{E}_{\alpha_2}) \to p^*(\mathcal{E}_\gamma) \to q^*(\mathcal{E}_{\alpha_1}) \to 0$$

Hence $p^*(\mathrm{ch}(\mathcal{E}_\gamma)) = q^*(\mathrm{ch}(\mathcal{E}_{\alpha_1}) + \mathrm{ch}(\mathcal{E}_{\alpha_2}))$. We deduce that

$$\begin{aligned} p_*(p^*(\mathrm{ev}_\gamma(x)) \cap q^!(c_1 \otimes c_2)) &= p_* q^! \left((\mathrm{ev}_{\alpha_1} \otimes \mathrm{ev}_{\alpha_2})(\Delta(x)) \cap (c_1 \otimes c_2) \right) \\ &= \sum (-1)^{|c_1| \cdot |x_i^{(2)}|} (x_i^{(1)} \bullet c_1) \cdot (x_i^{(2)} \bullet c_2). \end{aligned}$$

The compatibility between ϕ_S and the action of $\Lambda(S)$ results from the projection formula in hyperbolic or Borel-Moore homology. $\square$

The semi-direct product $\widetilde{\mathbf{H}}_0(S) = \mathbf{H}_0(S) \rtimes \Lambda(S)$ is the algebra generated by $\mathbf{H}_0(S)$ and $\Lambda(S)$ modulo the relations

$$x \cdot c = \sum (-1)^{|c| \cdot |x_i^{(2)}|} (x_i^{(1)} \bullet c) \cdot x_i^{(2)} \quad , \quad x \in \Lambda(S), \ c \in \mathbf{H}_0(S). \tag{1.15}$$

The multiplication map $\Lambda(S) \otimes \mathbf{H}_0(S) \to \widetilde{\mathbf{H}}_0(S)$ is an isomorphism of graded vector spaces. We define the semi-direct product $\widetilde{\mathbf{H}}_0^c(S) = \mathbf{H}_0^c(S) \rtimes \Lambda(S)$ similarly.

We finish with the following observation. The degree one piece of $\mathbf{H}_0(S)$ is

$$\mathbf{H}_0(S)[1, -] = H_*(\mathcal{C}oh_\delta, \mathbb{Q}) = H_*(S \times B\mathbb{G}_m, \mathbb{Q}) = H_*(S, \mathbb{Q})[u].$$

We consider the linear map

$$\omega_\delta : \Lambda(S) \to H_*(\mathcal{C}oh_\delta, \mathbb{Q}) \quad , \quad x \mapsto x \bullet [\mathcal{C}oh_\delta]$$

Lemma 1.19. *The map ω_δ is surjective.*

Proof. Since the restriction map $H_*(\overline{S},\mathbb{Q}) \to H_*(S,\mathbb{Q})$ is surjective, so is the restriction map $H_*(\mathcal{C}oh_\delta(\overline{S}),\mathbb{Q}) \to H_*(\mathcal{C}oh_\delta,\mathbb{Q})$. As the latter is a morphism of $\Lambda(\overline{S})$-module, it is enough to prove the statement for S projective. By (1.4), we have

$$\mathrm{ch}(\mathcal{E}_\delta) = \mathrm{Td}_S^{-1} e^u \Delta_S = \mathrm{Td}_S^{-1} e^u \sum_\lambda \lambda \otimes \lambda^*$$

where $\{\lambda\}$, $\{\lambda^*\}$ are dual bases of $H^*(S,\mathbb{Q})$. Since Td_S is invertible, the result follows. □

When S is not pure, the map ev_δ may still be defined but it cannot be surjective since $\mathcal{E}_\delta$ extends to the compactification $\mathcal{C}oh_\delta(\overline{S}) \times \overline{S}$.

Remark 1.20. The definition of the action of $\Lambda(S)$ on $\mathbf{H}_0(S)$ as well as Proposition 1.18 and its proof extend *mutatis mutandis* from $\mathbf{H}_0(S)$ to $\mathbf{H}(S)$.

2. Derived Hecke correspondences

In this section we consider and describe the simplest type of Hecke correspondence.

2.1. **Hecke correspondences.** From (1.10), we can derive the following induction diagrams:

$$\begin{gathered}\mathfrak{Coh}_{n\delta} \times \mathfrak{Coh}_\alpha \xleftarrow{q_{n\delta,\alpha}} \widetilde{\mathfrak{Coh}}_{n\delta;\alpha} \xrightarrow{p_{n\delta,\alpha}} \mathfrak{Coh}_{\alpha+n\delta},\\ \mathfrak{Coh}_\alpha \times \mathfrak{Coh}_{n\delta} \xleftarrow{\overline{q}_{\alpha,n\delta}} \widetilde{\mathfrak{Coh}}_{n\delta;\alpha-n\delta} \xrightarrow{\overline{p}_{\alpha,n\delta}} \mathfrak{Coh}_{\alpha-n\delta}.\end{gathered} \tag{2.1}$$

For the compactly supported COHA it is useful to factor the maps $p_{n\delta,\alpha}$, $\overline{p}_{\alpha,n\delta}$ as follows:

$$\begin{gathered}\widetilde{\mathfrak{Coh}}_{n\delta;\alpha} \xrightarrow{p'_{n\delta,\alpha}} \mathfrak{Coh}_{\alpha+n\delta} \times \mathrm{Sym}^n(S) \xrightarrow{p''_{n\delta,\alpha}} \mathfrak{Coh}_{\alpha+n\delta},\\ \widetilde{\mathfrak{Coh}}_{n\delta;\alpha-n\delta} \xrightarrow{\overline{p}'_{\alpha,n\delta}} \mathfrak{Coh}_{\alpha-n\delta} \times \mathrm{Sym}^n(S) \xrightarrow{\overline{p}''_{\alpha,n\delta}} \mathfrak{Coh}_{\alpha-n\delta}.\end{gathered}$$

We call the first/second diagram in (2.1) the positive/negative length n Hecke correspondences. Since $\mathfrak{Coh}^{\geqslant d}$ is stable under taking subobjects, the Hecke correspondences restrict to $\mathfrak{Coh}^{\geqslant d}$, yielding the following restricted induction diagrams

$$\begin{gathered}\mathfrak{Coh}_{n\delta} \times \mathfrak{Coh}^{\geqslant d}_\alpha \xleftarrow{q_{n\delta,\alpha}} \widetilde{\mathfrak{Coh}}^{\geqslant d}_{n\delta;\alpha} \xrightarrow{p_{n\delta,\alpha}} \mathfrak{Coh}^{\geqslant d}_{\alpha+n\delta},\\ \mathfrak{Coh}^{\geqslant d}_\alpha \times \mathfrak{Coh}_{n\delta} \xleftarrow{\overline{q}_{\alpha,n\delta}} \widetilde{\mathfrak{Coh}}^{\geqslant d}_{n\delta;\alpha-n\delta} \xrightarrow{\overline{p}_{\alpha,n\delta}} \mathfrak{Coh}^{\geqslant d}_{\alpha-n\delta},\end{gathered} \tag{2.2}$$

where we have defined

$$\widetilde{\mathfrak{Coh}}^{\geqslant d}_{\alpha;\beta} := \widetilde{\mathfrak{Coh}}_{\alpha;\beta} \underset{\mathfrak{Coh}_{\alpha+\beta}}{\times} \mathfrak{Coh}^{\geqslant d}_{\alpha+\beta}.$$

2.2. **Locally free resolutions.** The Hecke correspondences enjoy much better properties when the tautological sheaf $\mathcal{E}$ has perfect amplitude in $[-1,0]$ and admits locally a two-step locally free resolution. This is true in our situation after we restrict to the open substack $\mathfrak{Coh}^{\geqslant 1}$.

Lemma 2.1. *Let $\gamma \in K_0^c(S)_\mathbb{Q}$ and let $\mathfrak{U} \subset \mathfrak{Coh}^{\geqslant 1}_\gamma$ be any finite type open substack. The tautological sheaf $\mathcal{E}_\gamma|_{\mathfrak{U}\times\overline{S}}$ admits a 2-step resolution by locally free sheaves $0 \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{E}_\gamma \to 0$.*

Proof. Having a 2-step resolution by locally free sheaves is a local condition on any stack of finite type. Indeed, let $\mathcal{F} \in Coh(X)$, where X is a stack of finite type, and assume that $\mathcal{F}|_{U_i}$ admits 2-step resolutions by locally free sheaves for some open cover $\bigcup_i U_i = X$. Since X is of finite type, there exists a short exact sequence $0 \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{F} \to 0$ with $\mathcal{E}_0$ being a locally free sheaf. It is enough to check that $\mathcal{E}_{-1}$ is locally free, which may be done locally. Next, as $\mathcal{E}_\gamma|_{\mathfrak{U}\times\overline{S}}$ is $\mathfrak{U}$-flat, it is enough to check that for any $\mathbb{C}$-point $x \in \mathfrak{U}(\mathbb{C})$ the sheaf $\mathcal{E}_\gamma|_{\{x\}\times\overline{S}}$ admits a 2-step resolution by locally free sheaves over $\overline{S}$. This in turn follows from the fact that $\mathcal{E}_\gamma|_{\{x\}\times\overline{S}}$ is of dimension $\geqslant 1$ and that $\overline{S}$ is smooth, so that $\mathcal{E}_\gamma|_{\{x\}\times\overline{S}}$ has perfect amplitude in $[-1, 0]$. □

In the remainder of this section, we describe the length one Hecke correspondences and compute their action on tautological classes. Until §2.4, we let S be an arbitrary smooth connected surface. A general framework for derived Hecke correspondences has recently been worked out by Q. Jiang in [19, §8], in the language of derived algebraic geometry, following the work of Negut [32].

2.3. **Length one Hecke correspondences and operators.** Fix α and set $\gamma = \alpha + \delta$. In this section we consider length one Hecke correspondences given by the diagrams in (2.1) with $n = 1$ restricted to $\mathfrak{Coh}^{\geqslant 1}$. To unburden the notation, we will drop the indices of the maps p, q, etc. Let K_S be the canonical bundle of S and set $\mathcal{F}_\alpha = \mathcal{E}_\alpha^\vee \otimes K_S[1]$, a complex over $\mathfrak{Coh}_\alpha \times S$. By (1.3), the Serre duality gives an isomorphism of complexes over $\mathfrak{Coh}_\alpha \times \mathfrak{Coh}_\delta$

$$\begin{aligned}(\mathrm{RHom}_S(\mathcal{E}_\delta, \mathcal{E}_\alpha)[1])^\vee &= \mathrm{RHom}_S(\mathcal{E}_\alpha, K_S \otimes \mathcal{E}_\delta)[1] \\ &= Rp_{12*}(\mathcal{E}_\alpha^\vee \otimes K_S \otimes \mathcal{O}_{\Delta_S} \otimes \rho)[1] \\ &= \mathcal{E}_\alpha^\vee \otimes K_S \otimes \rho[1]\end{aligned}$$

where $p_{12} : \mathfrak{Coh}_\alpha \times \mathfrak{Coh}_\delta \times S \to \mathfrak{Coh}_\alpha \times \mathfrak{Coh}_\delta$ is the projection. In particular, the complex $\mathcal{F}_\alpha$ is the restriction to $\mathfrak{Coh}_\alpha \times S$ of $(\mathrm{RHom}_S(\mathcal{E}_\delta, \mathcal{E}_\alpha)[1])^\vee$. Let

$$\tau : \mathbb{P}(\mathcal{E}_\gamma) \to \mathbb{P}(\mathcal{E}_\gamma) \times S \quad , \quad \overline{\tau} : \mathbb{P}(\mathcal{F}_\alpha) \to \mathbb{P}(\mathcal{F}_\alpha) \times S$$

be the diagonal morphisms, i.e., the morphisms making the following diagrams commutative:

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}_\gamma) \arrow[r, "\tau"] \arrow[d, "\pi"'] & \mathbb{P}(\mathcal{E}_\gamma) \times S \arrow[d, "pr_S"] \\ \mathfrak{Coh}_\gamma \times S \arrow[r, "pr_S"] & S \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}(\mathcal{F}_\alpha) \arrow[r, "\overline{\tau}"] \arrow[d, "\overline{\pi}"'] & \mathbb{P}(\mathcal{F}_\alpha) \times S \arrow[d, "pr_S"] \\ \mathfrak{Coh}_\alpha \times S \arrow[r, "pr_S"] & S \end{tikzcd} \tag{2.3}$$

Here, pr_S is the projection to S.

Let $\mathcal{C}oh_\delta$ be the classical truncation of $\mathfrak{Coh}_\delta$, and let us put

$$\widetilde{\mathcal{C}oh}_{\delta;\alpha} := \widetilde{\mathfrak{Coh}}_{\delta;\alpha} \underset{\mathfrak{Coh}_\delta}{\times} \mathcal{C}oh_\delta, \qquad \widetilde{\mathcal{C}oh}^{\geqslant 1}_{\delta;\alpha} := \widetilde{\mathfrak{Coh}}^{\geqslant 1}_{\delta;\alpha} \underset{\mathfrak{Coh}_\delta}{\times} \mathcal{C}oh_\delta.$$

The motivation to consider this partial classical truncation will become clear in § 2.4. We have the following important result:

Proposition 2.2 ([19, §8 , prop. 4.33],[32, §2])**.**

(a) *There is a canonical isomorphism of derived stacks $\widetilde{\mathcal{C}oh}^{\geqslant 1}_{\delta;\alpha} = \mathbb{P}(\mathcal{E}_\gamma)$ which identifies the tautological sheaf $(q \times \mathrm{Id})^*(\mathcal{E}_\delta)$ with $\tau_*(\mathcal{O}_{\mathbb{P}(\mathcal{E}_\gamma)}(1))$;*

(b) *There is a canonical isomorphism of derived stacks $\widetilde{\mathcal{C}oh}^{\geqslant 1}_{\delta;\alpha} = \mathbb{P}(\mathcal{F}_\alpha)$ which identifies the tautological sheaf $(\overline{q} \times \mathrm{Id})^*(\mathcal{E}_\delta)$ with $\overline{\tau}_*(\mathcal{O}_{\mathbb{P}(\mathcal{F}_\alpha)}(-1))$.* □

Recall that the classical truncation map $\mathcal{C}oh_\delta \to \mathfrak{Coh}_\delta$ induces an isomorphism in Borel-Moore homology (and likewise in relative Borel-Moore homology). To carry out the computation of length one Hecke operators, we may and will therefore use the partially truncated induction diagrams

$$\mathcal{C}oh_\delta \times \mathfrak{Coh}_\alpha^{\geqslant 1} \xleftarrow{q_{\delta,\alpha}} \widetilde{\mathcal{C}oh}^{\geqslant 1}_{\delta;\alpha} \xrightarrow{p_{\delta,\alpha}} \mathfrak{Coh}^{\geqslant 1}_{\alpha+n\delta},$$
$$\mathfrak{Coh}_\alpha^{\geqslant 1} \times \mathcal{C}oh_\delta \xleftarrow{\overline{q}_{\alpha,\delta}} \widetilde{\mathcal{C}oh}^{\geqslant 1}_{\delta;\alpha-\delta} \xrightarrow{\overline{p}_{\alpha,\delta}} \mathfrak{Coh}^{\geqslant 1}_{\alpha-n\delta},$$

obtained by base change from (2.2).

Corollary 2.3. *The restrictions of the maps p, p', $\overline{p}$ and $\overline{p}'$ to $\widetilde{\mathcal{C}oh}^{\geqslant 1}$ are proper and representable. The restrictions of the maps q, $\overline{q}$ to $\widetilde{\mathcal{C}oh}^{\geqslant 1}$ are quasi-smooth.*

Proof. We already know that p is proper. Since both $\mathcal{E}_\gamma$ and $\mathcal{F}_\alpha$ have perfect amplitude in $[-1,0]$ over $\mathfrak{Coh}^{\geqslant 1} \times \overline{S}$, by [19, Lem. 5.4] the maps p' and $\overline{p}'$ are proper and quasi-smooth. Hence $\overline{p}$ is proper as well when S is proper. Since $\mathfrak{Coh}_\alpha$ is an open substack of $\mathfrak{Coh}_\alpha(\overline{S})$, we may deduce the case of an arbitrary S by base change. Indeed, note that there is a cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathcal{C}oh}^{\geqslant 1}_{\delta;\alpha}(S) \arrow[r] \arrow[d, "\overline{p}"'] & \widetilde{\mathcal{C}oh}^{\geqslant 1}_{\delta;\alpha}(\overline{S}) \arrow[d, "\overline{p}"'] \\
\mathfrak{Coh}^{\geqslant 1}_\alpha(S) \arrow[r] & \mathfrak{Coh}^{\geqslant 1}_\alpha(\overline{S})
\end{tikzcd}$$

since for any extension $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_x \to 0$ with $\mathcal{F}, \mathcal{E}$ of dimension $\geqslant$1, we have $\mathrm{supp}(\mathcal{E}) = \mathrm{supp}(\mathcal{F})$. Next, we claim that the maps q and $\overline{q}$ are the restrictions to suitable open substacks of the projections

$$\mathbb{V}(\mathrm{RHom}_S(\mathcal{E}_\delta, \mathcal{E}_\alpha)[1]) \to \mathcal{C}oh_\delta \times \mathfrak{Coh}_\alpha \quad , \quad \mathbb{V}(\mathrm{RHom}_S(\mathcal{E}_\gamma, \mathcal{E}_\delta)) \to \mathfrak{Coh}_\gamma \times \mathcal{C}oh_\delta.$$

Indeed, the condition for a sheaf to be supported on $S \subset \overline{S}$ is open, as is the condition for a morphism $\mathcal{E}_\gamma \to \mathcal{E}_\delta$ to be surjective. As a consequence, $q, \overline{q}$ are both quasi-smooth. $\square$

We will prove a more general version of Corollary 2.3 in §6.1. Thanks to Corollary 2.3, the truncated induction diagrams yield two types of Hecke operators

$$T_+ = p_! \circ q^* : H_*(\mathcal{C}oh_\delta, \mathbb{Q}) \otimes H_*(\mathcal{C}oh^{\geqslant 1}_\alpha, \mathbb{Q}) \to H_*(\mathcal{C}oh^{\geqslant 1}_\gamma, \mathbb{Q}),$$
$$T_- = \overline{p}_! \circ \overline{q}^* : H_*(\mathcal{C}oh_\delta, \mathbb{Q}) \otimes H_*(\mathcal{C}oh^{\geqslant 1}_\gamma, \mathbb{Q}) \to H_*(\mathcal{C}oh^{\geqslant 1}_\alpha, \mathbb{Q}).$$

Considering S-hyperbolic homology, we also define

$$\begin{aligned} T^c_+ &= r \circ p'_! \circ q^* : H^c_*(\mathcal{C}oh_\delta, \mathbb{Q}) \otimes H_*(\mathcal{C}oh^{\geqslant 1}_\alpha, \mathbb{Q}) \to H_*(\mathcal{C}oh^{\geqslant 1}_\gamma, \mathbb{Q}), \\ T^c_- &= r \circ \overline{p}'_! \circ \overline{q}^* : H^c_*(\mathcal{C}oh_\delta, \mathbb{Q}) \otimes H_*(\mathcal{C}oh^{\geqslant 1}_\gamma, \mathbb{Q}) \to H_*(\mathcal{C}oh^{\geqslant 1}_\alpha, \mathbb{Q}), \end{aligned} \tag{2.4}$$

where $r : H^c_*(S, \mathbb{Q}) \to \mathbb{Q}$ is the canonical degree 0 map. We will next use Proposition 2.2 to compute their action on tautological classes over suitable open substacks of $\mathfrak{Coh}^{\geqslant 1}$.

2.4. **Atlases for length one correspondences.** We assume until §2.7 that $S = \overline{S}$ is a projective surface. Let us fix a finite type open derived substack $\mathfrak{U} \subset \mathfrak{Coh}^{\geqslant 1}_\gamma$ and a locally free resolution

$$0 \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{E}_\gamma|_{\mathfrak{U} \times \overline{S}} \to 0. \tag{2.5}$$

We keep the notations from the previous section. Let us explicitly describe the projectivization $\mathbb{P}(\mathcal{E}_\gamma)$ in terms of the complex $\mathcal{E}_{-1} \to \mathcal{E}_0$. Let $\mathbb{P}(\mathcal{E}_0)$ be the total space of the projective bundle associated to $\mathcal{E}_0$ and $\pi : \mathbb{P}(\mathcal{E}_0) \to \mathfrak{U} \times \overline{S}$ be the projection. The points of $\mathbb{P}(\mathcal{E}_0)$ parametrize triples

$(\mathcal{W}, y, \lambda)$ where $\mathcal{W} \in \mathfrak{U}$ is a coherent sheaf on $\overline{S}$ of dimension $\geqslant 1$, $y \in \overline{S}$ and $\lambda : \mathcal{E}_0|_{(\mathcal{W},y)} \to \mathbb{C}$ is a nontrivial linear form, defined up to multiplication by a scalar. The morphism $\pi^*(\mathcal{E}_0^\vee) \to \pi^*(\mathcal{E}_{-1}^\vee)$ yields a map $\mathcal{O}_{\mathbb{P}(\mathcal{E}_0)}(-1) \to \pi^*(\mathcal{E}_{-1}^\vee)$. This map can be viewed as a section

$$s \in H^0(\mathbb{P}(\mathcal{E}_0), \pi^*(\mathcal{E}_{-1}^\vee)(1)).$$

The zero locus $Z(s)$ of s parametrizes the triples $(\mathcal{W}, y, \lambda) \in \mathbb{P}(\mathcal{E}_0)$ for which the map λ descends to $\mathcal{W}|_y$. By Proposition 2.2, we have

$$\widetilde{\mathcal{C}oh}_{\delta;\alpha} \underset{\mathfrak{Coh}_\gamma}{\times} \mathfrak{U} = \mathbb{P}(\mathcal{E}_\gamma|_{\mathfrak{U}\times\overline{S}}) \simeq \mathbb{P}(\mathcal{E}_0) \underset{\mathbb{V}(\pi^*(\mathcal{E}_{-1}^\vee)(1))}{\times} \mathbb{P}(\mathcal{E}_0).$$

The derived fiber product is taken with respect to the sections s and 0 of $\pi^*(\mathcal{E}_{-1}^\vee)(1)$. Let $\mathbb{P}(\mathcal{E}_0^{cl})$ be the classical projective bundle of $\mathcal{E}_0^{cl}$ over $\mathfrak{U}^{cl} \times \overline{S}$. By [19, prop. 4.21], the classical truncation of the derived stack $\mathbb{P}(\mathcal{E}_\gamma|_{\mathfrak{U}\times\overline{S}})$ is isomorphic to the zero locus in $\mathbb{P}(\mathcal{E}_0^{cl})$ of the section s^{cl}, i.e., we have

$$\mathbb{P}(\mathcal{E}_\gamma|_{\mathfrak{U}\times\overline{S}})^{cl} = Z(s^{cl}).$$

On the other hand, over the partial truncation $\mathcal{C}oh_\delta \times \mathfrak{Coh}_\alpha^{\geqslant 1}$, the complex $\mathrm{RHom}(\mathcal{E}_\delta, \mathcal{E}_\alpha)[1]$ has perfect amplitude in $[0,1]$. We fix a finite type open derived substack $\mathfrak{U}'$ of $\mathfrak{Coh}_\alpha^{\geqslant 1}$ with a presentation of the complex $\mathrm{RHom}(\mathcal{E}_\delta, \mathcal{E}_\alpha)[1]|_{\mathcal{C}oh_\delta\times\mathfrak{U}'}$

$$0 \to \mathcal{V}_0 \to \mathcal{V}_1 \to 0 \tag{2.6}$$

Let $\rho : \mathbb{V}(\mathcal{V}_0) \to \mathcal{C}oh_\delta \times \mathfrak{U}'$ be the projection. The map $\mathcal{V}_0 \to \mathcal{V}_1$ yields a section

$$s' \in H^0(\mathbb{V}(\mathcal{V}_0), \rho^*(\mathcal{V}_1)).$$

By §1.3 there is an isomorphism

$$\widetilde{\mathcal{C}oh}_{\delta;\alpha} \underset{\mathfrak{Coh}_\alpha}{\times} \mathfrak{U}' = \mathbb{V}(\mathcal{V}_0) \underset{\mathbb{V}(\rho^*(\mathcal{V}_1))}{\times} \mathbb{V}(\mathcal{V}_0)$$

where the derived fiber product is taken with respect to the sections s' and 0. By [19, prop. 4.10], the classical truncation is isomorphic to the zero locus $Z((s')^{cl}) \subset \mathbb{V}(\mathcal{V}_0^{cl})$. We thus get the following isomorphisms of derived and classical stacks over any open set on which both presentations (2.5) and (2.6) exist:

$$\begin{gathered}\mathbb{V}(\mathcal{V}_0) \underset{\mathbb{V}(\rho^*(\mathcal{V}_1))}{\times} \mathbb{V}(\mathcal{V}_0) \simeq \widetilde{\mathfrak{Coh}}_{\delta;\alpha} \simeq \mathbb{P}(\mathcal{E}_0) \underset{\mathbb{V}(\pi^*(\mathcal{E}_{-1}^\vee)(1))}{\times} \mathbb{P}(\mathcal{E}_0),\\ \mathbb{V}(\mathcal{V}_0^{cl}) \supset Z((s')^{cl}) = Z(s^{cl}) \subset \mathbb{P}(\mathcal{E}_0^{cl}).\end{gathered} \tag{2.7}$$

The case of $\mathbb{P}(\mathcal{F}_\alpha)$ is similar, let us briefly sketch it. Observe that

$$\mathrm{RHom}_S(\mathcal{E}_\gamma, \mathcal{E}_\delta) = Rp_{12*}(\mathcal{E}_\gamma^\vee \otimes \mathcal{E}_\delta) = Rp_{12*}(\mathcal{E}_\gamma^\vee \otimes \mathcal{O}_{\Delta_{\overline{S}}} \otimes \rho) = \mathcal{E}_\gamma^\vee \otimes \rho.$$

Hence there is a factorization

$$\begin{tikzcd} \mathbb{V}(\mathcal{E}_\gamma^\vee \otimes \rho) \arrow[d] & \widetilde{\mathcal{C}oh}_{\delta;\alpha}^{\geqslant 1} \arrow[l, "j"'] \arrow[dl, "\overline{q}"] \\ \mathfrak{Coh}_\gamma^{\geqslant} \times \mathcal{C}oh_\delta & \end{tikzcd}$$

where j is an open embedding. Let us restrict everything to an open substack of finite type $\mathfrak{U} \subset \mathfrak{Coh}_\gamma^{\geqslant 1}$. Consider locally free resolution (2.5), and let $\overline{\pi} : \mathbb{V}(\mathcal{E}_\gamma^\vee \otimes \rho) \to \mathfrak{U} \times \mathcal{C}oh_\delta$ be the projection. We have an obvious section

$$\overline{s} \in H^0(\mathbb{V}(\mathcal{E}_0^\vee \otimes \rho), \overline{\pi}^*(\mathcal{E}_{-1}^\vee \otimes \rho)).$$

The zero locus of $\overline{s}^{cl}$ in $\mathbb{V}(\mathcal{E}_0^\vee \otimes \rho^{cl})$ is

$$\mathbb{V}(\mathcal{E}_\gamma^\vee \otimes \rho|_{\mathfrak{U}\times\mathcal{C}oh_\delta})^{cl} = Z(\overline{s}^{cl})$$

Likewise, fix an open substack of finite type $\mathfrak{U}' \subset \mathfrak{Coh}_\alpha^{\geqslant 1}$ as in (2.6), so that $\mathcal{F}_\alpha|_{\mathfrak{U}'\times\overline{S}}$ has a presentation

$$0 \to \mathcal{V}_1^\vee \to \mathcal{V}_0^\vee \to 0.$$

We have an obvious section $\overline{s}' \in H^0(\mathbb{P}(\mathcal{V}_0^\vee), \overline{p}^*(\mathcal{V}_1(1)))$, and the zero locus of $(\overline{s}')^{cl}$ in $\mathbb{P}(\mathcal{V}_0^{\vee,cl})$ is $\mathbb{P}(\mathcal{F}_\alpha|_{\mathfrak{U}\times\overline{S}})^{cl} = Z((\overline{s}')^{cl})$. Therefore, over any open set over which both presentations (2.5) and (2.6) exist, we have the following isomorphisms of derived and classical stacks

$$\begin{aligned}\mathbb{P}(\mathcal{V}_0^\vee) \underset{\mathbb{V}(\overline{p}^*(\mathcal{V}_1\otimes\rho))}{\times} \mathbb{P}(\mathcal{V}_0^\vee) \simeq \widetilde{\mathcal{C}oh}_{\delta;\alpha} \simeq \mathbb{V}(\mathcal{E}_0^\vee \otimes \rho) \underset{\mathbb{V}(\overline{\pi}^*(\mathcal{E}_{-1}^\vee\otimes\rho))}{\times} \mathbb{V}(\mathcal{E}_0^\vee \otimes \rho),\\ \mathbb{P}(\mathcal{V}_0^{\vee,cl}) \supset Z((\overline{s}')^{cl}) \simeq Z(\overline{s}^{cl}) \subset \mathbb{V}(\mathcal{E}_0^{\vee,cl} \otimes \rho).\end{aligned} \tag{2.8}$$

For the future use, note that the image of the map j^{cl} is the complement of the zero section in $\mathbb{V}(\mathcal{E}_0^{\vee,cl} \otimes \rho)$. Therefore, the section $\overline{s}^{cl}$ is regular over this complement if and only if the section s^{cl} is regular. Similarly, the section $(\overline{s}')^{cl}$ is regular if and only if the section $(s')^{cl}$ is regular.

2.5. **Computation of Hecke operators on fundamental classes.** We are now in position to compute the action of Hecke operators on the fundamental classes $[\mathcal{C}oh_\alpha^{\geqslant 1}], [\mathcal{C}oh_\gamma^{\geqslant 1}]$ and on the virtual fundamental classes $[\mathfrak{Coh}_\alpha^{\geqslant 1}], [\mathfrak{Coh}_\gamma^{\geqslant 1}]$. We keep the notation of the previous sections. Fix finite type open substacks $\mathfrak{U} \subset \mathfrak{Coh}_\gamma^{\geqslant 1}$ and $\mathfrak{U}' \subset \mathfrak{Coh}_\alpha^{\geqslant 1}$ such that

$$q^{-1}(\mathfrak{Coh}_\delta(\overline{S}) \times \mathfrak{U}') \supseteq p^{-1}(\mathfrak{U}). \tag{2.9}$$

Recall that $\gamma = \alpha + \delta$. We will carry out the computation of the action on the non-virtual fundamental classes first, under the following assumption:

$$\boxed{\text{the sections } s^{cl} \text{ and } (s')^{cl} \text{ are regular.}} \tag{2.10}$$

This condition implies that the sections $\overline{s}^{cl}$ and $(\overline{s}')^{cl}$ are regular as well. This means that $q_{\delta,\alpha}^{cl}$ and $p_{\delta,\alpha}^{cl}$ are of the expected dimension over each irreducible component of $\mathfrak{U}^{cl}$ and $(\mathfrak{U}')^{cl}$ respectively. Recall from §1.2 the isomorphisms

$$\mathcal{C}oh_\delta = \overline{S} \times B\mathbb{G}_m, \qquad H^*(\mathcal{C}oh_\delta, \mathbb{Q}) = H^*(\overline{S}, \mathbb{Q}) \otimes \mathbb{Q}[u],$$

where $u = c_1(\rho)$ is the Chern class of the linear character $\rho \in Coh(B\mathbb{G}_m)$. Let

$$[\Delta_{\overline{S}}] \in H^*(\mathcal{C}oh_\delta, \mathbb{Q}) \otimes H^*(\overline{S}, \mathbb{Q})$$

be the fundamental class of the diagonal. Let h_n be the complete symmetric function of degree n. Recall the notation $h_n(\mathcal{E}_\gamma)$ in (0.4).

Proposition 2.4. *Assume that* (2.10) *holds. Let* $r = \mathrm{rk}(\alpha)$.

(a) *For any* $l \geqslant 0$ *the following equality holds in* $H_*(\mathfrak{U}, \mathbb{Q}) \otimes H^*(\overline{S}, \mathbb{Q})$:

$$T_+\Big(\big(u^l[\Delta_S] \cap [\mathcal{C}oh_\delta]\big) \otimes [\mathcal{C}oh_\alpha^{\geqslant 1}]\Big)\Big|_{\mathfrak{U}} = h_{l+1-r}(\mathcal{E}_\gamma) \cap [\mathfrak{U}] \tag{2.11}$$

(b) *For any* $\mu \in H^*(\overline{S}, \mathbb{Q})$ *we have*

$$T_+\Big(\big(\mu u^l \cap [\mathcal{C}oh_\delta]\big) \otimes [\mathcal{C}oh_\alpha^{\geqslant 1}]\Big)\Big|_{\mathfrak{U}} = h_{l+1-r}(\mu) \bullet [\mathfrak{U}]. \tag{2.12}$$

Proof. Let j be the open immersion $p^{-1}(\mathfrak{U}) \subset q^{-1}(\mathcal{C}oh_\delta \times \mathfrak{U}')$. We will work with classical stacks, but we will omit the superscript *cl* in the notation. Under the assumption (2.10), we have

$$j^* q^!([\mathcal{C}oh_\delta] \otimes [\mathcal{C}oh_\alpha]) = j^*([Z(s')]) = [Z(s)].$$

We abbreviate $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E}_0)}(1)$. By Proposition 2.2, we have the following isomorphism of coherent sheaves over $p^{-1}(\mathfrak{U}) \times \overline{S}$:

$$(j \times \mathrm{Id})^*(q \times \mathrm{Id})^* \mathcal{E}_\delta = \tau_* \mathcal{O}(1)$$

We deduce that

$$(j \times \mathrm{Id})^*(q \times \mathrm{Id})^!((\mathrm{ch}(\mathcal{E}_\delta) \cap [\mathcal{C}oh_\delta \times \overline{S}]) \otimes [\mathcal{C}oh_\alpha]) = \mathrm{ch}(\tau_* \mathcal{O}(1)) \cap ([Z(s) \times \overline{S}]) \tag{2.13}$$

Now, we consider the commutative diagram

$$\begin{tikzcd}
Z(s) \arrow[r, "\iota"] \arrow[d, "\tau"'] & \mathbb{P}(\mathcal{E}_0) \arrow[d, "\tau'"'] \arrow[dr, "\pi"] & \\
Z(s) \times \overline{S} \arrow[r, "\iota \times \mathrm{Id}"] & \mathbb{P}(\mathcal{E}_0) \times \overline{S} \arrow[r, "p' \times \mathrm{Id}"] & \mathfrak{U} \times \overline{S}
\end{tikzcd}$$

The maps ι, ι' are the obvious closed immersions, the map π is the projection to $\mathfrak{U} \times \overline{S}$, the map p' is the projection to $\mathfrak{U}$, and the maps τ, τ' are defined as in (2.3). The bottom row of the diagram clearly composes to $p \times \mathrm{Id}$.

Let $i : \mathfrak{U} \to \mathfrak{Coh}_\gamma^{\geqslant 1}$ be the open immersion. Applying $(p \times \mathrm{Id})_*$ to (2.13) and restricting to the open subset $\mathfrak{U} \times \overline{S}$, we get

$$(i \times \mathrm{Id})^* T_+\big((\mathrm{ch}(\mathcal{E}_\delta) \cap [\mathcal{C}oh_\delta \times \overline{S}]) \otimes [\mathcal{C}oh_\alpha]\big) = (p' \times \mathrm{Id})_*(\iota \times \mathrm{Id})_*\big(\mathrm{ch}(\tau_* \mathcal{O}(1)) \cap ([Z(s) \times \overline{S}])\big)$$

We claim that

$$\tau_* \iota^* \mathcal{O}(1) = (\iota \times \mathrm{Id})^* \tau'_* \mathcal{O}(1). \tag{2.14}$$

Indeed, since s is regular, the maps $\iota \times \mathrm{Id}$ and τ' are Tor-independent, and so we may use the proper base change theorem. Applying Chern character to (2.14), we get

$$\begin{aligned}
&(i \times \mathrm{Id})^* T_+\big((\mathrm{ch}(\mathcal{E}_\delta) \cap [\mathcal{C}oh_\delta \times \overline{S}]) \otimes [\mathcal{C}oh_\alpha])\big) \\
&\quad = (p' \times \mathrm{Id})_*\big(\mathrm{ch}(\tau'_* \mathcal{O}(1)) \cap (\iota \times \mathrm{Id})_*([Z(s) \times \overline{S}])\big) \\
&\quad = (p' \times \mathrm{Id})_*\big(\mathrm{ch}(\tau'_* \mathcal{O}(1)) \cap \mathrm{eu}\left(\pi^* \mathcal{E}_{-1}^\vee(1)\right) \cap [\mathbb{P}(\mathcal{E}_0)]\big) \\
&\quad = (p' \times \mathrm{Id})_*\big(\mathrm{eu}\left(\pi^* \mathcal{E}_{-1}^\vee(1)\right) \cap \tau'_* \mathrm{ch}(\mathcal{O}(1)) \cap \mathrm{Td}_{\overline{S}}^{-1} \cap [\mathbb{P}(\mathcal{E}_0)]\big) \\
&\quad = \mathrm{Td}_{\overline{S}}^{-1} \cap (p' \times \mathrm{Id})_* \tau'_*\big(\mathrm{eu}\left(\pi^* \mathcal{E}_{-1}^\vee(1)\right) \cap \mathrm{ch}(\mathcal{O}(1)) \cap [\mathbb{P}(\mathcal{E}_0)]\big) \\
&\quad = \mathrm{Td}_{\overline{S}}^{-1} \cap \pi_*\big(\mathrm{eu}\left(\pi^* \mathcal{E}_{-1}^\vee(1)\right) \cap \mathrm{ch}(\mathcal{O}(1)) \cap [\mathbb{P}(\mathcal{E}_0)]\big)
\end{aligned}$$

where we have successively used the fact that s is regular and the Grothendieck-Riemann-Roch formula for the proper morphism τ. The following formula is well-known.

Lemma 2.5. *Let X be a stack. Let $\mathcal{E}_{-1}$, $\mathcal{E}_0$ be vector bundles over X. Set $r = \mathrm{rk}(\mathcal{E}_0) - \mathrm{rk}(\mathcal{E}_{-1})$. Let $\pi : \mathbb{P}(\mathcal{E}_0) \to X$ be the projection. The following formula holds in $H^*(X, \mathbb{Q})$:*

$$\pi_*\big(\mathrm{ch}(\mathcal{O}(1)) \cup \mathrm{eu}(\pi^* \mathcal{E}_{-1}^\vee(1))\big) = \sum_{n \in \mathbb{N}} \frac{1}{n!} h_{n-r+1}(\mathcal{E}_0 - \mathcal{E}_{-1}). \qquad \square$$

Using the above lemma, we deduce that

$$i^*T_+\Big(\big(\mathrm{ch}(\mathcal{E}_\delta)\cap[\mathcal{C}oh_\delta]\big)\otimes[\mathcal{C}oh_\alpha]\Big)=\mathrm{Td}_{\overline{S}}^{-1}\cap\sum_n\frac{1}{n!}h_{n+1-r}(\mathcal{E}_\gamma)\cap[\mathfrak{U}]. \tag{2.15}$$

The proof of (2.11) now follows by multiplying throughout by the Todd class $\mathrm{Td}_{\overline{S}}$ and equating the terms of fixed homological degrees. Finally, the formula (2.12) is obtained by taking the intersection pairing with μ. □

A similar analysis can be made for negative Hecke correspondences. Once again, we fix finite type open substacks $\mathfrak{U}\subset\mathfrak{Coh}_\gamma^{\geqslant 1}$ and $\mathfrak{U}'\subset\mathfrak{Coh}_\alpha^{\geqslant 1}$ satisfying

$$\overline{q}^{-1}(\mathfrak{Coh}_\delta(\overline{S})\times\mathfrak{U})\supseteq\overline{p}^{-1}(\mathfrak{U}')\qquad\Leftrightarrow\qquad q^{-1}(\mathfrak{Coh}_\delta(\overline{S})\times\mathfrak{U}')\subseteq p^{-1}(\mathfrak{U}). \tag{2.16}$$

Recall the shift operation $\tau_c:\Lambda'\to\Lambda'[c]$, defined by (0.3). For any symmetric function $f\in\Lambda$ and $\lambda\in H^*(\overline{S},\mathbb{Q})$, we write

$$\widetilde{f}(\lambda):=\int_S p(\tau_{c_1}f)\cup\lambda\in\Lambda(\overline{S})$$

and extend this to an algebra automorphism $x\mapsto\widetilde{x}$ of $\Lambda(\overline{S})$. Note that we have, for any $\gamma\in K_0^c(\overline{S})_{\mathbb{Q}}$,

$$\mathrm{ev}_\gamma(\widetilde{f}(\lambda))=\int_S f(\mathcal{E}_\gamma\otimes K^\vee)\cup\lambda.$$

Proposition 2.6. *Assume that* (2.10) *holds. Let* $r=\mathrm{rk}(\alpha)$.

(a) *For any* $l\geqslant 0$ *we have the following equality in* $H_*(\mathfrak{U}',\mathbb{Q})\otimes H^*(\overline{S},\mathbb{Q})$

$$T_-\Big(\big(u^l[\Delta_{\overline{S}}]\cap[\mathcal{C}oh_\delta]\big)\otimes[\mathcal{C}oh_\gamma]\Big)\Big|_{\mathfrak{U}'}=(-1)^l e_{l+1+r}(\mathcal{E}_\alpha\otimes K_{\overline{S}}^\vee)\cap[\mathfrak{U}'] \tag{2.17}$$

(b) *For any* $\mu\in H^*(S,\mathbb{Q})$ *we have*

$$T_-\Big(\big(\mu u^l\cap[\mathcal{C}oh_\delta]\big)\otimes[\mathcal{C}oh_\gamma]\Big)\Big|_{\mathfrak{U}'}=(-1)^l\widetilde{e}_{l+1+r}(\mu)\bullet[\mathfrak{U}']. \tag{2.18}$$ □

Remark 2.7. We can express $\widetilde{e}_i$'s in terms of e_i's:

$$\widetilde{e}_n(\mu)=\sum_{i=0}^n\binom{\mathbf{r}-n+i}{i}e_{n-i}(\mu\cup c_1^i).$$

In the absence of an equivariant parameter we have $c_1^3=0$, and so only the first three terms of the sum above do not vanish. In particular, when S has trivial canonical bundle we have $\widetilde{e}_n(\mu)=e_n(\mu)$ for all $n\geqslant 0$, μ. Of course, in the situation of Proposition 2.6 we have $\mathbf{r}=r$.

In this paper, we will check the assumption (2.10) in two situations of interest: the Hilbert schemes of points $\mathfrak{Hilb}_n(\overline{S})$ in §7.3, and the stacks of Higgs bundles $\mathfrak{Higgs}_{r,d}$ over a smooth projective curve in §8.2.

Let us now turn our attention to the action of Hecke operators on *virtual* fundamental classes.

Proposition 2.8. *For any* α, *we have*

$$T_+\Big(\big(\mu u^l\cap[\mathcal{C}oh_\delta]\big)\otimes[\mathfrak{Coh}_\alpha]\Big)\Big|_{\mathfrak{Coh}_\gamma^{\geqslant 1}}=h_{l+1-r}(\mu)\bullet[\mathfrak{Coh}_\gamma^{\geqslant 1}],$$

$$T_-\Big(\big(\mu u^l\cap[\mathcal{C}oh_\delta]\big)\otimes[\mathfrak{Coh}_\gamma]\Big)\Big|_{\mathfrak{Coh}_\alpha^{\geqslant 1}}=(-1)^l\widetilde{e}_{l+1+r}(\mu)\bullet[\mathfrak{Coh}_\alpha^{\geqslant 1}].$$

Proof. The proof follows along exactly the same lines as the proof of Propositions 2.4 and 2.6. We use the fact that the Gysin pullback by a quasi-smooth morphism preserves virtual fundamental classes, see §A.1 for more details, and we use the projection formula of Proposition A.1(d). □

Remark 2.9. Given that Proposition 2.8 holds without any regularity assumption, one might wonder why one should bother considering non-virtual fundamental classes at all. The answer we give is that the virtual fundamental classes $[\mathfrak{Coh}_\alpha^{\geqslant 1}]$ and $[\mathfrak{Coh}_\gamma^{\geqslant 1}]$ typically lie in a homological degree less than that of their non-virtual cousins, and hence generate a different (and in many cases strictly smaller) space of tautological classes.

2.6. **Length one Hecke operators on tautological classes.** Recall that $S = \overline{S}$ is proper. In particular, the map $\omega_\delta : \Lambda(S) \to H_*(\mathcal{C}oh_\delta, \mathbb{Q})$ is surjective by Lemma 1.19. Hence Proposition 2.4 allows us to describe the action of the full subspace $\mathbf{H}_0(S)[1, -]$ on the fundamental class $[\mathcal{C}oh_\alpha]$ after restriction to suitable open substacks $\mathfrak{U}_\alpha$. Using the $\Lambda(S)$-module algebra structure of $\mathbf{H}(S)$, we will now deduce a formula for the action of $\mathbf{H}_0(S)[1, -]$ on the subspace $\Lambda(S) \bullet [\mathfrak{U}_\alpha]$ of $H^*(\mathfrak{U}_\alpha, \mathbb{Q})$ spanned by the tautological classes. To do so, we consider the linear map

$$\mathcal{L}_r^+ : H^*(\mathcal{C}oh_\delta, \mathbb{Q}) \otimes \Lambda(S) \to \Lambda(S) \quad , \quad r \geqslant 0$$

satisfying the following conditions:

(a) $\mathcal{L}_r^+(\lambda u^l \otimes 1) = h_{l+1-r}(\lambda)$ for any $\lambda \in H^*(S, \mathbb{Q})$ and $l \in \mathbb{N}$,

(b) $x \cdot \mathcal{L}_r^+(\xi \otimes y) = \sum \mathcal{L}_r^+(\mathrm{ev}_\delta(x^{(1)}) \cup \xi \otimes x^{(2)} \cdot y)$ for any $x, y \in \Lambda(S)$ and $\xi \in H^*(\mathcal{C}oh_\delta, \mathbb{Q})$.

Here we have used Sweedler's notation $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$. By (2.12) and Proposition 1.18, we deduce that for any stacks $\mathfrak{U}$ and $\mathfrak{U}'$ as above and for $r = \mathrm{rk}(\alpha)$ we have

$$i^*\big((\xi \cap [\mathcal{C}oh_\delta]) \star (x \bullet [\mathcal{C}oh_\alpha])\big) = i^*\big(\mathcal{L}_r^+(\xi \otimes x) \bullet [\mathcal{C}oh_\gamma]\big) \tag{2.19}$$

Hoping that this will not create any confusion, we will write

$$\mathcal{L}_r^+(\xi) : \Lambda(S) \to \Lambda(S) \quad , \quad x \mapsto \mathcal{L}_r^+(\xi \otimes x).$$

Then (b) translates into the following relation:

$$[p_n(\lambda), \mathcal{L}_r^+(\xi)] = \mathcal{L}_r^+(\mathrm{ev}_\delta(p_n(\lambda)) \cup \xi). \tag{2.20}$$

A direct computation using (1.1) and (1.4) gives, for any $n \geqslant 0$,

$$\mathrm{ev}_\delta(p_n(\lambda)) = f_n(u) \cup \lambda, \qquad f_n(u) := \frac{u^n - (u - t_1)^n - (u - t_2)^n + (u - t_1 - t_2)^n}{t_1 t_2}. \tag{2.21}$$

We define an algebra homomorphism R^+ and a $\Lambda(S)$-linear map Q^+ such that

$$R^+ : \Lambda(S) \to \Lambda(S) \otimes H^*(\mathcal{C}oh_\delta, \mathbb{Q}) \quad , \quad p_n(\lambda) \mapsto p_n(\lambda) \otimes 1 - 1 \otimes f_n(u) \cup \lambda,$$
$$Q^+ : \Lambda(S) \otimes H^*(\mathcal{C}oh_\delta, \mathbb{Q}) \to \Lambda(S) \quad , \quad x \otimes \lambda u^l \mapsto x \cdot h_l(\lambda).$$

It will be convenient to extend Q^+ to a map

$$Q^+ : \Lambda(S) \otimes H^*(S, \mathbb{Q})[u, u^{-1}] \to \Lambda(S)$$

by setting $Q^+(x \otimes u^l) = 0$ for $l < 0$. Using these notations, the following formula holds:

$$\mathcal{L}_r^+(\lambda u^l \otimes x) = Q^+(\lambda u^{l+1-r} R^+(x)). \tag{2.22}$$

Again, it will be convenient to formally extend this definition to any $r \in \mathbb{Z}$.

Using Proposition 2.6, we can write down similar formulas for negative correspondences. The analog of the relation (b) is

$$x \cdot \mathcal{L}_r^-(\xi \otimes y) = \sum \mathcal{L}_r^-(\mathrm{ev}_\delta(\upsilon(x')) \cup \xi \otimes x'' \cdot y) \quad , \quad x, y \in \Lambda(S),$$

where υ is the involution (1.11). This translates to the relation

$$[\widetilde{p}_n(\lambda), \mathcal{L}_r^-(\xi)] = -\mathcal{L}_r^-(\mathrm{ev}_\delta(\widetilde{p}_n(\lambda)) \cup \xi).$$

Note that

$$\mathrm{ev}_\delta(\widetilde{p}_n(\lambda)) = \int_S p_n(\mathcal{E}_\delta \otimes K^\vee) \cup \lambda = f_n(u+c_1) \cup \lambda = (-1)^n f_n(-u) \cup \lambda.$$

Thus we obtain the following formula:

$$\mathcal{L}_r^-(\lambda u^l \otimes x) = (-1)^{r+1}\, Q^-(\lambda u^{l+1+r} R^-(x))$$

where R^- and Q^- are the algebra homomorphism and the $\Lambda(S)$-linear map such that

$$R^- : \Lambda(S) \to \Lambda(S) \otimes H^*(\mathcal{C}oh_\delta, \mathbb{Q}) \quad , \quad \widetilde{p}_n(\lambda) \mapsto \widetilde{p}_n(\lambda) \otimes 1 + (-1)^n \otimes f_n(-u) \cup \lambda,$$
$$Q^- : \Lambda(S) \otimes H^*(\mathcal{C}oh_\delta) \to \Lambda(S) \quad , \quad x \otimes \lambda u^l \mapsto x \cdot (-1)^l \widetilde{e}_l(\lambda).$$

One can thus recast Propositions 2.4, 2.6 by saying that for any $\xi \in H^*(\mathcal{C}oh_\delta)$ the operators $T_\pm((\xi \cap [\mathcal{C}oh_\delta]) \otimes -)$ act on $\Lambda(S) \bullet [\mathfrak{U}_\alpha]$ via the action of $\mathcal{L}_r^\pm$ on $\Lambda(S)$.

For future use, we record here the following easily deduced formulas, valid for any n and λ:

$$\begin{aligned} R^+(\widetilde{p}_n(\lambda)) &= \widetilde{p}_n(\lambda) \otimes 1 - (-1)^n \otimes f_n(-u) \cup \lambda \\ R^-(p_n(\lambda)) &= p_n(\lambda) \otimes 1 + 1 \otimes f_n(u) \cup \lambda. \end{aligned} \tag{2.23}$$

Remark 2.10. Operators defined by conditions like (a) and (b) above are typically given by vertex operators. In our situation, these take the following form:

$$\sum_{\substack{l \geqslant 0 \\ \gamma}} \mathcal{L}_r^+(u^l\gamma) \otimes \gamma^* s^{-l+r-1} = \left\{ \exp\Big(\sum_{\gamma; k \geqslant 1} \frac{p_k}{k}(\gamma) \otimes \gamma^* s^{-k} \Big)_{[s^{<r}]} \exp\Big(- \sum_{\gamma; n \geqslant 0} \frac{\partial}{\partial \kappa_n(\gamma)} \otimes \gamma s^n \Big) \right\}_{[s^{<r}]}$$

$$\sum_{\substack{l \geqslant 0 \\ \gamma}} \mathcal{L}_r^-(u^l\gamma) \otimes \gamma^* s^{-l-r-1} = (-1)^{r+1} \left\{ \exp\Big(- \sum_{\gamma; k \geqslant 1} \frac{\tau_{c_1} p_k}{k}(\gamma) \otimes \gamma^* s^{-k} \Big)_{[s^{<-r}]} \exp\Big(\sum_{\gamma; n \geqslant 0} \frac{\partial}{\partial \kappa_n(\gamma)} \otimes \gamma s^n \Big) \right\}_{[s^{<-r}]}$$

where $\{\gamma\}$, $\{\gamma^*\}$ are dual bases of $H^*(S, \mathbb{Q})$ and the elements $\{\kappa_n(\lambda)\}$ are related to the $\{p_k(\lambda)\}$ through the relation

$$\sum_{\gamma; n \geqslant 0} \frac{x^{n+2}}{n!} \kappa_n(\gamma) \otimes \gamma^* = \Big(\mathrm{Td}_S(x) \cdot \sum_{\gamma; n \geqslant 1} \frac{x^n}{n!} p_n(\gamma) \otimes \gamma^* \Big)_{[s^{>1}]}. \tag{2.24}$$

2.7. **Hecke operators on open surfaces.**

Let us now return to the situation when the surface S is cohomologically pure, but not necessarily proper. Pick a smooth compactification $\iota : S \to \overline{S}$, and fix open substacks $\mathfrak{U} \subset \mathfrak{Coh}_\gamma^{\geqslant 1}$ and $\mathfrak{U}' \subset \mathfrak{Coh}_\alpha^{\geqslant 1}$ satisfying[2]

$$q^{-1}(\mathfrak{Coh}_\delta(S) \times \mathfrak{U}') \supseteq p^{-1}(\mathfrak{U}), \qquad \overline{q}^{-1}(\mathfrak{Coh}_\delta(S) \times \mathfrak{U}) \supseteq \overline{p}^{-1}(\mathfrak{U}'). \tag{2.25}$$

These conditions imply that the correspondences (2.1) restrict to Hecke operators

$$T_+ : H_*(\mathcal{C}oh_\delta(S), \mathbb{Q}) \otimes H_*(\mathfrak{U}', \mathbb{Q}) \to H_*(\mathfrak{U}, \mathbb{Q}),$$
$$T_- : H_*(\mathcal{C}oh_\delta(S), \mathbb{Q}) \otimes H_*(\mathfrak{U}, \mathbb{Q}) \to H_*(\mathfrak{U}', \mathbb{Q})$$

The following simple lemma, which follows from open base change, relates these operators to the analogous operators for $\overline{S}$. Let us denote by j both inclusions $\mathfrak{U} \to \mathfrak{Coh}_\gamma^{\geqslant 1}$ and $\mathfrak{U}' \to \mathfrak{Coh}_\alpha^{\geqslant 1}$.

Lemma 2.11. *For $x \in H_*(\mathcal{C}oh_\delta(\overline{S}), \mathbb{Q})$ and $c \in H_*(\mathfrak{U}, \mathbb{Q})$, $c' \in H_*(\mathfrak{U}', \mathbb{Q})$ we have*

$$j^* T_+(x \otimes c') = T_+(\iota^* x \otimes j^* c') \in H_*(\mathfrak{U}, \mathbb{Q}),$$
$$j^* T_-(x \otimes c) = T_-(\iota^* x \otimes j^* c) \in H_*(\mathfrak{U}', \mathbb{Q}).$$

[2] note the difference between these conditions and (2.9, 2.16)

Next, let us consider the compactly supported Hecke correspondences. Assume that the open substacks $\mathfrak{U} \subset \mathfrak{Coh}_\gamma^{\geqslant 1}$, $\mathfrak{U}' \subset \mathfrak{Coh}_\alpha^{\geqslant 1}$ satisfy

$$q^{-1}(\mathfrak{Coh}_\delta(S) \times \mathfrak{U}') \supseteq (p')^{-1}(\mathfrak{U} \times S), \qquad \overline{q}^{-1}(\mathfrak{Coh}_\delta(S) \times \mathfrak{U}) \supseteq (\overline{p}')^{-1}(\mathfrak{U}' \times S). \tag{2.26}$$

In this case we can define restrictions of Hecke operators (2.4):

$$\begin{aligned} T_+^c &: H_*^c(\mathcal{C}oh_\delta(S), \mathbb{Q}) \otimes H_*(\mathfrak{U}', \mathbb{Q}) \to H_*(\mathfrak{U}, \mathbb{Q}), \\ T_-^c &: H_*^c(\mathcal{C}oh_\delta(S), \mathbb{Q}) \otimes H_*(\mathfrak{U}, \mathbb{Q}) \to H_*(\mathfrak{U}', \mathbb{Q}). \end{aligned}$$

Proposition 2.12. *For $x \in H_*^c(\mathcal{C}oh_\delta(S), \mathbb{Q})$ and $c \in H_*(\mathfrak{U}, \mathbb{Q})$, $c' \in H_*(\mathfrak{U}', \mathbb{Q})$ we have*

$$j^* T_+(\iota(x) \otimes c') = T_+^c(x \otimes j^* c'), \qquad j^* T_-(\iota(x) \otimes c) = T_-^c(x \otimes j^* c).$$

Proof. The two cases being identical, we will prove the statement for T_+^c. Consider the following diagram (in which we omit the obvious indices):

$$\begin{tikzcd}
\mathcal{C}oh_\delta(\overline{S}) \times \mathfrak{Coh}_\alpha(\overline{S}) & \widetilde{\mathcal{C}oh}_{\delta,\alpha}(\overline{S}) \arrow[l, "q"'] \arrow[r, "p'"] & \mathfrak{Coh}_\gamma(\overline{S}) \times \overline{S} \arrow[r, "p''"] & \mathfrak{Coh}_\gamma(\overline{S}) \\
\mathcal{C}oh_\delta \times \mathfrak{Coh}_\alpha(\overline{S}) \arrow[u, "\iota \times \mathrm{Id}"] & \widetilde{\mathcal{C}oh}_{\delta,\alpha}(\overline{S}) \times_{\overline{S}} S \arrow[l, "t"'] \arrow[r, "s'"] \arrow[u, "\tilde{\iota}"] & \mathfrak{Coh}_\gamma(\overline{S}) \times S \arrow[r, "s''"] \arrow[u, "\mathrm{Id} \times \iota"] & \mathfrak{Coh}_\gamma(\overline{S}) \arrow[u, equal] \\
\mathcal{C}oh_\delta \times \mathfrak{U}' \arrow[u, "\mathrm{Id} \times j"] & \widetilde{\mathcal{C}oh}_{\delta,\alpha}(\overline{S}) \times_{\mathfrak{Coh}_\gamma(\overline{S}) \times \overline{S}} (\mathfrak{U} \times S) \arrow[l, "q^\circ"'] \arrow[r, "(p')^\circ"] \arrow[u, "\tilde{j}"] & \mathfrak{U} \times S \arrow[r, "(p'')^\circ"] \arrow[u, "j \times \mathrm{Id}"] & \mathfrak{Coh}_\gamma \arrow[u, "j"]
\end{tikzcd}$$

Observe that apart from the rightmost column, the second row is obtained from the top row by base change $- \times_{\overline{S}} S$. In addition, all of the vertical arrows are open embeddings and the middle square in the bottom is cartesian. By Proposition A.6 (a), we have $q^!(\iota_! \otimes \mathrm{Id}) = \tilde{\iota}_! t^!$. Hence, using Proposition A.6 (b), we get

$$p''_* p'_* q^!(\iota_!(x) \otimes c') = p''_* p'_* \tilde{\iota}_! t^!(x \otimes c') = p''_*(\mathrm{Id} \otimes \iota_!) s'_* t^!(x \otimes c').$$

Note further that $p''_*(\mathrm{Id} \otimes \iota_!) = r : H_*^c(S, \mathbb{Q}) \to \mathbb{Q}$, see Example A.5. It follows that $T_+(\iota_!(x) \otimes c') = r s'_* t^!(x \otimes c')$. On the other hand, by base change and functoriality of Gysin pullbacks, we have

$$j^* s'_* t^! = (p')^\circ_* \tilde{j}^* t^! = (p')^\circ_* (q^\circ)^! (\mathrm{Id} \otimes j^*).$$

Thus, we have

$$r s'_* t^!(x \otimes c') = r (p')^\circ_* (q^\circ)^! (x \otimes j^*(c')) = T_+^c(x \otimes j^*(c')). \qquad \square$$

One particular instance when both restrictions of Hecke operators are well defined is when $\mathfrak{U} = \mathcal{C}oh_\gamma^{\geqslant 1}$, $\mathfrak{U}' = \mathcal{C}oh_\alpha^{\geqslant 1}$. If we further restrict to an appropriate open substack of $\mathcal{C}oh^{\geqslant 1}$ where the regularity conditions (2.10) hold, Proposition 2.12 and Lemma 2.11 imply that the formulas of § 2.6 apply verbatim to the restricted operators $T_\pm$, $T_\pm^c$ over $\mathfrak{U}'$, $\mathfrak{U}$. Note that proper support implies $r = 0$. Since the evaluation map ev factors through $\Lambda(S)$ by Lemma 1.16, in this case we can interpret the operators $\mathcal{L}_0^\pm$ as acting on $\Lambda(S)$.

Remark 2.13. Proposition 2.12 and Lemma 2.11 continue to hold if we replace the embedding $S \subset \overline{S}$ by any open immersion.

3. Deformed W-algebras (projective surfaces)

In this section we introduce and study a class of associative algebras which are associated to our surface S (more precisely, to its cohomology ring). As these bear a resemblance to the deformed $W_{1+\infty}$-algebra studied, e.g., in [39] –which corresponds to the case $S=\mathbb{A}^2$ with an action of the torus $(\mathbb{C}^\times)^2$– we will refer to these as *deformed W-algebras*. In this section, we assume that S is proper, in which case there is only one such type of W-algebra. The case of open surfaces will be addressed in § 5.

We fix a smooth projective surface S. Recall that c_1, c_2 are the Chern classes of S and that $s_2=c_1^2-c_2$. We let $P_S(z)=\sum_n \dim(H^n(S,\mathbb{Q}))(-z)^n=h_S(z^{-1})z^4$ be the Poincaré polynomial of S.

3.1. Positive halves of deformed W-algebras.

Definition 3.1. Let $W^{\geqslant}(S)$ be the $\mathbb{N}\times\mathbb{Z}$-graded associative algebra generated by

$$\psi_n(\lambda)\quad,\quad T_n(\lambda)\quad,\quad n\geqslant 0\quad,\quad \lambda\in H^*(S,\mathbb{Q})$$

together with a central element $\mathbf{c}$ modulo the following set of relations for $a,b\in\mathbb{C}$, $n,m\geqslant 0$ and $\lambda,\mu\in H^*(S,\mathbb{Q})$:

$$\psi_m(a\lambda+b\mu)=a\psi_m(\lambda)+b\psi_m(\mu), \tag{a}$$

$$T_n(a\lambda+b\mu)=aT_n(\lambda)+bT_n(\mu), \tag{b}$$

$$[\psi_m(\lambda),\psi_n(\mu)]=0, \tag{c}$$

$$[\psi_m(\lambda),T_n(\mu)]=mT_{m+n-1}(\lambda\mu),\qquad (m\geqslant 0) \tag{d}$$

$$[T_m(\lambda\mu),T_n(\nu)]=[T_m(\lambda),T_n(\mu\nu)], \tag{e}$$

$$\begin{aligned}&[T_m(\lambda),T_{n+3}(\mu)]-3[T_{m+1}(\lambda),T_{n+2}(\mu)]+3[T_{m+2}(\lambda),T_{n+1}(\mu)]-[T_{m+3}(\lambda),T_n(\mu)]\\&\qquad-[T_m(\lambda),T_{n+1}(s_2\mu)]+[T_{m+1}(\lambda),T_n(s_2\mu)]+\{T_m,T_n\}(c_1\Delta_S\lambda\mu)=0,\end{aligned} \tag{f}$$

$$\sum_{w\in S_3} w\cdot[T_{m_3}(\lambda_3),[T_{m_2}(\lambda_2),T_{m_1+1}(\lambda_1)]]=0, \tag{g}$$

$$\psi_n(\lambda)=0 \text{ if } 2n-2+\deg(\lambda)<0. \tag{h}$$

The expression $\{T_m,T_n\}(c_1\Delta_S\lambda\mu)$ above is the super-commutator of T_m and T_n, whose arguments are taken from the symmetric 2-tensor $c_1\Delta_S\lambda\mu\in H^*(S,\mathbb{Q})\otimes H^*(S,\mathbb{Q})$. The generators have the following degrees:

$$\deg(T_n(\lambda))=(1,2n-2+\deg(\lambda))\quad,\quad\deg(\psi_n(\lambda))=(0,2n-2+\deg(\lambda)),\quad,\quad\deg(\mathbf{c})=(0,0). \tag{3.1}$$

Let $W^{\geqslant}(S)[m,n]$ be the subspace spanned by all bidegree (m,n) elements, and define $W^+(S)$, resp. $W^0(S)$ to be the graded subalgebra generated by $\{T_n(\lambda)\,;\,n\geqslant 0\,,\,\lambda\in H^*(S,\mathbb{Q})\}$, resp. by $\{\mathbf{c},\psi_n(\lambda)\,;\,n\geqslant 0\,,\,\lambda\in H^*(S,\mathbb{Q})\}$.

Like $\Lambda(S)$, $W^0(S)$ is just a supercommutative algebra generated by elements indexed by pairs (n,λ). It will be convenient to identify[3] them as follows. Set

$$\psi_\lambda(x)=\sum_{n\geqslant 0}\psi_n(\lambda)\frac{x^n}{n!},$$

$$\underline{\psi}(x)=x^{-1}\mathbf{c}\otimes 1+\sum_\lambda \psi_\lambda(x)\otimes\lambda^*\in W^0(S)\otimes H^*(S,\mathbb{Q})[[x]].$$

[3] This identification is why we included the central element $\mathbf{c}$ in $W^{\geqslant}(S)$ rather than just in $W(S)$.

There is a unique graded algebra isomorphism $i : W^0(S) \simeq \Lambda(S)$

$$i : \underline{\psi}(x) \mapsto x^{-1}\mathbf{r} \otimes 1 + x^{-1}(\underline{\mathrm{ch}}(x) - \mathbf{r} \otimes 1)\mathrm{Td}_S(x).$$

Through this identification, we may consider elements $p(\lambda) \in W^0(S)$ for a symmetric function p and $\lambda \in H^*(S, \mathbb{Q})$.

Theorem 3.2. *The following hold:*

(a) *The graded character of $W^+(S)$ is given by*

$$P_{W^+(S)}(z, w) = \sum_{l,n} \dim(W^+(S)[l, n])(-z)^n w^l = \mathrm{Exp}\left(\frac{P_S(z)z^{-2}w}{(1-z^2)(1-w)}\right). \tag{3.2}$$

(b) *The elements $D_{m,0}(\lambda)$ for $m \geqslant 1, \lambda \in H^*(S, \mathbb{Q})$ generate a free graded commutative polynomial algebra $\mathfrak{h}_S^+$.*

Definition 3.3. Assume that $c_1 = 0$ and that there exists $q \in H^2(S, \mathbb{Q})$ such that $q^2 = s_2$. We define the Lie algebra $\mathfrak{w}^{\geqslant}(S)$ to be spanned by $z^m D^n \lambda$ with $m, n \in \mathbb{N}$ and $\lambda \in H^*(S, \mathbb{Q})$, and a central element $\mathbf{c}$, with the Lie bracket given by

$$[z^m D^n \lambda\,,\, z^{m'} D^{n'} \mu] = z^{m+m'} \frac{(D + m'q)^n D^{n'} - D^n (D + mq)^{n'}}{q} \lambda\mu. \tag{3.3}$$

We let $\mathfrak{w}^+(S)$ be the subalgebra spanned by $z^m D^n \lambda$ for all $m \geqslant 1, n \geqslant 0$ and $\lambda \in H^*(S, \mathbb{Q})$.

Remark 3.4. More explicitly, the right hand side of (3.3) is given by

$$\sum_{i=1}^{a} \binom{a}{i} n^i z^{m+n} D^{a+b-i} q^{i-1} \lambda\mu - \sum_{j=1}^{b} \binom{b}{j} m^j z^{m+n} D^{a+b-j} q^{j-1} \lambda\mu.$$

Note that the q^1-term of this sum only depends on $(m + n)$ and $(mn' - nm')$ in accordance with the last claim of Proposition 3.8 below.

When $s_2 = q = 0$, we write $D_{m,n}(\lambda) = z^m D^n \lambda$. The Lie bracket the degenerates as follows:

$$[D_{m,n}(\lambda), D_{m',n'}(\mu)] = (nm' - mn') D_{m+m',n+n'-1}(\lambda\mu). \tag{3.4}$$

We will denote this degenerate Lie algebra by $\mathfrak{w}_0^{\geqslant}(S)$. This Lie algebra is well-defined for *any* S.

Theorem 3.5. *Assume that $c_1 = 0$ and $q^2 = s_2$. Then the assignment $T_l(\lambda) \mapsto zD^l\lambda$ extends to an isomorphism $W^{\geqslant}(S) \simeq U(\mathfrak{w}^{\geqslant}(S))$, which restricts to an isomorphism $W^+(S) \simeq U(\mathfrak{w}^+(S))$.*

We will prove Theorem 3.2 in §§ 3.7, 3.5, and Theorem 3.5 in § 3.8.

3.2. **Structure of $W^{\geqslant}(S)$ in the non-deformed case.** In this section we assume that $s_2 = 0$ and $c_1 = 0$, so that the relation (f) simplifies to

$$[T_m(\lambda), T_{n+3}(\mu)] - 3[T_{m+1}(\lambda), T_{n+2}(\mu)] + 3[T_{m+2}(\lambda), T_{n+1}(\mu)] - [T_{m+3}(\lambda), T_n(\mu)] = 0. \tag{3.5}$$

Theorem 3.6. *Assume that $s_2 = 0$ and $c_1 = 0$. Then the assignment*

$$\psi_n(\lambda) \mapsto D_{0,n}(\lambda) \quad , \quad T_n(\lambda) \mapsto D_{1,n}(\lambda) \quad , \quad \mathbf{c} \mapsto \mathbf{c} \tag{3.6}$$

extends to an algebra isomorphism $\Phi : W^{\geqslant}(S) \to U(\mathfrak{w}_0^{\geqslant}(S))$.

Proof. For simplicity, we will denote the elements $T_n(1), \psi_n(1)$ simply by T_n, ψ_n. The elements $D_{0,n}(\lambda)$, $D_{1,n}(\lambda)$ satisfy the relations (a)-(g). Hence the assignment above yields an algebra homomorphism. We claim that the following defines an inverse homomorphism Ψ to the map Φ

$$\Psi : D_{m,n}(\lambda) \mapsto \widetilde{D}_{m,n}(\lambda) := \frac{n!}{(m+n)!}(-\operatorname{Ad}_{T_0})^m \psi_{m+n}(\lambda). \tag{3.7}$$

To prove the Theorem, we need to check that the elements $\widetilde{D}_{m,n}(\lambda)$ satisfy the defining relations (3.4). Note that they are trivially satisfied for $m = n = 0$. We begin by explicitly computing a few commutators of low rank.

Let us first observe that by relation (e), for any k and any λ, we have

$$[T_1, T_1(\lambda)] = [T_1(\lambda), T_1] = 0$$

since T_1 is even. In particular, unraveling the definition of $\widetilde{D}_{m,n}(\mu)$ we obtain

$$[T_0(\lambda), \widetilde{D}_{m,n}(\mu)] = -n\widetilde{D}_{m+1,n-1}(\lambda\mu), \quad n \geqslant 1. \tag{3.8}$$

In the same way, the relations (d), (c) imply that $[\psi_1(\lambda), T_0(\mu)] = T_0(\lambda\mu)$ and $[\psi_1(\lambda), \psi_n(\mu)] = 0$. So we get

$$[\psi_1(\lambda), \widetilde{D}_{m,n}(\mu)] = m\widetilde{D}_{m,n}(\lambda\mu). \tag{3.9}$$

Let us now consider the commutators $[T_k(\lambda), T_l(\mu)]$. By (e), it's enough to assume that $\lambda = 1$. As above,

$$[T_k, T_k(\mu)] = [T_k(\mu), T_k] = -[T_k, T_k(\mu)] = 0$$

for any k, μ. Likewise,

$$[T_k, T_{k+1}(\mu)] = -[T_{k+1}(\mu), T_k] = -[T_{k+1}, T_k(\mu)].$$

Using (3.5) first for $m \in \{n, n \pm 1\}$ and then successively for $m > n$ and $m < n$ we deduce that for any n, the commutators $[T_k, T_{n-k}(\mu)]$ for $k = 0, 1, \ldots, n$ are all proportional to one another, hence, say to $[T_n, T_0(\mu)] = n\widetilde{D}_{2,n-1}(\mu)$. But since $\Phi(\widetilde{D}_{2,l}(\mu)) = D_{2,l}(\mu)$, we deduce that

$$[T_k(\lambda), T_{n-k}(\mu)] = [T_k, T_{n-k}(\lambda\mu)] = (2k - n)\widetilde{D}_{2,n-1}(\lambda\mu)$$

as expected. Further taking commutators with $\psi_k(\mu) = \widetilde{D}_{0,k}(\mu)$, we get

$$[\widetilde{D}_{2,j}(\lambda), \widetilde{D}_{0,k}(\mu)] = -2k\widetilde{D}_{2,j+k-1}(\lambda\mu). \tag{3.10}$$

Next, we consider commutators with $\widetilde{D}_{2,0}$. Relation (g) implies that $[\widetilde{D}_{2,0}, T_0] = [[T_1, T_0], T_0] = 0$, and more generally one has $[\widetilde{D}_{2,0}(\lambda), T_0(\mu)] = 0$. Using (3.8), we deduce from (3.10) by induction on m that

$$\begin{aligned} [\widetilde{D}_{2,0}(\lambda), \widetilde{D}_{m,n}(\mu)] &= -\frac{1}{n+1}[T_0, [\widetilde{D}_{2,0}(\lambda), \widetilde{D}_{m-1,n+1}(\mu)]] \\ &= 2[T_0, \widetilde{D}_{m+1,n}(\lambda\mu)] \\ &= -2n\widetilde{D}_{m+2,n-1}(\lambda\mu) \end{aligned} \tag{3.11}$$

for any $n \geqslant 1$.

Finally, from relations (3.11), (3.8), (3.9) and the definition of $\widetilde{D}_{m,n}$ we deduce in turn the following equalities

$$[T_1(\lambda), \widetilde{D}_{m,n}(\mu)] = (m - n)\widetilde{D}_{m+1,n}(\lambda\mu), \tag{3.12}$$

$$[\psi_2(\lambda), \widetilde{D}_{m,n}(\mu)] = 2m\widetilde{D}_{m,n+1}(\lambda\mu) \tag{3.13}$$

for all $m, n \geqslant 0$. Assume now that the relations (3.4) hold for a given pair (m,n) and *all* pairs (m',n') with $n' \geqslant 1$. Then, by applying Ad_{T_1} or Ad_{T_0}, resp. Ad_{ψ_2} and using (3.12) or (3.8), resp. (3.13), we deduce that the same is true for the pairs $(m+1,n)$, resp. $(m,n+1)$. Starting from $m=1, n=0$, we deduce that (3.4) holds for any pair (m,n) with $m \geqslant 1$ and $n' \geqslant 1$, and thus, by symmetry, whenever $m' \geqslant 1$ and $n \geqslant 1$. The only remaining cases occur when $m = m' = 0$ (for which (3.4) trivially holds) and when $n = n' = 0$, which we now deal with. We'll prove by induction on $m+m'$ that

$$[\widetilde{D}_{m,0}(\lambda), \widetilde{D}_{m',0}(\mu)] = 0. \tag{3.14}$$

Fix s and assume that (3.14) holds whenever $m + m' \leqslant s$. Note that the above calculations show that this is indeed the case for $s = 3$. If $m, m' > 0$ satisfy $m + m' = s+1$ then by the induction hypothesis and what we've already established we have

$$[T_0, \widetilde{D}_{m-1,1}(\lambda)] = -\widetilde{D}_{m,0}(\lambda), \qquad [T_0, \widetilde{D}_{m'-1,1}(\mu)] = -\widetilde{D}_{m',0}(\mu),$$
$$[T_0, \widetilde{D}_{m,0}(\lambda)] = 0, \qquad [T_0, \widetilde{D}_{m',0}(\mu)] = 0,$$
$$\frac{1}{m'}[\widetilde{D}_{m-1,1}(\lambda), \widetilde{D}_{m',0}(\mu)] = \widetilde{D}_{m+m',0}(\lambda\mu) = -\frac{1}{m}[\widetilde{D}_{m,0}(\lambda), \widetilde{D}_{m'-1,1}(\mu)].$$

Applying Ad_{T_0} to this last equation we deduce $[\widetilde{D}_{m,0}(\lambda), \widetilde{D}_{m',0}(\mu)] = 0$. We are done. $\square$

3.3. **Order filtration.** Let us return to the case of an arbitrary projective S. We will introduce a filtration and provide a set of linear generators of $W^{\geqslant}(S)$. Let $F_\bullet$ be the smallest filtration of $W^{\geqslant}(S)$ such that $\mathbf{c} \in F_0$ and

- $\psi_n(\lambda), T_n(\lambda) \in F_n$ for all $n \in \mathbb{N}$, $\lambda \in H^*(S,\mathbb{Q})$,
- $F_n F_{n'} \subset F_{n+n'}$,
- $[F_n, F_{n'}] \subset F_{n+n'-1}$.

We call $F_\bullet$ the *order filtration*. The algebra $W^{\geqslant}(S)$ is bigraded by (3.1). Set

$$F_{m,n} = F_n \cap W^{\geqslant}(S)[m,-].$$

The filtration $F_\bullet$ can be given more explicitly as follows. By a Lie word we mean a combination of Lie brackets applied to the generators of $W^{\geqslant}(S)$. By a monomial we mean a product of Lie words. We assign a weight to any Lie word by summing up the indices of the generators and subtracting the number of brackets. We assign a weight to any monomial by adding up the weights of the Lie words. Then F_n is the span of expressions of weight $\leqslant n$. Note that the relations of $W^{\geqslant}(S)$ are not homogeneous for the weight.

Lemma 3.7. *We have $F_{0,-1} = 0$, and $F_{m,-m} = 0$ for any $m > 0$.*

Proof. We must show that the weight n of any non-zero monomial of degree m is $n > -m$ if $m > 0$ and $n > -1$ if $m = 0$. Since degrees and weights are additive for products, it is enough to prove the statement for generators and Lie words. The generators have weights $\geqslant 0$, so they satisfy the claim. Next, take a non zero Lie word of the form $[f,g]$ for two Lie words f, g of degrees m, m' and weights n, n' respectively. We prove the claim by induction. The weight of $[f,g]$ is $n+n'-1$, its degree is $m+m'$. If $m, m' > 0$, then $n > -m$ and $n' > -m'$, hence $n + n' - 1 > -m - m'$. Now suppose $m = 0$. We have $W^{\geqslant}(S)[0,-] = \Lambda(S)$. Hence, the algebra $W^{\geqslant}(S)[0,-]$ is super-commutative. Thus f cannot be a Lie bracket. So we have $f = \psi_n(\lambda)$ for some $\lambda \in H^*(S,\mathbb{Q})$. Since $\psi_0(\lambda)$ is central, we have $n > 0$. Hence $n + n' - 1 \geqslant n' > -m' = -m - m'$. $\square$

Proposition 3.8. *There are elements $D_{m,n}(\lambda) \in W^{\geqslant}(S)$ for each $m, n \in \mathbb{N}$ and $\lambda \in H^*(S,\mathbb{Q})$, such that*

(a) $D_{0,n}(\lambda) = \psi_n(\lambda)$, $D_{1,n}(\lambda) = T_n(\lambda)$,

(b) $F_{-1}=0$, *and* F_n *is spanned by all products* $D_{m_1,n_1}(\lambda_1)\cdots D_{m_k,n_k}(\lambda_k)$ *with* $\sum_i n_i \leqslant n$,
(c) *the relation* (3.4) *holds modulo* $F_{n+n'-3}$.

Proof. The graded vector space $\mathrm{Gr}_\bullet\, W^{\geqslant}(S)=\sum_{n\in\mathbb{Z}} F_n/F_{n-1}$ has two operations: a graded (commutative) multiplication and a Lie bracket of degree -1. Consider the Lie algebra $\mathfrak{g}$ generated by the elements $\psi_n(\lambda), T_n(\lambda)$ for $n\geqslant 0, \lambda\in H^*(S,\mathbb{Q})$. The relations (a)-(e), (g) hold in $\mathfrak{g}$, as well as the simplified version (3.5) of the relation (f). By Theorem 3.6 the Lie algebra $\mathfrak{g}$ is spanned by the elements $D_{m,n}(\lambda)$, and therefore $\mathrm{Gr}_\bullet\, W^{\geqslant}(S)$ is generated by these elements as an algebra. In particular, we have $F_n/F_{n-1}=0$ for $n<0$, hence Lemma 3.7 implies that $F_{-1}=0$. Further F_n/F_{n-1} is spanned by all products $D_{m_1,n_1}(\lambda_1)\cdots D_{m_k,n_k}(\lambda_k)$ with $\sum_i n_i=n$. Therefore the first two claims of the theorem are satisfied with any lift of the elements $D_{m,n}(\lambda)$ in $\mathfrak{g}$ to $W^{\geqslant}(S)$. We also get a weak version of last claim, i.e., the identity (3.4) holds modulo $F_{n+n'-2}$. In order to get this identity modulo $F_{n+n'-3}$, we repeat the argument above with the (possibly non commutative) algebra $\sum_{n\in\mathbb{Z}} F_n/F_{n-2}$. $\square$

Definition 3.9. The algebra $W_0^{\geqslant}(S)$ is generated by $\psi_n(\lambda)$, $T_n(\lambda)$ for all n,λ subject to the relations (a)-(e), (3.5) and (g).

By the proof of Theorem 3.6 we have $W_0^{\geqslant}(S)=U(\mathfrak{w}_0^{\geqslant}(S))$, where $\mathfrak{w}_0^{\geqslant}(S)$ is the degenerate Lie algebra (3.4). Proposition 3.8 thus implies the existence of a canonical surjective algebra homomorphism

$$\rho:\mathrm{Sym}(\mathfrak{w}_0^{\geqslant}(S))\twoheadrightarrow \mathrm{Gr}_\bullet W^{\geqslant}(S) \tag{3.15}$$

as well as a morphism of Lie algebras (with the Lie bracket on $\mathrm{Gr}_\bullet W^{\geqslant}(S)$ being of degree -1)

$$\rho':\mathfrak{w}_0^{\geqslant}(S)\twoheadrightarrow \mathrm{Gr}_\bullet W^{\geqslant}(S). \tag{3.16}$$

3.4. **Deformed W-algebras.** We now proceed to define a 'doubled' version $W^{(\mathbf{c})}(S)$ of $W^{\geqslant}(S)$. Let us set $\delta(z)=\sum_{n\in\mathbb{Z}} z^n$ and put $\theta(z)=\sum_{n\geqslant 0} h_n z^n$. Recall that $\widetilde{\theta}(x)=\tau_{c_1}(\theta(x))$.

Definition 3.10. The algebra $W^{(\mathbf{c})}(S)$, called *the deformed W-algebra of S*, is generated by elements $\psi_n(\lambda), T_n^{\pm}(\lambda)$ for $n\geqslant 0, \lambda\in H^*(S,\mathbb{Q})$ and a central element $\mathbf{c}$ subject to the following relations:

- Relations (a), (b), (c) and (e) of §3.1 with $T_n^{\pm}(\lambda)$ in place of $T_n(\lambda)$;
- $[\psi_m(\lambda), T_n^{\pm}(\mu)]=\pm m T^{\pm}_{n+m-1}(\lambda\mu)$;
- The assignment $T_n(\lambda)\mapsto T_n^+(\lambda)$, resp. $T_n(\lambda)\mapsto T_n^-(\lambda)$, extends to a homomorphism, resp. to an *anti*-homomorphism $W^+(S)\to W^{(\mathbf{c})}(S)$;
- The double relation, which is best expressed in terms of generating series:

$$\left[T_\lambda^+(x),T_\mu^-(y)\right]=-\exp(i\pi\mathbf{c})\left[\frac{1}{c_1x}\left(1-\frac{\theta(x)}{\widetilde{\theta}(x)}\right)\delta\Big(\frac{y}{x}\Big)(\lambda\mu)\right]_{++} \tag{3.17}$$

where $T_\lambda^{\pm}(z)=\sum_{n\geqslant 0} T_n^{\pm}(\lambda)z^n$ and where $A(x,y)_{++}$ stands for the truncation of a power series to its terms x^ay^b with $a,b\geqslant 0$. Explicitly, one may rewrite this relation as follows:

$$[T_m^+(\lambda),T_n^-(\mu)]=-\exp(i\pi\mathbf{c})\sum_{0\leqslant i\leqslant j\leqslant m+n}(-1)^j\binom{\mathbf{c}-j+i}{i+1}h_{m+n-j}e_{j-i}(c_1^i\lambda\mu). \tag{3.18}$$

We will denote by $W^+(S), W^-(S)$ and $W^0(S)$ the subalgebras generated by $T_n^+(\lambda)$, resp. by $T_n^-(\lambda)$, resp. by $\psi_n(\lambda)$, $\mathbf{c}$ for $n\geqslant 0$ and $\lambda \in H^*(S,\mathbb{Q})$. We likewise define $W^{\geqslant}(S), W^{\leqslant}(S)$. The algebra $W^{(\mathbf{c})}(S)$ is naturally $\mathbb{Z}^2$-graded, with

$$\deg(T_n^{\pm}(\lambda)) = (\pm 1, 2n-2+\deg(\lambda)), \qquad \deg(\psi(n)) = (0, 2n-2+\deg(\lambda)), \qquad \deg(\mathbf{c}) = (0,0).$$

We will write $W^{(e)}(S)$ for the central specialization of $W^{(\mathbf{c})}(S)$ to $\mathbf{c}=e$.

Remark 3.11.

(i) The elements $T_n^-(\lambda)$ satisfy the following sign-corrected version of (f):

$$\text{(f')} \quad \begin{aligned} &[T_m(\lambda), T_{n+3}(\mu)] - 3[T_{m+1}(\lambda), T_{n+2}(\nu)] + 3[T_{m+2}(\lambda), T_{n+1}(\mu)] - [T_{m+3}(\lambda), T_n(\mu)] \\ &\qquad - [T_m(\lambda), T_{n+1}(s_2\mu)] + [T_{m+1}(\lambda), T_n(s_2\mu)] - \{T_m, T_n\}(c_1\Delta_S\lambda\mu) = 0, \end{aligned}$$

(ii) When $c_1 = 0$, the r.h.s of (3.18) reduces to (up to the factor $-\exp(i\pi\mathbf{c})$)

$$\sum_{0\leqslant k\leqslant n+m} (\mathbf{c}-k)(-1)^k h_{n+m-k} e_k(\lambda\mu) = \mathbf{c}\delta_{n+m,0}(\lambda\mu) - \sum_{0\leqslant k\leqslant n+m} k(-1)^k h_{n+m-k}e_k(\lambda\mu) = p_{n+m}(\lambda\mu).$$

If, in addition $s_2 = 0$ (equivalently, $c_2 = 0$) then we also have $\mathrm{Td}_S(x) = 1$ and $p_l(\lambda) = l\psi_{l-1}(\lambda)$.

(iii) Note that the relation (3.18) has a formal term $\exp(i\pi\mathbf{c})$. For any central specialization $\mathbf{c} = e$ it becomes just a complex number. Moreover, in all cases of interest for us, $\mathbf{c}$ will be a non-negative integer, so that $\exp(i\pi\mathbf{c}) = (-1)^e$, and $W^{(e)}(S)$ is defined over $\mathbb{Q}$.

Proposition 3.12. *The natural maps $W^{\geqslant}(S)^{\mathrm{op}} \to W^{(\mathbf{c})}(S)$, $W^{\geqslant}(S) \to W^{(\mathbf{c})}(S)$ are embeddings of algebras, and the multiplication map $W^-(S)\otimes W^0(S)\otimes W^+(S) \to W^{(\mathbf{c})}(S)$ is an isomorphism of vector spaces.*

Proof. Analogous to [42, Appendix A]. □

Remark 3.13. The same proof as in [42, Appendix A] also shows that $W^+(S)$ is isomorphic to the algebra generated by the $T_l(\lambda)$ subject to the relations (b), (e), (f) and (g).

In the undeformed case, we have a presentation of $W^{(\mathbf{c})}(S)$ as the enveloping algebra of a Lie algebra again. More precisely, consider the Lie algebra $\mathfrak{w}(S)$ generated by elements $D_{m,n}(\lambda)$, $m\in\mathbb{Z}$, $n\in\mathbb{N}$, $\lambda\in H^*(S,\mathbb{Q})$, whose Lie bracket is given by (3.4), where we set $D_{i,-1} := \delta_{i,0}\mathbf{c}$.

Theorem 3.14. *Assume that $s_2=0$ and $c_1=0$. There is an algebra isomorphism $\Phi : W^{(\mathbf{c})}(S) \to U(\mathfrak{w}(S))$, which sends $\psi_n(\lambda)$ to $D_{0,n}(\lambda)$, $T_n^+(\lambda)$ to $D_{1,n}(\lambda)$ and $T_n^-(\lambda)$ to $\exp(i\pi\mathbf{c})D_{-1,n}(\lambda)$.*

Proof. Denote by $\mathfrak{w}^+(S)$, $\mathfrak{w}^-(S)$ the Lie subalgebras of $\mathfrak{w}(S)$ spanned by $D_{m,n}(\lambda)$ with $m>0$, $m<0$ respectively. The restriction of Φ to $W^{\pm}$ defines isomorphisms of algebras $W^{\pm}(S)\simeq U(\mathfrak{w}^{\pm}(S))$ by Theorem 3.6. An inductive argument analogous to the proof of Proposition 3.15 below shows that the commutation relations between $\mathfrak{w}^+(S)$ and $\mathfrak{w}^-(S)$ hold in $W^{(\mathbf{c})}(S)$ as well. We conclude by Proposition 3.12 and the PBW theorem for universal enveloping algebras. □

3.5. **Heisenberg subalgebra.** The *Heisenberg algebra* $\mathfrak{h}_S$ of S is the Lie algebra generated by elements

$$\{\mathfrak{q}_n(\lambda), C \mid n\neq 0, \lambda\in H^*(S,\mathbb{Q})\},$$

(with, as usual, the relation $\mathfrak{q}_n(\lambda+\mu) = \mathfrak{q}_n(\lambda)+\mathfrak{q}_n(\mu)$ for any n and λ,μ) with Lie bracket given by

$$[\mathfrak{q}_m(\lambda), \mathfrak{q}_l(\mu)] = m\delta_{-m,l}C\int_S \lambda\cup\mu, \qquad C \text{ is central.} \tag{3.19}$$

Recall the elements $D_{m,n}(\lambda) \in W^{\geqslant}(S)$ from Proposition 3.8. Let us consider the homomorphism $\Theta : W^+(S) \to W^-(S)$ defined by $\Theta(T_n^+(\lambda)) = (-1)^n T_n^-(\lambda)$. We set $D_{-m,n}(\lambda) = \Theta(D_{m,n}(\lambda))$. The same theorem implies that elements $D_{m,0}(\lambda)$ with $m \geqslant 1$ and $\lambda \in H^*(S, \mathbb{Q})$ super-commute with each other, and are uniquely determined by the following formula:

$$D_{m+1,0}(\lambda) = \frac{1}{m}[D_{1,1}(1), D_{m,0}(\lambda)]. \tag{3.20}$$

Proposition 3.15. *The assignment $C \mapsto \mathbf{c}$ and*

$$\mathfrak{q}_n(\lambda) \mapsto D_{n,0}(\lambda), \qquad \mathfrak{q}_{-n}(\lambda) \mapsto -e^{-i\pi\mathbf{c}} D_{-n,0}(\lambda) \qquad (n > 0)$$

defines a morphism of algebras $U(\mathfrak{h}_S) \to W^{(\mathbf{c})}(S)$. In particular, Theorem 3.2(b) holds.

Proof. Denote $L_\pm = D_{\pm1,1}(1)$, $L_0 = \psi_1(1)$. By definition, we have $[L_\pm, \mathfrak{q}_{\pm m}(\lambda)] = m\mathfrak{q}_{\pm(m+1)}(\lambda)$. Moreover, the following equalities are easy consequences of relation (3.18):

$$[\mathfrak{q}_1(\lambda), \mathfrak{q}_{-1}(\mu)] = \mathbf{c}\int_S \lambda \cup \mu, \qquad [L_\pm, \mathfrak{q}_{\mp1}(\lambda)] = \pm\psi_0(\lambda) \mp \binom{\mathbf{c}}{2}\int_S \lambda \cup c_1, \tag{3.21}$$

$$[L_0, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_0 - \mathbf{c}\psi_0(c_1) + \binom{\mathbf{c}}{3}\int_S c_1^2.$$

In view of Theorem 3.8, it suffices to check the relations $[\mathfrak{q}_{-m}(\lambda), \mathfrak{q}_n(\mu)] = n\mathbf{c}\delta_{m,n}(\lambda, \mu)$ for $m, n > 0$. We proceed by induction. First, note that $[L_0, \mathfrak{q}_{\pm n}] = \pm n\mathfrak{q}_{\pm n}$:

$$\begin{aligned}[L_0, \mathfrak{q}_{n+1}] &= \frac{1}{n}[L_0, [L_+, \mathfrak{q}_n]] = -\frac{1}{n}\left([\mathfrak{q}_n, [L_0, L_+]] + [L_+, [\mathfrak{q}_n, L_0]]\right) = \mathfrak{q}_{n+1} + [L_+, \mathfrak{q}_n] \\ &= (n+1)\mathfrak{q}_{n+1}.\end{aligned}$$

Next, $[L_\pm, \mathfrak{q}_{\mp n}(\lambda)] = \mp n\mathfrak{q}_{\mp(n-1)}$ for $n > 1$:

$$\begin{aligned}[L_+, \mathfrak{q}_{-(n+1)}(\lambda)] &= \frac{1}{n}[L_+, [L_-, \mathfrak{q}_{-n}(\lambda)]] = -\frac{1}{n}\left([\mathfrak{q}_{-n}, [L_+, L_-]] + [L_-, [\mathfrak{q}_{-n}, L_+]]\right) \\ &= -2\mathfrak{q}_{-n} - [L_-, \mathfrak{q}_{-(n-1)}] = -(n+1)\mathfrak{q}_{-n}.\end{aligned}$$

It is easy to see that $[\mathfrak{q}_{\pm1}(\lambda), \mathfrak{q}_{\mp n}(\mu)] = 0$ for $n > 1$:

$$\begin{aligned}[\mathfrak{q}_{-1}(\lambda), \mathfrak{q}_{n+1}(\mu)] &= \frac{1}{n}[\mathfrak{q}_{-1}(\lambda), [L_+, \mathfrak{q}_n(\mu)]] \\ &= -\frac{1}{n}\left([L_+, [\mathfrak{q}_n(\mu), \mathfrak{q}_{-1}(\lambda)]] + [\mathfrak{q}_n(\mu), [\mathfrak{q}_{-1}(\lambda), L_+]]\right) = 0.\end{aligned}$$

Finally, for any positive m, n we have by induction

$$\begin{aligned}[\mathfrak{q}_{-(m+1)}(\lambda), \mathfrak{q}_{n+1}(\mu)] &= \frac{1}{n}[\mathfrak{q}_{-(m+1)}(\lambda), [L_+, \mathfrak{q}_n(\mu)]] \\ &= \frac{1}{n}[L_+, [\mathfrak{q}_{-(m+1)}(\lambda), \mathfrak{q}_n(\mu)]] + \frac{m+1}{n}[\mathfrak{q}_{-m}(\lambda), \mathfrak{q}_n(\mu)] \\ &= \mathbf{c}(m+1)\delta_{m,n}\int_S \lambda \cup \mu,\end{aligned}$$

which proves the desired statement. □

Remark 3.16. Note that (3.21) contains central terms which do not appear in [23, Theorem 3.3]. The reason is these terms vanish for the Hilbert schemes (see § 7), since in this case $\mathbf{c} = 1$ and $\psi_0(\lambda) = 0$ for all $\lambda \in H^{\geqslant 2}(S)$.

The relations (c-d) imply that $D_{0,0}(\lambda) = \psi_0(\lambda)$ is central in $W^{(\mathbf{c})}(S)$ for any $\lambda \in H^{\geqslant 2}(S)$. Denote by $Z(S)$ the super-commutative subalgebra generated by $D_{0,0}(\lambda)$'s, and by $W^{(\mathbf{c})}_{\mathrm{red}}(S)$ the quotient of $W^{(\mathbf{c})}(S)$ by the (two-sided) ideal generated by $Z(S)$.

Lemma 3.17. *If $I \subset W^{(\mathbf{c})}_{\mathrm{red}}(S)$ is a two-sided ideal such that $I \cap U(\mathfrak{h}_S) = \{0\}$, then $I = \{0\}$. Similarly, if $I^+ \subset W^+(S)$ is a two-sided ideal with $I^+ \cap U(\mathfrak{h}_S^+) = \{0\}$, then $I^+ = \{0\}$.*

Proof. The proofs of the two claims being completely analogous, we will only prove the first one. We follow the proof of [39, Lemma F.7].

The algebra $W^{(\mathbf{c})}_{\mathrm{red}}(S)$ admits the order filtration as in § 3.3. Recall that $\mathrm{Gr}_\bullet W^{(\mathbf{c})}_{\mathrm{red}}(S)$ is a graded super-commutative algebra, equipped with a Lie bracket of degree -1. Let $\mathrm{Gr}_\bullet I \subset \mathrm{Gr}_\bullet W^{(\mathbf{c})}_{\mathrm{red}}(S)$ be the associated graded of I with respect to the induced filtration. Using Theorem 3.14 instead of Theorem 3.6, we can repeat the proof of Theorem 3.8 to get an algebra surjection $\nu : \mathbb{Q}[D_{m,n}(\lambda)] \to \mathrm{Gr}_\bullet W^{(\mathbf{c})}_{\mathrm{red}}(S)$. Set $J = \nu^{-1}(\mathrm{Gr}_\bullet I)$. It is enough to show that $J = \{0\}$.

The ideal J is graded by the weight. Let $x \in J$ be a non-zero element of minimal weight n. Since $U(\mathfrak{h}_S) \subset \mathrm{Gr}_0 W^{(r)}_{\mathrm{red}}(S)$, we have $J \cap U(\mathfrak{h}_S) = \{0\}$. Hence $n > 0$. We write

$$x = \sum_i c_i \prod_j D_{m_{ij},n_{ij}}(\lambda_{ij}) \quad , \quad \sum_j n_{ij} = n.$$

Assume that for each i, j we have either $n_{ij} > n_{i,j+1}$, or $n_{ij} = n_{i,j+1}$ and $m_{ij} \geqslant m_{i,j+1}$. Let $n_i = \{n_{i1} \geqslant n_{i2} \geqslant \ldots\}$. Let $\bar{\imath}$ be the index of the maximal tuple among all n_i's, with respect to the lexicographic order. We write $n_{\bar{\imath}} = \{\overline{n}_1 \geqslant \ldots \geqslant \overline{n}_s\}$ and $m_{\bar{\imath}} = \{\overline{m}_1 \geqslant \ldots \geqslant \overline{m}_s\}$.

The space $\mathbb{Q}[D_{m,n}(\lambda)]$ is equipped with the Lie bracket given by (3.5) and Leibniz rule. Consider the operators $\sigma_l = \mathrm{Ad}(D_{l,0}(1))$ of degree -1. We have

$$\sigma_l(D_{m_1,n_1}(\lambda_1) \cdots D_{m_k,n_k}(\lambda_k)) = -l \sum_{i=1}^{k} n_i D_{m_1,n_1}(\lambda_1) \cdots D_{m_i+l,n_i-1}(\lambda_i) \cdots D_{m_k,n_k}(\lambda_k). \tag{3.22}$$

The ideal J is preserved by the action of the operators σ_l. Fix $l > \max\{m_{ij}\}$. Let us compute the coefficient in $\sigma_l(x)$ of the monomial

$$D_{\overline{m}_1+l,\overline{n}_1-1}(\lambda_1) D_{\overline{m}_2,\overline{n}_2}(\lambda_2) \cdots D_{\overline{m}_s,\overline{n}_s}(\lambda_s)$$

The condition on l implies that the only monomial in x which can contribute to this coefficient is the monomial corresponding to $\bar{\imath}$. Using the formula (3.22), we obtain that this coefficient is $-l\, c_{\bar{\imath}}\, \overline{n}_1\, t$, where t is the maximal number with $\overline{n}_t = \overline{n}_1$ and $\overline{m}_t = \overline{m}_1$. This coefficient is non-zero. Hence, we have $\sigma_l(x) \neq 0$. However, we have $\sigma_l(x) \in J$ and $\deg \sigma_l(x) < \deg x$. This contradicts the minimality of the weight n. □

3.6. **Virasoro subalgebra.** Let us introduce another Lie subalgebra of $W^{(\mathbf{c})}(S)$. The results of this section will not be used anywhere, but seem to be of independent interest.

Definition 3.18. Let $\eta : H^*(S, \mathbb{Q})^{\otimes 2} \to Z(S)[\mathbf{c}]$ be a bilinear map. The *Virasoro algebra* $\mathrm{Vir}_S(\eta)$ of S of central charge η is the Lie algebra generated by

$$\{\mathfrak{L}_n(\lambda), \gamma, \mathbf{c} \mid n \in \mathbb{Z}, \lambda \in H^*(S, \mathbb{Q}), \gamma \in Z(S)\},$$

where $\gamma \in Z(S)$ and $\mathbf{c}$ are central, and the Lie bracket is given by

$$[\mathfrak{L}_m(\lambda), \mathfrak{L}_n(\mu)] = (n-m)\mathfrak{L}_{m+n}(\lambda\mu) - \frac{n^3 - n}{12}\delta_{-m,n}\eta(\lambda, \mu). \tag{3.23}$$

Remark 3.19. Our definition differs from the standard conventions by a sign; in other words, we are considering the opposite Lie algebra of the usual definition.

Let us fix a specific choice of elements $D_{\pm n,1}$, $n\geqslant 2$ in $W^{(\mathbf{c})}(S)$:

$$D_{\pm 2,1} := \pm\frac{1}{2}[D_{\pm 1,2}, D_{\pm 1,0}], \qquad D_{\pm(n+1),1} := \frac{\pm 1}{n-1}[D_{\pm 1,1}, D_{\pm n,1}].$$

Let us also define the following elements:

$$\mathfrak{L}_n(\lambda) = D_{n,1}(\lambda) + \frac{(n-1)\mathbf{c}}{2} D_{n,0}(c_1\lambda) + \frac{1}{2}\delta_{n,0}\binom{\mathbf{c}}{3}\int_S c_1^2\lambda, \qquad n\geqslant 0,$$

$$\mathfrak{L}_n(\lambda) = \exp(i\pi\mathbf{c})\left(D_{n,1}(\lambda) - \frac{(n+1)\mathbf{c}}{2} D_{n,0}(c_1\lambda)\right), \qquad n<0.$$

Proposition 3.20. *The assignment $\mathfrak{L}_n(\lambda)\mapsto\mathfrak{L}_n(\lambda)$, $\gamma\mapsto\gamma$, $\mathbf{c}\mapsto\mathbf{c}$ defines a morphism of algebras $U(\mathrm{Vir}_S(\eta))\to W^{(\mathbf{c})}(S)$, where the central charge η is given by*

$$\eta(\lambda,\mu) = \mathbf{c}\left(\int_S c_2\lambda\mu - (1-\mathbf{c}^2)\int_S c_1^2\lambda\mu + 2\psi_0(c_1\lambda\mu)\right).$$

Proof. Note that the relation (3.23) for m, n of the same sign follows from Proposition 3.8. For other commutators, note that $\mathrm{Vir}_S(\eta)$ is generated over $Z(S)[\mathbf{c}]$ by the elements $\mathfrak{L}_n$, $|n|\leqslant 2$. Once we check the commutation relations between these elements, the rest of the relations can be deduced by an inductive argument as in Proposition 3.15. The computation for the five elements above is straightforward, albeit laborious; we leave it to the interested reader. It is performed using the definitions of elements $\mathfrak{L}_n(\lambda)$ and the defining relations of $W^{(\mathbf{c})}(S)$. Let us briefly comment on the appearance of c_2 in the formula η. While (3.18) does not manifestly depend on c_2, after writing out its r.h.s. in terms of ψ_i's for $m+n=4$ we obtain

$$-\exp(i\pi\mathbf{c})\left(4\psi_3(\lambda\mu) - 3\mathbf{c}\psi_2(c_1\lambda\mu) - 2(\psi_0\psi_1)(c_1\lambda\mu) + (\mathbf{c}^2-\mathbf{c}+2)\psi_1(c_1^2\lambda\mu) - \underline{2\psi_1(c_2\lambda\mu)} + \cdots\right)$$

where the omitted summands belong to the center of $W^{(\mathbf{c})}(S)$. The underlined term is precisely the one which gives rise to $\int_S c_2\lambda\mu$. □

3.7. **Proof of Theorem 3.2(a).** Recall that we have obtained an upper bound on the graded dimension of $W^{\geqslant}(S)$ in § 3.3. In order to obtain a lower bound, we consider the descending algebra filtration $G^\bullet$ of $W^{(1)}(S)$ obtained by putting the generators $T_n^{\pm}(\lambda), \psi_l(\lambda)$ in degree $\deg(\lambda)$. From the defining relations and Proposition 3.12 it follows that G^N be spanned by all monomials

$$T^-_{i_1}(\mu^-_1)\cdots T^-_{i_{l_-}}(\mu^-_{l_-})\psi_{j_1}(\lambda_1)\cdots\psi_{j_{l_0}}(\lambda_{l_0})T^+_{k_1}(\mu^+_1)\cdots T^+_{k_{l_+}}(\mu^+_{l_+})$$

with $\sum\deg(\lambda_i)+\sum\deg(\mu_j^{\pm})\geqslant N$. In particular, the restriction of this filtration to $W^{\geqslant}(S)$ is given by the same definition without T^-'s. It is clear that each G^N is a two-sided ideal in $W^{(1)}(S)$.

One defines a double $W_0^{(1)}(S) = U(\mathfrak{w}_0^{(1)}(S))$ of $W_0^{\geqslant}(S)$ in an obvious way, using relations as in Theorem 3.14. The images of $\psi_n(\lambda)$'s and $T_n^{\pm}(\lambda)$'s in $\mathrm{Gr}^\bullet W^{\geqslant}(S)$ satisfy the relations (a)-(e), (g). As the last three summands in (f) have higher G-degree than the first four, the relation (3.4) holds in $\mathrm{Gr}^\bullet W^{\geqslant}(S)$ as well. For the same reason, relation (3.18) with $c_1=s_2=0$ holds too. We deduce that there is an algebra homomorphism $\zeta: W_0^{(1)}(S)\to\mathrm{Gr}^\bullet W^{(1)}(S)$ which maps $\psi_n(\lambda)$ to $\mathrm{Gr}\,\psi_n(\lambda)$ and $T_n^{\pm}(\lambda)$ to $\mathrm{Gr}\,T_n^{\pm}(\lambda)$.

Lemma 3.21. *The morphism $\zeta: W_0^{(1)}(S)\to\mathrm{Gr}^\bullet W^{(1)}(S)$ is an isomorphism.*

Proof. The morphism is surjective by definition of $G^\bullet$. Its restriction to $U(\mathfrak{h}(S))$ is non-zero (and of central charge $\mathbf{c}=1$); the kernel of this map is a two-sided ideal of $U(\mathfrak{h}(S))$, which has to be zero by a standard argument. We conclude by Lemma 3.17. □

Proof of Theorem 3.2(a). The map ζ restricts to an isomorphism $W_0^{\geqslant}(S) \simeq \mathrm{Gr}^\bullet W^{\geqslant}(S)$, compatible with the grading. It suffices to observe that the Hilbert series of $W_0^{\geqslant}(S) = U(\mathfrak{w}_0^{\geqslant}(S))$ is given by (3.2). □

3.8. **Structure of $W^{\geqslant}(S)$ in the semi-deformed case.** Suppose that $c_1 = 0$, and $q \in H^2(S, \mathbb{Q})$ is such that $q^2 = s_2$. Recall the Lie algebra $\mathfrak{w}^{\geqslant}(S)$ from Definition 3.3. The following proposition is a slightly more precise version of Theorem 3.5.

Proposition 3.22. *Assume that $c_1 = 0$ and $s_2 = q^2$ for $q \in H^2(S, \mathbb{Q})$. There exists an algebra isomorphism $\Phi : W^{\geqslant}(S) \simeq U(\mathfrak{w}^{\geqslant}(S))$ such that*

$$\Phi(T_n^+(\lambda)) = zD^n\lambda, \qquad \Phi(\psi_n(\lambda)) = D^n\lambda + \sum_{i>0} a_{ni} D^{n-i} q^i \lambda$$

for some explicit rational numbers (a_{ij}).

Proof. It is straightforward to check that the relations (b), (e)-(g) hold between $\Phi(T_n^+(\lambda))$. Note that we have

$$[D^m\lambda, zD^n\mu] = mzD^{m+n-1}\lambda\mu + \sum_{i=2}^{m} \binom{m}{i} zD^{m+n-i} q^{i-1}\lambda\mu$$

Let $A = (a_{ij})$ be the inverse of the matrix $B = (b_{ij})$, where $b_{ij} = \frac{1}{i}\binom{i}{j-1}$ for $i \geqslant j$ and 0 otherwise, and define $\Phi(\psi_n(\lambda)) = \sum_{i \geqslant 0} a_{n,n-i} D^{n-i} q^i \lambda$. Since B is upper-triangular with 1's on the diagonal, $\Phi(\psi_n(\lambda))$ has the required form, and the relation (d) holds by definition, as well as the tautological relations (a), (c). We have thus obtained a well-defined homomorphism $\Phi : W^{\geqslant}(S) \to U(\mathfrak{w}^{\geqslant}(S))$.

Observe that the Lie algebra $\mathfrak{w}^{\geqslant}(S)$ is generated by the elements $D^n\lambda$, $zD^n\lambda$. In particular, Φ is surjective. Finally, the graded dimension $W^{\geqslant}(S)$ is equal to the graded dimension of $U(\mathfrak{w}^{\geqslant}(S))$ by Theorem 3.5(a), so we may conclude. □

Remark 3.23. In [25], an action of a certain Lie algebra $\mathcal{W}_S$ on the cohomology of Hilbert schemes of points on S was constructed via vertex algebra methods. Its basis is given by elements $\mathfrak{J}_m^p(\lambda)$, $m \in \mathbb{Z}$, $p \in \mathbb{Z}_{\geqslant 0}$, $\lambda \in H^*(S, \mathbb{Q})$ and the Lie bracket is, up to central charge,

$$[\mathfrak{J}_m^p(\lambda), \mathfrak{J}_n^q(\mu)] = (qm - pn)\mathfrak{J}_{m+n}^{p+q-1}(\lambda\mu) - \frac{\Omega_{m,n}^{p,q}}{12}\mathfrak{J}_{m+n}^{p+q-3}(c_2\lambda\mu),$$

where $\Omega_{m,n}^{p,q}$ is given by formula (5.2) in *loc. cit.* This Lie algebra by definition lives in a certain completion of $W^{(1)}(S)$, but in the case when $c_1 = 0$ it is an actual subalgebra of $W^{(1)}(S)$ by Lemma 5.2 in *loc. cit.* Let us set

$$T_n^{\pm}(\lambda) = \mathfrak{J}_{\pm 1}^n(\lambda), \qquad \psi_n(\lambda) = \mathfrak{J}_0^n(\lambda).$$

One can check by a direct computation that the relations of $W^{(1)}(S)$ with $c_1 = 0$ hold. Therefore we obtain a homomorphism of algebras $W^{(1)}(S) \to U(\mathcal{W}_S)$, which may be shown to be an isomorphism by a simple dimension check. We want to emphasize that the existence of this homomorphism crucially relies on the fact that $c_2^2 = 0$, which is true in $H^*(S, \mathbb{Q})$ for degree reasons. In the presence of a torus action this vanishing typically fails (hence the results of [25] do not apply) and one obtains instead the semi-deformed algebra $U(\mathfrak{w}^{\geqslant}(S))$. Note that our presentation does not involve the factor $\Omega_{m,n}^{p,q}$.

4. Fock space representations of $W^{(r)}(S)$

In this section, we construct a Fock space representation of $W^{(r)}(S)$ for any $r\geqslant 0$ by considering the action of Hecke correspondences on tautological cohomology rings. We still assume that S is projective.

4.1. **The algebra of universal Hecke operators.** Recall from (2.22) the algebra homomorphisms $R^{\pm}$ and the $\Lambda(S)$-linear maps $Q^{\pm}$. We define the elements $\phi_n(\lambda)$, $n\in\mathbb{N}$ and $c(\lambda)$ in $\Lambda(S)$ by the following generating series:

$$\phi_\lambda(x)=x^{-1}c(\lambda)+\sum_{n\geqslant 0}\frac{x^n}{n!}\phi_n(\lambda):=\sum_{n\geqslant 0}\frac{x^{n-1}}{n!}p_n\big(\mathrm{Td}_S(x)\cup\lambda\big)\in\Lambda(S)((x)).$$

Note that $c(\lambda)=0$ if $\lambda\notin\mathbb{C}[\mathrm{pt}]$ while $c([\mathrm{pt}])=p_0([\mathrm{pt}])=\mathbf{r}$. Let us denote by the same symbols $\phi_n(\lambda),c(\lambda)$ the operators of left multiplication in $\Lambda(S)$. For $n\in\mathbb{Z}$, we put, following § 2.6

$$L_n^{\pm}(\lambda):\Lambda(S)\to\Lambda(S),\quad f\mapsto Q^{\pm}(\lambda u^nR^{\pm}(f)).\tag{4.1}$$

We recall $\theta(x)=\sum_{m\geqslant 0}h_mx^m$ from § 3.4 and we set

$$L_\lambda^{\pm}(x)=\sum_{n\in\mathbb{Z}}L_n^{\pm}(\lambda)x^n.$$

Proposition 4.1. *The assignment $\psi_n(\lambda)\mapsto\phi_n(\lambda),\mathbf{c}\mapsto\mathbf{r},T_n^{\pm}(\lambda)\mapsto L_n^{\pm}(\lambda)$ for $n\geqslant 0$ and $\lambda\in H^*(S,\mathbb{Q})$ extends to respective actions of $W^{\geqslant}(S)$ and $W^{\leqslant}(S)$ on $\Lambda(S)$.*

Proof. We will deal with the positive operators only, the second case being identical. To unburden the notation, we suppress $+$ from the notation. We have to show that the operators $\phi_n(\lambda),L_n(\lambda)$ satisfy the defining relations (a)-(g) of $W^{\geqslant}(S)$. The relations (a), (b), (c) being immediate, we concentrate on the remaining ones. We deduce from (2.21) that

$$\sum_{m\geqslant 0}\frac{x^m}{m!}\,\mathrm{ev}_\delta(p_m(\lambda))=x^2e^{ux}\mathrm{Td}_S^{-1}(x)\lambda.$$

In particular, we get

$$\begin{aligned}[\phi_\lambda(x),L_n(\mu)]&=\sum_{m\geqslant 0}\frac{x^{m-1}}{m!}[p_m(\mathrm{Td}_S(x)\lambda),L_n(\mu)]\\&=\sum_{m\geqslant 0}\frac{x^{m-1}}{m!}Q\Big(\mu u^n(p_m-R(p_m))(\mathrm{Td}_S(x)\lambda)R(-)\Big)\\&=xQ\Big(\mu u^n\sum_{m\geqslant 0}\frac{x^{m-2}}{m!}\,\mathrm{ev}_\delta(p_m)(\mathrm{Td}_S(x)\lambda)R(-)\Big)\\&=xQ\Big(u^ne^{ux}\lambda\mu R(-)\Big)=\sum_{m\geqslant 0}\frac{x^{m+1}}{m!}Q\Big(\lambda\mu u^{m+n}R(-)\Big)=\sum_{m\geqslant 1}\frac{x^m}{(m-1)!}L_{m+n-1}(\lambda\mu)\end{aligned}$$

which proves the relation (d). Note that thanks to this, it suffices to check relations (e), (f) and (g) when evaluated at 1.

Let us write

$$\Omega(x,y)=\frac{y^2}{(1-y(x^{-1}-t_1))(1-y(x^{-1}-t_2))}\in H^*(S,\mathbb{Q})[x,x^{-1}][[y]].$$

Lemma 4.2. *We have the following identity:*

$$L_\lambda(x)L_\mu(y)(1) = m(\theta(x)\otimes\theta(y))(1-\Delta\Omega(x,y))(\lambda\otimes\mu)$$

where we denote by $m : \Lambda(S)\otimes\Lambda(S)\to\Lambda(S)$ *the multiplication map.*

Proof. Let us first note that

$$L_\mu(y)(1) = \sum_{n\geqslant 0} y^n h_n(\mu) = \exp\Big(\sum_{k\geqslant 1}\frac{y^k}{k}p_k\Big)(\mu).$$

Using formula (2.21), we get after applying R:

$$\begin{aligned}
R(L_\mu(y)(1)) &= R\left(\int_S \exp\Big(\sum_k \frac{y^k}{k}\underline{\mathrm{ch}}_k\Big)\mu_2\right) = \int_S \exp\Big(\sum_{k\geqslant 1}\frac{y^k}{k}((\underline{\mathrm{ch}}_k)_{12} - (\Delta f_k(u))_{23})\Big)\mu_2 \\
&= \int_S \exp\Big(\sum_{k\geqslant 1}\frac{y^k}{k}((\underline{\mathrm{ch}}_k)\Big)_{12}\exp\Big(-\sum_k\frac{y^k}{k}\Delta f_k(u)\Big)_{23}\,\mu_2 \\
&= \int_S p(\theta(y))_{12}\ \exp\Big(-\Delta\sum_{k\geqslant 1}\frac{y^k}{k}f_k(u)\Big)_{23}\,\mu_2.
\end{aligned}$$

In the above, we use indices to specify the position in the tensor product, i.e. $\mu_2 = 1\otimes\mu\otimes 1$, etc. Since $\Delta^2 = t_1t_2\Delta$, we can compute the exponential term:

$$\begin{aligned}
&\exp\Big(-\Delta\sum_{k\geqslant 1}y^k\frac{u^k-(u-t_1)^k-(u-t_2)^k+(u-t_1-t_2)^k}{kt_1t_2}\Big) \\
&= 1+\frac{\Delta}{t_1t_2}\left(\exp\Big(-\sum_{k\geqslant 1}y^k\frac{u^k-(u-t_1)^k-(u-t_2)^k+(u-t_1-t_2)^k}{k}\Big)-1\right) \\
&= 1-\frac{\Delta}{t_1t_2}\left(\frac{(1-yu)(1-yu+yt_1+yt_2)}{(1-yu+yt_1)(1-yu+yt_2)}-1\right) \\
&= 1-\frac{y^2\Delta}{(1-yu+yt_1)(1-yu+yt_2)}.
\end{aligned}$$

By the definition of the operator Q, we have the following equality

$$\sum_{m\in\mathbb{Z}} Q(x^m u^{m+i}\lambda) = x^{-i}\sum_{m\geqslant 0}x^m h_m(\lambda).$$

Putting everything together, we conclude that

$$\begin{aligned}
L_\lambda(x)L_\mu(y)(1) &= Q\Big(\lambda\sum_{n\in\mathbb{Z}}(xu)^n RL_\mu(y)(1)\Big) \\
&= \int_S Q\Big(\lambda_2\sum_{n\in\mathbb{Z}}(xu)^n p(\theta(y))_{13}(1-\frac{y^2\Delta_{23}}{(1-yu+yt_1)(1-yu+yt_2)})\mu_3\Big) \\
&= \int_S Q\Big(\lambda_2\sum_{n\in\mathbb{Z}}(xu)^n p(\theta(y))_{13}(1-\frac{y^2\Delta_{23}}{(1-yx^{-1}+yt_1)(1-yx^{-1}+yt_2)})\mu_3\Big) \\
&= \int_{S\times S} p(\theta(x))_{12}\,p(\theta(y))_{13}\,\Big(1-\frac{y^2\Delta}{(1-yx^{-1}+yt_1)(1-yx^{-1}+yt_2)}\Big)\,\lambda_2\,\mu_3 \\
&= m(\theta(x)\otimes\theta(y))(1-\Delta\Omega(x,y))(\lambda\otimes\mu)).
\end{aligned}$$

In the above, we have made use of the relation $\sum_{n\in\mathbb{Z}}(ux)^n A(u) = \sum_{n\in\mathbb{Z}}(ux)^n A(x^{-1})$ which is valid for a Laurent series $A(u)$. □

Now, by Lemma 4.2, we have

$$\begin{aligned}[L_\lambda(x), L_\mu(y)](1) &= -\int_{S\times S} p(\theta(x))_{12}\, p(\theta(y)))_{13}\, (\Delta\Omega(x,y) - \Delta\Omega(y,x))_{23}\, \lambda_2\, \mu_3 \\ &= -\theta(x)\theta(y)(\Omega(x,y) - \Omega(y,x))(\lambda\cup\mu)\end{aligned}$$

which implies relation (e).

Next, let us put $z = y^{-1} - x^{-1}$. The following relation results from a direct computation:

$$\begin{aligned}&(z^3 - z(s_2\otimes 1 + 1\otimes s_2)/2 - s_1\Delta)(1 - \Delta\Omega(x,y)) \\ &\qquad = (z^3 - z(s_2\otimes 1 + 1\otimes s_2)/2 + s_1\Delta)(1 - \Delta\Omega(y,x)).\end{aligned}$$

Unpacking the generating series and using Lemma 4.2, we see that the relation (f) holds when evaluated at 1 (hence it holds in general).

Let's finally turn to (g). Similarly to Lemma 4.2, we have

$$L_{\lambda_1}(x_1)L_{\lambda_2}(x_2)L_{\lambda_3}(x_3)(1) = m(\theta(x_1)\otimes\theta(x_2)\otimes\theta(x_3))\prod_{i<j}(1 - \Delta\Omega(x_i,x_j))_{ij}(\lambda_1\otimes\lambda_2\otimes\lambda_3).$$

Note that the product of Ω functions is well-defined as a Laurent series. Using this formula and expanding the products, one can show that

$$[L_{\lambda_1}(x_1), [L_{\lambda_2}(x_2), L_{\lambda_3}(x_3)]](1) = m(\theta(x_1)\otimes\theta(x_2)\otimes\theta(x_3))\Delta_{123}K(x_1,x_2,x_3)(\lambda_1\otimes\lambda_2\otimes\lambda_3),$$

where $\Delta_{123} = \Delta_{12}\Delta_{23} = \Delta_{12}\Delta_{13} = \Delta_{13}\Delta_{23}$, and

$$K(x_1,x_2,x_3) = (1-\sigma_{23})(1+\sigma_{13})\Gamma(x_1,x_2,x_3)$$

where we have set

$$\begin{aligned}\Gamma(x_1,x_2,x_3) = \Big(&\Omega(x_1,x_2)\Omega(x_2,x_3) + \Omega(x_1,x_2)\Omega(x_1,x_3) + \Omega(x_1,x_3)\Omega(x_2,x_3) \\ &- t_1t_2\Omega(x_1,x_2)\Omega(x_2,x_3)\Omega(x_1,x_3)\Big)\end{aligned}$$

and where σ_{ij} stands for the transposition of the indices (i,j). In order to prove the relation (g), we need to show that the following expression vanishes, as a (Laurent) series:

$$\sum_{w\in S_3} w[L_{\lambda_1}(x_1), [L_{\lambda_2}(x_2), x_3^{-1}L_{\lambda_3}(x_3)]](1) = (\theta(x_1)\otimes\theta(x_2)\otimes\theta(x_3))\Delta_{123}K'(x_1,x_2,x_3)(\lambda_1\otimes\lambda_2\otimes\lambda_3),$$

where $K'(x_1,x_2,x_3) = \sum_{w\in S_3} w(x_3^{-1}K(x_1,x_2,x_3))$. We will show that K' vanishes. Using the observation that

$$\begin{aligned}\sum_{w\in S_3} wx_3^{-1}(1-\sigma_{23})(1+\sigma_{13}) &= \sum_w w(x_3^{-1} - \sigma_{23}x_2^{-1} + \sigma_{13}x_1^{-1} - \sigma_{23}\sigma_{13}x_2^{-1}) \\ &= \sum_w w(x_1^{-1} - 2x_2^{-1} + x_3^{-1})\end{aligned}$$

we obtain

$$K'(x_1,x_2,x_3) = \sum_{w\in S_3} w\Big((x_1^{-1} - 2x_2^{-1} + x_3^{-1})\Gamma(x_1,x_2,x_3)\Big).$$

We verify by a direct computation that

$$\begin{aligned}&(x_1^{-1} - 2x_2^{-1} + x_3^{-1})(\Omega(x_1,x_2)\Omega(x_2,x_3) - t_1t_2\Omega(x_1,x_2)\Omega(x_2,x_3)\Omega(x_1,x_3)) \\ &\qquad = (x_3^{-1} - x_1^{-1})(\Omega(x_1,x_2)\Omega(x_1,x_3) - \Omega(x_1,x_3)\Omega(x_2,x_3)).\end{aligned}$$

Therefore, we can express K' as follows:

$$K'(x_1,x_2,x_3)=2\sum_{w\in S_3} w\left((x_3^{-1}-x_2^{-1})\Omega(x_1,x_2)\Omega(x_1,x_3)+(x_1^{-1}-x_2^{-1})\Omega(x_1,x_3)\Omega(x_2,x_3)\right).$$

Note that the first term in parentheses is antisymmetric is x_2 and x_3, and the second one is antisymmetric in x_1 and x_2. Therefore their respective symmetrizations vanish, and we obtain $K'=0$. Thus the relation (g) holds, and the proof is complete. Note that we considered in the above calculations elements $L_n(\lambda)$ with $n<0$, but the only relations which we are interested in are those involving only the generators $L_n(\lambda)$ for non negative values of n. Proposition 4.1 is proved. □

4.2. **Level r Fock space representation of $W^{(\mathbf{c})}$.** Let us fix an integer $r\geqslant 0$, set

$$\Lambda(S)_r=\Lambda(S)_{|\mathbf{r}=r},\qquad {}_rL^{\pm}_n(\sigma)=L_{n+1\mp r}(\sigma),\qquad (n\in\mathbb{Z},\sigma\in H^*(S,\mathbb{Q}))$$

and consider the normalized currents

$${}_rL^{\pm}_{\sigma}(z)=z^{-1\pm r}L_{\sigma}(z)\in \operatorname{End}(\Lambda(S)_{|r})[z,z^{-1}].$$

We recover the currents $L^{\pm}(\sigma)$ of the previous section for $r=1$.

Proposition 4.3. *The following relation holds in*

$$\left[{}_rL^+_{\lambda}(x),{}_rL^-_{\mu}(y)\right]_{++}=\left[\frac{1}{c_1x}\left(1-\frac{\theta(x)}{\widetilde{\theta}(x)}\right)\delta\Big(\frac{y}{x}\Big)(\lambda\mu)\right]_{++}\in \operatorname{End}(\Lambda(S)_{|r})[[x,y]]. \tag{4.2}$$

Proof. We begin by evaluating the l.h.s. of (4.2) on the element 1.

Lemma 4.4. *We have*

$$\left[{}_rL^+_{\lambda}(x),{}_rL^-_{\mu}(y)\right]_{++}\cdot 1=\left[\frac{1}{c_1x}\left(1-\frac{\theta(x)}{\widetilde{\theta}(x)}\right)\delta\Big(\frac{y}{x}\Big)(\lambda\mu)\right]_{++}.$$

Proof. The proof bears some resemblance to that of Lemma 4.2. We have

$$L^-_{\mu}(y)\cdot 1=\sum_l y^lQ^-(\mu u^l)=\sum_l(-y)^l\tilde{e}_l(\mu)=\widetilde{\theta}(y)^{-1}(\mu)=\exp\Big(-\sum_{k\geqslant 1}\frac{y^k}{k}\tilde{p}_k\Big)(\mu).$$

A computation using (2.23) yields

$$\begin{aligned}L^+_{\lambda}(x)L^-_{\mu}(y)\cdot 1&=\int_{S\times S}Q^+\left(\sum_{n\in\mathbb{Z}}x^nu^n\widetilde{\theta}(y)^{-1}_{13}\exp\Big(\sum_k\frac{(-y)^k}{k}f_k(-u)\Delta\Big)_{23}\lambda_2\mu_3\right)\\&=\int_{S\times S}Q^+\left(\sum_{n\in\mathbb{Z}}x^nu^n\widetilde{\theta}(y)^{-1}_{13}\exp\Big(\sum_k\frac{(-y)^k}{k}f_k(-x^{-1})\Delta\Big)_{23}\lambda_2\mu_3\right).\end{aligned}$$

Here we have used the classical result that for any Laurent series $F(v)\in A((v))$ with coefficients in a ring A we have $\delta(vw)A(v)=\delta(vw)A(w^{-1})$, where $\delta(u)=\sum_{n\in\mathbb{Z}}u^n$.

One computes

$$\exp\Big(\sum_{k\geqslant 1}\frac{a^k}{k}\Delta f_k(b)\Big)=1+\frac{\Delta a^2}{(1-ab)(1-a(b-c_1))}=:\overline{\Omega}(a,b).$$

Using this, one obtains

$$L^+_{\lambda}(x)L^-_{\mu}(y)\cdot 1=m(\theta(x)\otimes\widetilde{\theta}(y)^{-1})\overline{\Omega}(-y,-x^{-1})(\lambda\otimes\mu) \tag{4.3}$$

and likewise

$$L^-_\mu(y)L^+_\lambda(x)\cdot 1 = m(\theta(x)\otimes\widetilde{\theta}(y)^{-1})\overline{\Omega}(x,y^{-1})(\lambda\otimes\mu). \tag{4.4}$$

Observe that $\overline{\Omega}(-y,-x^{-1}) = \overline{\Omega}(x,y^{-1})$ as rational functions but (4.3) and (4.4) should be expanded out in $\Lambda(S)((x))[[y]]$ and $\Lambda(S)((y))[[x]]$ respectively since by construction $L^{\pm}_n(\sigma)\cdot 1 = 0$ for $n<0$. In other words, we have

$$\left[{}_rL^+_\lambda(x), {}_rL^-_\mu(y)\right]\cdot 1 = x^{r-1}y^{-1-r}m(\theta(x)\otimes\widetilde{\theta}(y)^{-1})\left(\overline{\Omega}(x,y^{-1})_+ - \overline{\Omega}(x,y^{-1})_-\right)(\lambda\otimes\mu)$$

where $+$ and $-$ subscript indicate expansion in $\Lambda(S)((x))[[y]]$ and $\Lambda(S)((y))[[x]]$ respectively. From the formal equality

$$\overline{\Omega}(x,y^{-1}) = 1+\frac{\Delta}{c_1}\left(\frac{1}{x^{-1}-y^{-1}} - \frac{1}{x^{-1}-(y^{-1}-c_1)}\right)$$

we deduce

$$(xy)^{-1}\left(\overline{\Omega}(x,y^{-1})_+ - \overline{\Omega}(x,y^{-1})_-\right) = \frac{\Delta}{c_1x}\left(\delta\left(\frac{x(1-c_1y)}{y}\right) - \delta\left(\frac{y}{x}\right)\right).$$

Thus we get

$$\left[{}_rL^+_\lambda(x), {}_rL^-_\mu(y)\right]\cdot 1 = \int_{S\times S}\frac{\Delta_{23}}{c_1x}\big(A(x,y)+B(x,y)\big)\lambda_2\mu_3 \tag{4.5}$$

where

$$\begin{aligned}
A(x,y) &= \Big(\frac{x}{y}\Big)^r m(\theta(x)\otimes\widetilde{\theta}(y)^{-1})\,\delta\Big(\frac{x(1-c_1y)}{y}\Big)\\
&= \Big(\frac{x(1-c_1y)}{y}\Big)^r m\Big(\theta(x)\otimes\theta\Big(\frac{y}{1-c_1y}\Big)^{-1}\Big)\,\delta\Big(\frac{x(1-c_1y)}{y}\Big),\\
&= m(\theta(x)\otimes\theta(x)^{-1})\,\delta\Big(\frac{x(1-c_1y)}{y}\Big)\\
B(x,y) &= -\Big(\frac{x}{y}\Big)^r m(\theta(x)\otimes\widetilde{\theta}(y)^{-1})\,\delta\Big(\frac{y}{x}\Big) = -m\Big(\theta(x)\otimes\widetilde{\theta}(x)^{-1}\Big)\,\delta\Big(\frac{y}{x}\Big).
\end{aligned}$$

In simplifying $A(x,y)$ we used the following calculation

$$\begin{aligned}
\widetilde{\theta}(y)(\lambda) &= \int_S \tau_{c_1}\exp\left(\sum_{k\geqslant 1}\frac{y^k}{k}p_k\right)\lambda = \int_S\exp\left(\sum_{k\geqslant 1}\frac{y^k}{k}\sum_{i=0}^k\binom{k}{i}p_ic_1^{k-i}\right)\lambda\\
&= \int_S\exp\left(r\sum_{k\geqslant 1}\frac{(c_1y)^k}{k}\right)\exp\left(\sum_{k\geqslant i\geqslant 1}\binom{k-1}{i-1}\frac{y^k}{i}p_ic_1^{k-i}\right)\lambda\\
&= (1-c_1y)^{-r}\int_S\exp\left(\sum_{i\geqslant 1}\frac{y^i}{i(1-c_1y)^i}p_i\right)\lambda\\
&= (1-c_1y)^{-r}\theta\left(\frac{y}{1-c_1y}\right)(\lambda).
\end{aligned}$$

Substituting in (4.5) and observing that

$$\left[\int_{S\times S}\frac{\Delta}{c_1x}A(x,y)\lambda_2\mu_3\right]_{++} = \left[\int_{S\times S}\frac{\Delta}{c_1x}\delta\Big(\frac{x(1-c_1y)}{y}\Big)\lambda_2\mu_3\right]_{++} = 0$$

and

$$\Big[\int_{S\times S}\frac{\Delta}{c_1 x}\delta\Big(\frac{y}{x}\Big)\lambda_2\mu_3\Big]_{++}=0$$

we easily deduce Lemma 4.4. □

In order to extend Lemma 4.4 to the whole of $\Lambda(S)_r$, we next consider the commutation relation between $C_{ij}(\lambda,\mu) := [{}_rL_i^+(\lambda), {}_rL_j^-(\mu)]$ and $\psi_k(\nu)$. Below, we drop the index r for simplicity.

$$\begin{aligned}\psi_k(\nu)C_{ij}(\lambda,\mu) &= \psi_k(\nu)L_i^+(\lambda)L_j^-(\mu) - \psi_k(\nu)L_j^-(\mu)L_i^+(\lambda)\\ &= \Big(L_i^+(\lambda)L_j^-(\mu)\psi_k(\nu) - kL_i^+(\lambda)L_{j+k-1}^-(\nu\mu) + kL_{i+k-1}^+(\lambda\nu)L_j^-(\mu)\Big)\\ &\quad - \Big(L_j^-(\mu)L_i^+(\lambda)\psi_k(\nu) + kL_j^-(\mu)L_{i+k-1}^+(\lambda\nu) - kL_{j+k-1}^-(\nu\mu)L_i^+(\lambda)\Big)\\ &= C_{ij}(\lambda,\mu)\psi_k(\nu) + k\left(C_{i+k-1,j}(\lambda\nu,\mu) - C_{i,j+k-1}(\lambda,\nu\mu)\right)\end{aligned}$$

We have therefore obtained

$$[\psi_k(\nu)/k, C_{ij}(\lambda,\mu)] = C_{i+k-1,j}(\lambda\nu,\mu) - C_{i,j+k-1}(\lambda,\nu\mu). \tag{4.6}$$

Applying (4.6) to 1 and using the fact that, thanks to Lemma 4.4, $C_{ij}(\lambda,\mu)\cdot 1$ only depends on $i+j$ and $\lambda\mu$ we see that

$$[\psi_k(\nu)/k, C_{ij}(\lambda,\mu)]\cdot 1 = 0. \tag{4.7}$$

Using (4.6) and (4.7) recursively, one gets that

$$[C_{ij}(\lambda,\mu), \psi_{k_1}(\nu_1)\cdots\psi_{k_l}(\nu_l)]\cdot 1 = 0$$

for any $(k_1,\nu_1),\ldots,(k_l,\nu_l)$. This proves Proposition 4.3. □

Unraveling formula (4.2), we obtain

$$\Big[{}_rL_i^+(\lambda), {}_rL_{n-i}^-(\mu)\Big] = \sum_{0\leqslant j\leqslant k\leqslant n}(-1)^k\binom{r-k+j}{j+1}h_{n-k}e_{k-j}(c_1^j\lambda\mu) \tag{4.8}$$

which highlights the dependence on r.

Remark 4.5. Note that the "same sign" commutators $[L_i^+(\lambda), L_j^+(\mu)]$ are independent of $i-j$ only in the non-deformed case, see Remark 3.4.

4.3. **Fock space representations of $W^{(r)}(S)$.**

Definition 4.6. The level r Fock space associated to S is the graded vector space

$$\mathbf{F}^{(r)}(S) := \Lambda(S)_{|\mathbf{r}=r}\otimes\mathbb{C}[s,s^{-1}].$$

We may restate Propositions 4.1 and 4.3 as follows:

Corollary 4.7. *The assignment*

$$\psi_n(\lambda)\mapsto \textit{mult. by } \psi_n(\lambda), \qquad T_n^+(\lambda) = {}_rL_n^+(\lambda)s, \qquad T_n^-(\lambda) = (-1)^{r+1}{}_rL_n(\lambda)s^{-1}$$

defines a graded $W^{(r)}(S)$-module structure on $\mathbf{F}^{(r)}(S)$. □

Proposition 4.8. *Assume $r>0$. Then the action of $W^{(r)}(S)$ on $\mathbf{F}^{(r)}(S)$ is faithful.*

Proof. Recall that $Z(S)$ denotes the subalgebra of $W^{(r)}(S)$ generated by $\psi_0(\lambda)$ for $\lambda \in H^*(S,\mathbb{Q})$. Consider the subspace $Z(S)\cdot \mathbf{F}^{(r)}(S) \subset \mathbf{F}^{(r)}(S)$. Since $Z(S)$ lies in the center of $W^{(r)}(S)$, this is a $W^{(r)}(S)$-submodule. Moreover, the action of $W^{(r)}(S)$ on $\mathbf{F}^{(r)}_{\mathrm{red}}(S) := \mathbf{F}^{(r)}(S)/Z(S)\mathbf{F}^{(r)}(S)$ factors through $W^{(r)}_{\mathrm{red}}(S)$ by definition, and we have isomorphisms of vector spaces

$$\mathbf{F}^{(r)}_{\mathrm{red}}(S) \simeq \mathrm{Sym}(H^*(S,\mathbb{Q})\otimes t^2\mathbb{Q}[t]), \qquad \mathbf{F}^{(r)}(S) \simeq \Lambda^1(S)\otimes \mathbf{F}^{(r)}_{\mathrm{red}}(S),$$

where $\Lambda^1(S) = \mathrm{Sym}(H^{\geqslant 2}(S,\mathbb{Q})\otimes t)$. Since $\Lambda^1(S)$ is the regular $Z(S)$-module, it suffices to prove that the action of $W^{(r)}_{\mathrm{red}}(S)$ on $\mathbf{F}^{(r)}_{\mathrm{red}}(S)$ is faithful. Using Lemma 3.17, we only need to prove the faithfulness of its restriction to $U(\mathfrak{h}_S)$. The central charge being non-zero, this last statement follows from a standard argument. $\square$

5. Deformed W-algebras (open surfaces)

In this section we do not assume that S is proper anymore, and define several versions of W-algebras, modeled on the cohomology resp. cohomology with compact support of S. Throughout, we fix a smooth compactification $\iota : S \to \overline{S}$. The W-algebras which we consider will end up being independent of this choice of compactification.

5.1. **Positive halves.** We begin with a general construction. Consider a graded ideal $I \subset H^*(\overline{S},\mathbb{Q})$. Note that $\Delta(I) \subset I \otimes I$ hence $I^{\perp}$ is also an ideal. We denote by J the quotient of $H^*(\overline{S},\mathbb{Q})$ by I.

Definition 5.1. Let $W^+_{\downarrow}(I)$ be the smallest graded subalgebra of $W^+(\overline{S})$ containing $D_{n,0}(\lambda)$ for all $n\geqslant 0$, $\lambda \in I$, and stable under operators $\mathrm{Ad}(\psi_l(\mu))$, for all $l > 0$ and $\mu \in H^*(\overline{S},\mathbb{Q})$. Likewise, let $W^+_{\uparrow}(J)$ be the quotient of $W^+(\overline{S})$ by the two-sided ideal $\mathcal{I}^+$ generated by $W^+_{\downarrow}(I)$. We define $W^-_{\downarrow}(I), W^-_{\uparrow}(J)$ in the same way. Finally, we let $W^0(J)$ be the quotient of $W^0(\overline{S})$ by the ideal generated by elements $\psi_l(\lambda)$ for $l > 0$ and $\lambda \in I$ (thus $\mathbf{c}$ descends to a non zero element of $W^0(J)$).

Remark 5.2.
(i) $W^+_{\downarrow}(I)$ is in general different from the subalgebra of $W^+(\overline{S})$ generated by $T^+_n(\lambda)$ with $n\geqslant 0$ and $\lambda \in I$. For instance, let $S = \mathbb{P}^2$ and $I = \mathbb{Q}[pt]$; then $W^+_{\downarrow}(I)$ is a commutative algebra with basis given by monomials in $D_{m,n}([\mathrm{pt}])$, which is not generated by $\{D_{1,n}([\mathrm{pt}])\}_n$.
(ii) However, it is easy to see that $\mathcal{I}^+$ is generated *as an ideal* by $T^+_n(\lambda)$ with $n\geqslant 0$ and $\lambda \in I$.

Recall the elements $D_{m,n}(\lambda) \in W^+(\overline{S})$ considered in Proposition 3.8 in connection to the order filtration $F_{\bullet}$. They are not canonically defined unless $m\leqslant 1$ or $n\leqslant 1$; in this subsection, we fix them to be

$$D_{m,n}(\lambda) := \frac{1}{m(n+1)}[\psi_{n+1}(1), D_{m,0}(\lambda)], \qquad \lambda \in H^*(\overline{S},\mathbb{Q}), \quad m\geqslant 1,\ n\geqslant 1.$$

By construction, the elements $D_{m,n}(\lambda)$ belong to $W^+_{\downarrow}(I)$ if $\lambda \in I$.

Our first result concerns the size of the algebras $W^+_{\downarrow}(I), W^+_{\uparrow}(J)$.

Proposition 5.3. *The Hilbert series of $W^+_{\downarrow}(I)$ and $W^+_{\uparrow}(J)$ are respectively equal to*

$$P_{W^+_{\downarrow}(I)}(z,w) = \mathrm{Exp}\left(\frac{P_I(z)z^{-2}w}{(1-z^2)(1-w)}\right), \qquad P_{W^+_{\uparrow}(J)}(z,w) = \mathrm{Exp}\left(\frac{P_J(z)z^{-2}w}{(1-z^2)(1-w)}\right),$$

where $P_I(z) = \sum_d \dim(I\cap H^d(\overline{S},\mathbb{Q}))z^d$ and $P_J(z) = P_{\overline{S}}(z) - P_I(z)$.

Proof. We begin with the first statement. Let $A \subset W^+(\overline{S})$ be the subalgebra generated by all Lie words in $T_{n_1}(\lambda_1), \ldots, T_{n_s}(\lambda_s)$ for which $\prod_i \lambda_i \in I$. From the inductive definition of $D_{m,0}(\lambda)$ we see that $W^+_\downarrow(I) \subset A$. We will later show that this is in fact an equality. Let $\widetilde{W}^+(\overline{S})$ be the free algebra generated by elements $\{\tilde{T}_n(\lambda)\ ;\ \lambda \in H^*(\overline{S}, \mathbb{Q}), n \geqslant 0\}$, modulo the sole relations $\tilde{T}_n(a\lambda + b\mu) = a\tilde{T}_n(\lambda) + b\tilde{T}_n(\mu)$. There is a canonical morphism $\pi : \widetilde{W}^+(\overline{S}) \to W^+(\overline{S})$ whose kernel R is generated by the collection of relations (a)-(g) of § 3. The order filtration $F_\bullet$ on $W^+(\overline{S})$ lifts to a filtration $\widetilde{F}_\bullet$ on $\widetilde{W}^+(\overline{S})$, in which a Lie word $L = L(\tilde{T}_{n_1}(\lambda_1), \ldots, \tilde{T}_{n_s}(\lambda_s))$ belongs to $\widetilde{F}_{o(L)}$ with $o(L) = 1 - s + \sum_i n_i$. Let $\widetilde{A} \subset \widetilde{W}^+(\overline{S})$ be the subalgebra generated by Lie words as above for which $\prod_i \lambda_i \in I$. Thus $\pi(\widetilde{A}) = A$. We claim that

$$\pi(\widetilde{F}_n \cap \widetilde{A}) = F_n \cap A \tag{5.1}$$

which is equivalent to the equality

$$(\widetilde{F}_n \cap \widetilde{A}) + R = (\widetilde{A} + R) \cap \widetilde{F}_n + R \tag{5.2}$$

The only relations which do not preserve the order are those of type (f). Observe that if the symbol (with respect to $\widetilde{F}_\bullet$) of such a relation belongs to $\widetilde{A}$ then so does in fact the relation itself (indeed, if $\lambda\mu \in I$ then $s_2\lambda\mu \in I$ and $c_1\Delta\lambda\mu \in I \otimes I$). Equations (5.2) and (5.1) follow.

From Lemma 3.21 we have $\mathrm{Gr}_\bullet W^+(\overline{S}) \simeq \mathrm{Sym}(\mathfrak{w}_0^+(\overline{S}))$ and there is a Lie algebra morphism $\mathfrak{w}_0^+(\overline{S}) \to \mathrm{Gr}_\bullet W^+(\overline{S})$. Since the symbol of a Lie word $L = L(T_{n_1}(\lambda_1), \ldots, T_{n_s}(\lambda_s))$ is a multiple of $D_{s,o(L)}(\prod_i \lambda_i)$, we deduce from (5.1) that

$$\mathrm{Gr}_n(A) = \mathrm{Span}\Big\{ D_{m_1,n_1}(\lambda_1) \cdots D_{m_s,n_s}(\lambda_s)\ ;\ \sum_i n_i \leqslant n, \lambda_1, \ldots, \lambda_s \in I \Big\} \subset F_n/F_{n-1}. \tag{5.3}$$

Because $D_{m,n}(\lambda)$ belongs to $W^+_\downarrow(I)$ if $\lambda \in I$ we deduce that $\mathrm{Gr}_\bullet A \subset Gr_\bullet W^+_\downarrow(I)$, from which we deduce that $\mathrm{Gr}_\bullet A = \mathrm{Gr}_\bullet W^+_\downarrow(I)$, hence in fact $A = W^+_\downarrow(I)$. The formula for the Hilbert series of $W^+_\downarrow(I)$ now follows from (5.3).

Let us now turn to the second equality. Let $W^+(J)$ denote the graded algebra defined by generators and relations as in § 3, but with J in place of $H^*(\overline{S}, \mathbb{Q})$. The results of § 3 may be repeated *mutatis mutandis* for $W^+(J)$. In particular, we have

$$P_{W^+(J)}(z, w) = \mathrm{Exp}\left(\frac{P_J(z) z^{-2} w}{(1 - z^2)(1 - w)} \right).$$

There is a canonical surjective morphism $W^+(\overline{S}) \to W^+(J)$, which factors to a surjection $W^+_\uparrow(J) \to W^+(J)$. In particular, $P_{W^+(J)}(z, w) \leqslant P_{W^+_\uparrow}(z, w)$ (coefficientwise). We will show that this map is an isomorphism by proving the reverse inequality. Let $\mathcal{I} \subset W^+(\overline{S})$ be the two-sided ideal generated by $W^+_\downarrow(I)$. Recall that $D_{m,n}(\mu) \in \mathrm{Gr}_\bullet W^+_\downarrow(I)$ for $\mu \in I$ and any $m \geqslant 1, n \geqslant 0$, hence $\mathrm{Sym}(\mathfrak{w}_0^+(I)) \subset \mathrm{Gr}_\bullet \mathcal{I}$, where

$$\mathfrak{w}_0^+(I) = \mathrm{Span}\{ D_{m,n}(\mu)\ ;\ m \geqslant 1, n \geqslant 0,\ \mu \in I \}.$$

We deduce that

$$P_{W^+_\uparrow(J)}(z, w) \leqslant P_{U(\mathfrak{w}_0^+(\overline{S}))}(z, w) / P_{U(\mathfrak{w}_0^+(I))}(z, w) = P_{U(\mathfrak{w}_0^+(J))}(z, w) = P_{W^+(J)}(z, w)$$

which gives the desired reverse inequality. □

In the course of the proof, we obtained the following useful characterization of $W^\pm_\downarrow(I)$:

Corollary 5.4. *For any ideal $I \subset H^*(\overline{S},\mathbb{Q})$, the subalgebra $W^{\pm}_{\downarrow}(I)$ is generated by all Lie words $L(T_{n_1}(\lambda_1),\ldots,T_{n_s}(\lambda_s))$ for which $\prod_i \lambda_i \in I$.* □

By definition, the action of $\mathrm{Ad}_{\psi_l(\mu)}$ preserves $W^{\pm}_{\downarrow}(I)$ for any l and any $\mu \in H^*(\overline{S},\mathbb{Q})$. This induces an action of $W^0(\overline{S})$ on both $W^{\pm}_{\downarrow}(I)$ and on $W^{\pm}_{\uparrow}(J)$.

Lemma 5.5. *The action of $W^0(\overline{S})$ on $W^{\pm}_{\uparrow}(J)$ factors through $W^0(J)$. The action of $W^0(\overline{S})$ on $W^{\pm}_{\downarrow}(I)$ factors through $W^0(H^*(\overline{S},\mathbb{Q})/I^{\perp})$.*

Proof. The first statement is a simple consequence of the fact that $[\psi_l(\mu)), W^{\pm}(\overline{S})] \subset W^{\pm}_{\downarrow}(I)$ if $\mu \in I$, which may be checked on the generators $T_n(\lambda)$. The second statement is a consequence of the following claim: for any $l \geqslant 0$ and $\mu \in I^{\perp}$ we have

$$[\psi_l(\mu)), W^{\pm}_{\downarrow}(I)] = \{0\}. \tag{5.4}$$

We sketch the proof of this claim, leaving the details to the reader. Let $\widetilde{W}^+(\overline{S}), \widetilde{F}_\bullet$ and $\pi : \widetilde{W}^+(\overline{S}) \to W^+(\overline{S})$ be as in the proof of Proposition 5.3, and let $R \subset \widetilde{W}^+(\overline{S})$ be the ideal of relations, $R_n = R \cap \widetilde{F}_n$. For a Lie word $L(\tilde{T}_{n_1}(\lambda_1),\ldots,\tilde{T}_{n_s}(\lambda_s))$ we put $c(L) = \prod_i \lambda_i \in H^*(\overline{S},\mathbb{Q})$. From relations (e), (f) and (g) one checks the following: for any $r \in R_n/R_{n-1}$ which is a linear combination of products of Lie words $L_1,\ldots,L_t$ such that $c(L_1) = \cdots = c(L_t) = \alpha$ there exists a lift $r' \in R_n$ of r which is a linear combination of products of Lie words $L'_1,\ldots,L'_s$ satisfying $c(L'_i) \in \alpha H^*(\overline{S},\mathbb{Q})$ for all i. In particular, for any two Lie words L_1, L_2 for which $c(L_1) = c(L_2) = 0$ and $L_1 - L_2 \in R_n/R_{n-1} \subset \widetilde{F}_n/\widetilde{F}_{n-1}$ we have

$$\pi(L_1 - L_2) \in \mathrm{Span}\Big\{\pi(L'_1\cdots L'_s) \mid \forall\, i,\ c(L'_i) = 0, \sum_i o(L_i) < n\Big\}. \tag{5.5}$$

For any Lie word $L \in \widetilde{W}^+(\overline{S})$, the symbol of $\pi(L)$ with respect to $F_\bullet$ is equal to a multiple of $D_{m,n}(\lambda)$ for some m, n and λ. As $D_{m,n}(0) = 0$, we deduce from (5.5) by induction on the order that for any Lie word L with $c(L) = 0$ we have $\pi(L) = 0$. This yields (5.4) as a particular case. □

We will be mostly interested in the ideals $I_S := H^*_c(S,\mathbb{Q}), I_S^{\perp} \simeq H^*(\overline{S}, S)$ and the quotient $J_S = H^*(S,\mathbb{Q}) \simeq H^*(\overline{S},\mathbb{Q})/I_S^{\perp}$. Recall that S being pure, the maps $\iota_! : H^*_c(S,\mathbb{Q}) \to H^*(\overline{S},\mathbb{Q})$ and $\iota^* : H^*(\overline{S},\mathbb{Q}) \to H^*(S,\mathbb{Q})$ are respectively injective and surjective.

Definition 5.6. We put

$$W^{\pm}_{\downarrow}(S) = W^{\pm}_{\downarrow}(I_S), \qquad W^{\pm}_{\uparrow}(S) = W^{\pm}_{\uparrow}(J_S), \qquad W^0(S) = W^0(J_S) \simeq \Lambda(S),$$
$$W^{\geqslant 0}_{\uparrow}(S) = W^0(S) \ltimes W^+_{\uparrow}(S), \qquad W^{\geqslant 0}_{\downarrow}(S) = W^0(S) \ltimes W^+_{\downarrow}(S),$$

and define $W^{\leqslant 0}_{\uparrow}(S), W^{\leqslant 0}_{\downarrow}(S)$ in the same way.

Composing the inclusion $W^{\pm}_{\downarrow}(S) \subset W^{\pm}(\overline{S})$ with the projection $W^{\pm}(\overline{S}) \to W^{\pm}_{\uparrow}(S)$ yields an algebra homomorphism

$$\varphi_S : W^{\pm}_{\downarrow}(S) \to W^{\pm}_{\uparrow}(S).$$

Corollary 5.7. *The Hilbert series of $W^+_{\downarrow}(S), W^+_{\uparrow}(S)$ are given by*

$$P_{W^+_{\downarrow}(S)}(z,w) = \mathrm{Exp}\left(\frac{P^c_S(z)z^{-2}w}{(1-z^2)(1-w)}\right), \tag{5.6}$$

$$P_{W^+_{\uparrow}(S)}(z,w) = \mathrm{Exp}\left(\frac{P_S(z)z^{-2}w}{(1-z^2)(1-w)}\right) \tag{5.7}$$

where $P_S^c(z) = \sum_n \dim(H_c^n(S,\mathbb{Q}))(-u)^n$ *and* $P_S(z) = \sum_n \dim(H^n(S,\mathbb{Q}))(-u)^n$.

5.2. **Full W-algebras.** There are *four* different types of W-algebras which one can associate to S, namely

$$W^{(\mathbf{c})}_{\uparrow\uparrow}(S), \qquad W^{(\mathbf{c})}_{\uparrow\downarrow}(S), \qquad W^{(\mathbf{c})}_{\downarrow\uparrow}(S), \qquad W^{(\mathbf{c})}_{\downarrow\downarrow}(S).$$

These depend on the choice of compactly supported vs. ordinary homology for each half, and are all naturally defined as subquotients of $W^{(\mathbf{c})}(\overline{S})$. For instance, $W^{(\mathbf{c})}_{\uparrow\uparrow}(S)$ is the quotient of $W^{(\mathbf{c})}(\overline{S})$ by the two-sided ideal generated by $W^{\pm}_{\downarrow}(I_S^{\perp})$ and $W^0(I_S^{\perp})$ while $W^{(\mathbf{c})}_{\uparrow\downarrow}(S)$ is the quotient of the subalgebra of $W^{(\mathbf{c})}(\overline{S})$ generated by $W^{\leqslant}(\overline{S}), W^+_{\downarrow}(I_S)$ by its two-sided ideal generated by $W^0(I_S^{\perp})$ and $W^-_{\downarrow}(I_S^{\perp})$. It is easy to see that $W^{(\mathbf{c})}_{\uparrow\uparrow}(S)$ may be presented just as $W^{(\mathbf{c})}(\overline{S})$, where we replace $H^*(\overline{S},\mathbb{Q})$ by $H^*(S,\mathbb{Q})$ everywhere.

For any $r \in \mathbb{Q}$, we let $W^{(r)}_{\uparrow\uparrow}(S), \ldots$ be the specialization of $W^{(\mathbf{c})}_{\uparrow\uparrow}(S), \ldots$ to $\mathbf{c} = r$.

Proposition 5.8. *For a pair $(a,b) \in \{\uparrow,\downarrow\}^2$, the multiplication map induces an isomorphism*

$$W^{(\mathbf{c})}_{ab}(S) \simeq \begin{cases} W^-_a(S) \otimes W^0(S) \otimes W^+_b(S) & (a,b) \neq (\uparrow,\uparrow), \\ W^-_{\uparrow}(S) \otimes W^0(S)_{|\mathbf{c}=0} \otimes W^+_{\uparrow}(S) & (a,b) = (\uparrow,\uparrow) \end{cases}.$$

Proof. Let $\mathcal{I}^{\geqslant}, \mathcal{I}^{\leqslant}$ be the two-sided ideals of $W^{\geqslant}(\overline{S})$ and $W^{\leqslant}(\overline{S})$ respectively generated by $W^{\pm}_{\downarrow}(I_S^{\perp})$ and $\{\psi_l(\lambda) \mid l > 0, \lambda \in I_S^{\perp}\}$. Arguing as in the proof of Lemma 5.5 and using Proposition 3.12, it is enough to check the following inclusions (and their dual versions):

$$\begin{aligned} \mathcal{I}^{\geqslant} \cdot W^-(\overline{S}) &\subseteq W^-(\overline{S}) \cdot \mathcal{I}^{\geqslant} + \mathbf{c}W^{(\mathbf{c})}(\overline{S}), \\ W^+(\overline{S}) \cdot \mathcal{I}^{\leqslant} &\subseteq \mathcal{I}^{\leqslant} \cdot W^+(\overline{S}) + \mathbf{c}W^{(\mathbf{c})}(\overline{S}), \\ W^+_{\downarrow}(S) \cdot W^-_{\downarrow}(S) &\subseteq W^-_{\downarrow}(S) \cdot W^0(\overline{S}) \cdot W^+_{\downarrow}(S), \\ W^+_{\downarrow}(S) \cdot \mathcal{I}^{\leqslant} &\subseteq \mathcal{I}^{\leqslant} \cdot W^+_{\downarrow}(S) \end{aligned}$$

These are in turn easily verified using (3.18), Corollary 5.4 together with the fact that $\mathcal{I}^{\pm}$ is generated by $T_n^{\pm}(\lambda)$ for $\lambda \in I_S^{\perp}$. Note that $[T_0^-(\lambda), T_0^+(\mu)] = \mathbf{c}$ when $\lambda\mu = [pt]$ so that $\mathbf{c}$ is indeed in the two-sided ideal of $W^{(\mathbf{c})}(\overline{S})$ generated by $W^{\pm}_{\downarrow}(S)$. □

To finish, we mention the following analogue of Theorem 3.14. For a pair $(a,b) \in \{\uparrow,\downarrow\}^2$, we define a Lie algebra $\mathfrak{w}^{(\mathbf{c})}_{ab}(S)$ as follows. It has a basis given by a central element $\mathbf{c}$ and elements $D_{m,n}(\lambda)$ where $m \in \mathbb{Z}$, $n \geqslant 0$, and λ belongs to $H^*(S,\mathbb{Q})$ or $H^*_c(S,\mathbb{Q})$ according to the following rule: for $m = 0$, $\lambda \in H^*(S,\mathbb{Q})$; for $m < 0$, resp. $m > 0$, $\lambda \in H^*(S,\mathbb{Q})$ if a, resp. b is equal to $\uparrow$ and $\lambda \in H^*_c(S,\mathbb{Q})$ otherwise. The Lie bracket is given by

$$[D_{m,n}(\lambda), D_{m',n'}(\mu)] = (nm' - m'n)D_{m+m',n+n'-1}(\lambda\mu),$$

where we have set $D_{m,-1} = \delta_{m,0}\mathbf{c}$, and where the product $\lambda\mu$ is either induced by the cup product on $H^*_{(c)}(S,\mathbb{Q})$ or by the product $H^*(S,\mathbb{Q}) \otimes H^*_c(S,\mathbb{Q}) \to H^*_c(S,\mathbb{Q})$ (or its composition with the natural map $H^*_c(S,\mathbb{Q}) \to H^*(S,\mathbb{Q})$).

Corollary 5.9. *Assume that $s_2 = 0$ and $c_1\Delta = 0$. There is an algebra isomorphism*

$$\Phi : W^{(\mathbf{c})}_{ab}(S) \simeq \begin{cases} U(\mathfrak{w}^{(\mathbf{c})}_{ab}(S)) & (a,b) \neq (\uparrow,\uparrow) \\ U(\mathfrak{w}^{(\mathbf{c})}_{\uparrow\uparrow}(S))_{|\mathbf{c}=0} & (a,b) = (\uparrow,\uparrow) \end{cases}$$

which sends $\psi_n(\lambda)$ to $D_{0,n}(\lambda)$ and $T_n^{\pm}(\lambda)$ to $D_{\pm 1,n}(\lambda)$.

Proof. This follows from the explicit identification $I_S^{\pm} = \mathrm{Span}\{D_{m,n}(\lambda) \mid \pm m > 0, n \geqslant 0, \lambda \in I_S\}$ (and similarly for $I_S^{\perp}$). Note that the condition $c_1(\overline{S}) = 0$ may be relaxed to $c_1(S)\Delta_S = 0$ in this context: indeed, the condition really is $c_1\lambda\mu = 0$ for any λ, μ in $H_c^*(S, \mathbb{Q})$ or $H^*(S, \mathbb{Q})$ (according to the situation). □

Example 5.10. Let $S = \mathbb{A}^2$ together with the natural action of a two-dimensional torus T. In this case the Chern roots t_1, t_2 are precisely the linear characters of T, and we have $\Delta = t_1t_2$. Identifying $H_T^*(\mathbb{A}^2)$ with $\mathbb{Q}[t_1, t_2]$, we have $H_{c,T}^*(\mathbb{A}^2) = t_1t_2\mathbb{Q}[t_1, t_2]$. In particular, the difference between $W_{\uparrow}^{+,T}(\mathbb{A}^2)$ and $W_{\downarrow}^{+,T}(\mathbb{A}^2)$ is just multiplication of generators T_i^+ by t_1t_2, so we will abuse the notation and omit the subscripts.

It is instructive to compare the output of our constructions with the affine Yangian $\ddot{Y}_{t_1,t_2}(\mathfrak{gl}_1)$ as considered in [42] (see also [2]). Recall that it is generated by elements $e_i, f_i, \widetilde{\psi}_i$ with $i \geqslant 0$, modulo certain relations (Y0-Y6), see *loc. cit.* for details. We can define a morphism of algebras $\ddot{Y}_{t_1,t_2}(\mathfrak{gl}_1) \to W_T^{(\mathbf{c})}(\mathbb{A}^2)$ by sending

$$e_i \mapsto T_i^+, \quad f_i \mapsto T_i^-,$$

and $\widetilde{\psi}_i$ maps to the expression on the right hand side of (4.8); in particular $\widetilde{\psi}_i$ is quite different from $\psi_i \in W_T^{(\mathbf{c})}(\mathbb{A}^2)$. The only relations that are not immediately obvious are (Y4-Y5). Since they are almost identical, let us concentrate on (Y4):

$$\begin{aligned}[][\widetilde{\psi}_{i+3}, e_j] - 3[\widetilde{\psi}_{i+2}, e_{j+1}] + 3[\widetilde{\psi}_{i+1}, e_{j+2}] - [\widetilde{\psi}_i, e_{j+3}] + (t_1^2 + t_1t_2 + t_2^2)([\widetilde{\psi}_{i+1}, e_j] - [\widetilde{\psi}_i, e_{j+1}])& \\ + t_1t_2(t_1 + t_2)\{\widetilde{\psi}_i, e_j\} = 0&\end{aligned}$$

The relation (Y1) is precisely the relation (f) in our Definition 3.1. Using the relation $[e_i, f_j] = \widetilde{\psi}_{i+j}$ and applying Ad_{f_i} to (Y1), we can see that proving (Y4) for (i, j) is equivalent to proving it for $(i - 1, j + 1)$. Thus it suffices to check (Y4) for $i = 0$. Using (Y4'), the formula we need to check is

$$[[T_3^+, T_0^-], T_i^+] = 6T_{i+1}^+ + 2\mathbf{c}t_1t_2(t_1 + t_2)T_i^+.$$

Unpacking the right hand side of formula (4.8) for $n = 3$, we can get an explicit formula for $[T_3^+, T_0^-]$ and check the relation above applying (2.20).

Using the triangular decomposition of both sides, it is easy to see that the map above is actually an isomorphism $\ddot{Y}_{t_1,t_2}(\mathfrak{gl}_1) \simeq W_T^{(\mathbf{c})}(\mathbb{A}^2)$.

5.3. **Fock space representations.** In § 4.3, we defined a representation $\mathbf{F}^{(r)}(\overline{S})$ of $W^{(\mathbf{c})}(\overline{S})$. We now consider variants of $\mathbf{F}^{(r)}(\overline{S})$ in the case of open surfaces.

First observe that we have a canonical projection $\iota^* : \mathbf{F}^{(r)}(\overline{S}) \to \mathbf{F}^{(r)}(S)$, induced by the surjection $H^*(\overline{S}, \mathbb{Q}) \to H^*(S, \mathbb{Q})$. The kernel of ι^* is generated by the tautological classes $\psi_n(\lambda)$ with $\lambda \in I_S^{\perp}$. The formulas (4.1) and the definition of the operators $R^{\pm}, Q^{\pm}$ imply that $\ker(\iota^*)$ is a $W^{(r)}(\overline{S})$-submodule of $\mathbf{F}^{(r)}(S)$ and that dually $T_n^{\pm}(\lambda)\mathbf{F}^{(r)}(\overline{S}) \subset \ker(\iota^*)$ for any $\lambda \in I_S^{\perp}$. From these observations, one deduces the following:

Proposition 5.11. *The action of $W^{(r)}(\overline{S})$ on the Fock space $\mathbf{F}^{(r)}(\overline{S})$ descends/restricts to an action of $W_{\uparrow\downarrow}^{(r)}(S)$, $W_{\downarrow\uparrow}^{(r)}(S)$, $W_{\downarrow\downarrow}^{(r)}(S)$ on $\mathbf{F}^{(r)}(S)$ for any r, and to an action of $W_{\uparrow\uparrow}^{(0)}(S)$ on $\mathbf{F}^{(0)}(S)$. These representations are faithful when $r \neq 0$.*

Proof. Let us only address the faithfulness. We only need to consider $W_{\uparrow\downarrow}^{(r)}(S)$, $W_{\downarrow\uparrow}^{(r)}(S)$, since $W_{\downarrow\downarrow}^{(r)}(S)$ is a subalgebra of $W^{(r)}(\overline{S})$. Similarly to Proposition 3.15, these algebras contain a copy of the Heisenberg algebra $\mathfrak{h}_S$ of central charge r, where half of the generators are labeled by $H_c^*(S,\mathbb{Q})$. Since $r>0$, we conclude by Lemma 3.17. $\square$

6. Hecke patterns

In this section we introduce a general framework to construct modules over the algebras $\mathbf{H}_0(S)$ and $\mathbf{H}_0^c(S)$ by using a compactification $\overline{S}$ of S and by restricting the multiplication in $\mathbf{H}(\overline{S})$ to suitable substacks. For this we fix a smooth compactification $\overline{S}$ of S.

6.1. **Hecke patterns and Hecke correspondences.** The following is a variation on the notion of a Hecke pattern which appears in [22, §5].

Definition 6.1. A *(two-sided) Hecke pattern* on $\overline{S}$ is a locally closed derived substack $X=\bigsqcup_\alpha X_\alpha$ of $\mathfrak{Coh}^{\geqslant 1}(\overline{S})$ satisfying the following properties:

(a) for any short exact sequence $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$ with $\mathcal{G}\in\mathfrak{Coh}^0(\overline{S})$ and $\mathcal{F}\in X$ we have $\mathcal{E}\in X$,

(b) for any short exact sequence $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$ with $\mathcal{G}\in\mathfrak{Coh}^0(\overline{S})$, $\mathcal{E}\in X$ and $\mathcal{F}\in\mathfrak{Coh}^{\geqslant 1}(\overline{S})$ we have $\mathcal{F}\in X$.

We call X an S-*weak* Hecke pattern if the conditions (a-b) only hold for $\mathcal{G}\in\mathfrak{Coh}^0(S)$. We say that a Hecke pattern is S-*strong* if both conditions (a-b) imply that $\mathcal{G}\in\mathfrak{Coh}^0(S)$.

We say that X is of rank r if X_α is nonempty only if $\mathrm{rk}(\alpha)=r$; every Hecke pattern is clearly a disjoint union of Hecke patterns of a well defined rank. Note that the conditions of being a usual/S-strong/S-weak Hecke pattern imply the conditions (2.9,2.16), (2.25), (2.26) respectively.

Remark 6.2. A substack $X\subset\mathfrak{Coh}^{\geqslant 1}(\overline{S})$ satisfying (a) alone may be called a *left* Hecke pattern, while a substack $X\subset\mathfrak{Coh}^{\geqslant 1}(\overline{S})$ satisfying (b) may be called a *right* Hecke pattern. We can consider the S-weak/strong versions of these notions separately.

Example 6.3. The property of being of dimension $\geqslant d$, $d=1,2$ is stable by passing to a subsheaf, and for any extension $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$ with $\dim\mathcal{G}=0$, $\mathcal{E}\in\mathfrak{Coh}^{\geqslant d}$ and $\mathcal{F}\in\mathfrak{Coh}^{\geqslant 1}$ we have $\mathcal{F}\in\mathfrak{Coh}^{\geqslant d}$. Therefore $\mathfrak{Coh}^{\geqslant d}(\overline{S})$, $d=1,2$ is a Hecke pattern. One can similarly see that $\mathfrak{Coh}^{\geqslant 1}(S)$ is an S-strong Hecke pattern. More generally, any S-weak Hecke pattern contained in $\mathfrak{Coh}^{\geqslant 1}(S)$ is automatically S-strong. The collection of Hilbert schemes of points on S (see § 7) is a left S-weak and right S-strong Hecke pattern.

Lemma 6.4. *A left Hecke pattern is S-strong if and only if $X\subset\mathfrak{Coh}^{\geqslant 1}(S)$. A right Hecke pattern is S-strong if and only if every sheaf in X is locally free at any point in $\overline{S}\setminus S$.*

Proof. The first claim follows from the fact that for any $x\in S$, any sheaf $\mathcal{E}$ with $\mathcal{E}_x\neq 0$ admits a surjection to $\mathcal{O}_x$. For the second claim, let $\mathcal{E}_1\subset\mathcal{E}$ be the maximal 1-dimensional subsheaf. Since $\mathrm{Ext}^1(\mathcal{O}_x,\mathcal{E}_1)\to\mathrm{Ext}^1(\mathcal{O}_x,\mathcal{E})$ is injective, $\mathcal{E}_1$ has to be supported away from $\overline{S}\setminus S$. We conclude by observing that the cokernel of the double dual map $\mathcal{E}|_{\overline{S}\setminus\mathrm{supp}(\mathcal{E}_1)}\hookrightarrow(\mathcal{E}|_{\overline{S}\setminus\mathrm{supp}(\mathcal{E}_1)})^{\vee\vee}$ cannot have support in $\overline{S}\setminus S$ either. $\square$

An S-weak two-sided Hecke pattern X yields two families of induction diagrams. We define

$$\widetilde{\mathfrak{Coh}}^{\circ}_{n\delta;\alpha}=\widetilde{\mathfrak{Coh}}_{n\delta;\alpha}(\overline{S})\underset{\mathfrak{Coh}_{n\delta}(\overline{S})}{\times}\mathfrak{Coh}_{n\delta}\quad,\quad\widetilde{X}_{n\delta;\alpha}=\widetilde{\mathfrak{Coh}}^{\circ}_{n\delta;\alpha}\underset{\mathfrak{Coh}_{n\delta+\alpha}(\overline{S})}{\times}X_{n\delta+\alpha}.$$

Conditions (a) and (b) yield the following diagrams with Cartesian squares

$$
\begin{tikzcd}
\mathfrak{Coh}_{n\delta} \times \mathfrak{Coh}_{\alpha}(\overline{S}) & \widetilde{\mathfrak{Coh}}^{\circ}_{n\delta;\alpha} \arrow[l, "q^{\circ}_{n\delta,\alpha}"'] \arrow[r, "p^{\circ}_{n\delta,\alpha}"] & \mathfrak{Coh}_{\alpha+n\delta}(\overline{S}) \\
\mathfrak{Coh}_{n\delta} \times X_{\alpha} \arrow[u] & \widetilde{X}_{n\delta;\alpha} \arrow[l, "\kappa_{n\delta,\alpha}"'] \arrow[r, "\pi_{n\delta,\alpha}"] \arrow[u] & X_{\alpha+n\delta} \arrow[u] \\
\mathfrak{Coh}^{\geqslant 1}_{\alpha}(\overline{S}) \times \mathfrak{Coh}_{n\delta} & \widetilde{\mathfrak{Coh}}^{\circ,\geqslant 1}_{n\delta;\alpha-n\delta} \arrow[l, "\overline{q}^{\circ}_{\alpha,n\delta}"'] \arrow[r, "\overline{p}^{\circ}_{\alpha,n\delta}"] & \mathfrak{Coh}^{\geqslant 1}_{\alpha-n\delta}(\overline{S}) \\
X_{\alpha} \times \mathfrak{Coh}_{n\delta} \arrow[u] & \widetilde{X}_{n\delta;\alpha-n\delta} \arrow[l, "\overline{\kappa}_{\alpha,n\delta}"'] \arrow[r, "\overline{\pi}_{\alpha,n\delta}"] \arrow[u] & X_{\alpha-n\delta} \arrow[u]
\end{tikzcd}
\tag{6.1}
$$

The maps $\pi_{n\delta,\alpha}$ and $\overline{\pi}_{\alpha,n\delta}$ may not be proper. Luckily, they factorise as

$$
\widetilde{X}_{n\delta;\alpha} \xrightarrow{\pi'_{n\delta,\alpha}} X_{\alpha+n\delta} \times \mathrm{Sym}^n(S) \xrightarrow{\pi''_{n\delta,\alpha}} X_{\alpha+n\delta},
$$
$$
\widetilde{X}_{n\delta;\alpha-n\delta} \xrightarrow{\overline{\pi}'_{\alpha,n\delta}} X_{\alpha-n\delta} \times \mathrm{Sym}^n(S) \xrightarrow{\overline{\pi}''_{\alpha,n\delta}} X_{\alpha-n\delta}.
$$

Lemma 6.5. *The following hold:*

(a) *the morphism $\kappa_{n\delta,\alpha}$ and $\overline{\kappa}_{\alpha,n\delta}$ are quasi-smooth,*
(b) *the morphisms $\pi'_{n\delta,\alpha}$ and $\overline{\pi}'_{\alpha,n\delta}$ are proper and representable.*

Proof. Quasi-smoothness is preserved by derived base change. Hence $\kappa_{n\delta,\alpha}$ is a quasi-smooth morphism. Next, by construction $\widetilde{\mathfrak{Coh}}^{\geqslant 1}_{n\delta;\alpha-n\delta}$ is open in $\mathbb{V}(\mathrm{RHom}(\mathcal{E}_{\alpha}, \mathcal{E}_{n\delta}))$ and as the sheaves over S parametrized by $\mathfrak{Coh}^{\geqslant 1}$ have no zero-dimensional subsheaves,

$$
\mathrm{Ext}^2_S(\mathcal{E}_{\alpha}, \mathcal{E}_{n\delta})|_{X_{\alpha}\times\mathfrak{Coh}_{n\delta}} = \mathrm{Hom}_S(\mathcal{E}_{n\delta}, \mathcal{E}_{\alpha} \otimes K_S)|_{X_{\alpha}\times\mathfrak{Coh}_{n\delta}} = \{0\}.
$$

Statement (a) follows. In order to prove (b), we introduce for all β a full flag version of $\widetilde{X}_{n\delta;\beta}$ as follows:

$$
\widetilde{X}_{\delta^n;\beta} = \{\mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \cdots \subset \mathcal{F}_0 \mid \mathcal{F}_0 \in X_{\beta+n\delta}, \forall\, i, \mathcal{F}_i/\mathcal{F}_{i+1} \in \mathcal{C}oh_{\delta}\}.
$$

Note that because X is a Hecke pattern, each $\mathcal{F}_i$ belongs to X. There are commutative diagrams

$$
\begin{tikzcd}
\widetilde{X}_{\delta^n;\alpha} \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & \widetilde{X}_{n\delta;\alpha} \arrow[d, "\pi'_{n\delta;\alpha}"] \\
X_{\alpha+n\delta} \times S^n \arrow[r, "\mathrm{Id}\times t"] & X_{\alpha+n\delta} \times \mathrm{Sym}^n(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{X}_{\delta^n;\alpha-n\delta} \arrow[r, "\overline{\gamma}"] \arrow[d, "\overline{\varphi}"'] & \widetilde{X}_{n\delta;\alpha-n\delta} \arrow[d, "\overline{\pi}'_{n\delta;\alpha}"] \\
X_{\alpha-n\delta} \times S^n \arrow[r, "\mathrm{Id}\times t"] & X_{\alpha-n\delta} \times \mathrm{Sym}^n(S)
\end{tikzcd}
$$

where $\gamma, \overline{\gamma}, \varphi, \overline{\varphi}$ are obvious forgetful maps and t is the projection. The map t being finite, it is proper and representable. Moreover, as any length n sheaf on S admits a full flag of subsheaves, the morphisms $\gamma, \overline{\gamma}$ are proper, representable and surjective. Thus to prove that $\pi'_{n\delta,\alpha}$ and $\overline{\pi}'_{\alpha,n\delta}$ are proper and representable, it suffices to show that the same holds for the maps $\varphi, \overline{\varphi}$. In order to show this we consider the chains of forgetful morphisms

$$
\widetilde{X}_{\delta^n;\beta} \xrightarrow{\overline{\varphi}_1} \widetilde{X}_{\delta^{n-1};\beta} \times S \xrightarrow{\overline{\varphi}_2} \cdots \xrightarrow{\overline{\varphi}_{n-1}} \widetilde{X}_{\delta;\beta} \times S^{n-1} \xrightarrow{\overline{\varphi}_n} X_{\beta} \times S^n
$$

and

$$
\widetilde{X}_{\delta^n;\beta} \xrightarrow{\varphi_0} \widetilde{X}_{\delta^{n-1};\beta+\delta} \times S \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-2}} \widetilde{X}_{\delta;\beta+(n-1)\delta} \times S^{n-1} \xrightarrow{\varphi_{n-1}} X_{\beta+n\delta} \times S^n .
$$

Observe that the composition of these chains for $\beta = \alpha - n\delta$ respectively yield φ and $\overline{\varphi}$. Thus we are reduced to checking that each φ_i and $\overline{\varphi}_i$ is proper and representable. By definition of a two-sided Hecke pattern, φ_i and $\overline{\varphi}_i$ are respectively obtained by base change from the maps

$$\rho' : \widetilde{\mathfrak{Coh}}^{\geqslant 1}_{\delta;\beta+i\delta} \to \mathfrak{Coh}^{\geqslant 1}_{\beta+(i+1)\delta} \times S, \qquad \overline{\rho}' : \widetilde{\mathfrak{Coh}}^{\geqslant 1}_{\delta;\beta+(n-i)\delta} \to \mathfrak{Coh}^{\geqslant 1}_{\beta+(n-i)\delta} \times S.$$

The maps $\rho', \overline{\rho}'$ above are respectively the projections from the projectivization $\mathbb{P}(\mathcal{E}_{\beta+(i+1)\delta})$ and $\mathbb{P}(\mathcal{E}^\vee_{\beta+(n-i)\delta} \otimes K_S[1])$. Since $\mathcal{E}$ and $\mathcal{E}^\vee \otimes K_S[1]$ are both of perfect amplitude $[0,1]$ over $X \times S$ it follows that $\rho', \overline{\rho}'$ are proper and representable (see [19, Lemma 5.4]). The same therefore holds for $\varphi_i, \overline{\varphi}_i$, and we are done. □

Let $X = \{X_\alpha\}$ be an S-weak two-sided Hecke pattern. By base change from the diagrams (6.1), we may analyze length one Hecke correspondences for X in the same fashion as for $\mathfrak{Coh}$. We will say that the two-sided Hecke pattern X is *regular* if there exist, for each α, an open cover $X_\alpha = \bigcup_i U^{(i)}_\alpha$ and locally free resolutions $\mathcal{E}^{(i)}_\alpha$ of length two for which the condition (2.10) holds when restricted to X. Equivalently, X is regular if and only if the morphisms $\kappa^{cl}_{\delta,\alpha}$ and $\overline{\kappa}^{cl}_{\alpha,\delta}$ are lci and of the same dimension as their derived enhancements.

6.2. **Hecke patterns and Hecke operators.** Let X be an S-weak two-sided Hecke pattern. We define

$$\mathbf{V}(X)_{\alpha+\mathbb{Z}\delta} = \bigoplus_{n\in\mathbb{Z}} H_*(X_{\alpha+n\delta}, \mathbb{Q}).$$

Let r be the projection

$$r : H^c_*(\mathrm{Sym}^n(S), \mathbb{Q}) \to H^c_0(\mathrm{Sym}^n(S), \mathbb{Q}) = \mathbb{Q}.$$

We define the maps

$$\begin{aligned} m_{\alpha,-n\delta} &= r \circ (\overline{\pi}'_{\alpha,n\delta})_* \circ (\overline{\kappa}_{\alpha,n\delta})^! : H_*(X_\alpha, \mathbb{Q}) \otimes H^c_*(\mathcal{C}oh_{n\delta}, \mathbb{Q}) \to H_*(X_{\alpha-n\delta}, \mathbb{Q}), \\ m_{n\delta,\alpha} &= r \circ (\pi'_{n\delta,\alpha})_* \circ (\kappa_{n\delta,\alpha})^! : H^c_*(\mathcal{C}oh_{n\delta}, \mathbb{Q}) \otimes H_*(X_\alpha, \mathbb{Q}) \to H_*(X_{\alpha+n\delta}, \mathbb{Q}). \end{aligned} \tag{6.2}$$

These maps have cohomological degree $2\langle \alpha, n\delta\rangle$ and $-2\langle n\delta, \alpha\rangle$ respectively. The proof of the following proposition, which is analogous to Theorem 1.9, is left to the reader.

Proposition 6.6. *The map $m_{n\delta,\alpha+m\delta}$ defines a left action Ψ^+_X, and the map $m_{\alpha+m\delta,-n\delta}$ a right action Ψ^-_X of $\mathbf{H}^c_0(S)$ on the space $\mathbf{V}(X)_{\alpha+\mathbb{Z}\delta}$. If X is just a left/right S-weak Hecke pattern, we have only the left/right action.* □

Remark 6.7.

(i) Suppose that X is a usual Hecke pattern. We can write diagrams (6.1) with $\mathfrak{Coh}_{n\delta}$ replaced by $\mathfrak{Coh}_{n\delta}(\overline{S})$. In this case the maps $\pi_{n\delta,\alpha}$, $\overline{\pi}_{\alpha,n\delta}$ are automatically proper for all n and α, and the actions $\Psi^\pm_X$ can be lifted to $\mathbf{H}_0(\overline{S})$.

(ii) Similarly, if X is an S-strong Hecke pattern, the maps $\pi_{n\delta,\alpha}$, $\overline{\pi}_{\alpha,n\delta}$ become proper, and so the actions $\Psi^\pm_X$ can be further lifted to $\mathbf{H}_0(S)$.

(iii) For $X = \mathfrak{Coh}^{\geqslant 1}$ and $n = 1$, the maps $m_{\alpha,-n\delta}$, $m_{n\delta,\alpha}$ are the negative and positive Hecke operators defined in §2.3.

By Example 6.3, we have both a left and a right $\mathbf{H}_0(S)$-module structure on the space

$$\mathbf{H}(S)^{\geqslant d}_{\alpha+\mathbb{Z}\delta} = \bigoplus_{n\in\mathbb{Z}} H_*(\mathcal{C}oh^{\geqslant d}_{\alpha+n\delta}, \mathbb{Q}).$$

Similarly, we have left and right $\mathbf{H}_0(\overline{S})$-actions on $\mathbf{H}(\overline{S})^{\geqslant d}_{\alpha+\mathbb{Z}\delta}$.

6.3. **Hecke patterns and tautological classes.** Let X be an S-weak two-sided Hecke pattern. Let

$$\mathrm{ev}_\alpha : \Lambda(\overline{S}) \to H^*(X_\alpha, \mathbb{Q})$$

denote the restriction of tautological classes on $\mathfrak{Coh}_\alpha(\overline{S})$ to X_α. This defines a representation

$$\bullet : \Lambda(\overline{S}) \times \mathbf{V}(X)_{\alpha+\mathbb{Z}\delta} \to \mathbf{V}(X)_{\alpha+\mathbb{Z}\delta}$$

This representation of $\Lambda(\overline{S})$ is compatible with the left and right $\mathbf{H}^c_0(S)$-actions: for any $c \in \Lambda(\overline{S})$, $u \in \mathbf{H}^c_0(S)$ and $z \in \mathbf{V}_X$ we have

$$c \bullet m_{n\delta,\alpha}(u \otimes z) = \sum (-1)^{|c_i^{(2)}|\cdot|u|} m_{n\delta,\alpha}(c_i^{(1)} \bullet u) \otimes (c_i^{(2)} \cdot z),$$
$$c \bullet m_{\alpha,-n\delta}(u \otimes z) = \sum (-1)^{|c_i^{(2)}|\cdot|u|} m_{\alpha,-n\delta}(\upsilon(c_i^{(1)}) \bullet u) \otimes (c_i^{(2)} \cdot z).$$

In particular, the left and right $\mathbf{H}^c_0(S)$-actions on $\mathbf{V}(X)_{\alpha+\mathbb{Z}\delta}$ extend to actions of $\widetilde{\mathbf{H}}^c_0(S)$. The proof is identical to that of Proposition 1.18. Analogous statements hold for usual and S-strong Hecke patterns, and the actions defined in Remark 6.7.

6.4. **The case of regular Hecke correspondences.** Let X be a two-sided Hecke pattern. Set

$$H^{\mathrm{taut}}_*(X_\alpha, \mathbb{Q}) = \Lambda(\overline{S}) \bullet [X^{cl}_\alpha] \quad , \quad \mathbf{V}^{\mathrm{taut}}(X)_{\alpha+\mathbb{Z}\delta} = \bigoplus_{n\in\mathbb{Z}} H^{\mathrm{taut}}_*(X_{\alpha+n\delta}, \mathbb{Q}).$$

We consider the linear map

$$\mathrm{ev}' : \mathbf{F}(\overline{S}) \to \mathbf{V}^{\mathrm{taut}}(X)_{\alpha+\mathbb{Z}\delta} \quad , \quad xu^n \mapsto x \bullet [X^{cl}_{\alpha+n\delta}] \quad , \quad x \in \Lambda(\overline{S}).$$

Let $\mathrm{End}_X(\mathbf{F}(\overline{S}))$ be the subspace of all endomorphisms of $\mathbf{F}(\overline{S})$ preserving the kernels of the maps ev'. Propositions 2.4, 2.6 and 4.1 yield the following.

Proposition 6.8. *Let X be a regular two-sided Hecke pattern of rank r. Then there is a commutative diagram of homomorphisms*

$$\begin{tikzcd}
W^{\pm}(\overline{S}) \arrow[r] \arrow[rr, bend left, "\Phi^{\pm}"] \arrow[dr, "\Phi^{\pm}_X"'] & \mathrm{End}_X(\mathbf{F}(\overline{S})) \arrow[r] \arrow[d, "\mathrm{ev}'"] & \mathrm{End}(\mathbf{F}(\overline{S})) \\
& \mathrm{End}(\mathbf{V}^{\mathrm{taut}}(X)_{\alpha+\mathbb{Z}\delta}) &
\end{tikzcd}$$

in which $\Phi^{\pm}, \Phi^{\pm}_X$ are algebra homomorphisms. Moreover, for any $\xi \in H^*(\overline{S})$ and $n\geqslant 0$, we have

$$\Phi^+_X(T_n(\xi)) = T_+(\xi u^{n-r+1}) = \Psi^+_X(\xi u^{n-r+1} \cap [\mathcal{C}oh_\delta]),$$
$$\Phi^-_X(T_n(\xi)) = T_-(\xi u^{n+r+1}) = \Psi^-_X(\xi u^{n+r+1} \cap [\mathcal{C}oh_\delta]),$$

and the maps $\Phi^{\pm}_X$ glue into an algebra homomorphism $\Phi_X : W^{(r)}(\overline{S}) \to \mathrm{End}(\mathbf{V}^{\mathrm{taut}}(X)_{\beta+\mathbb{Z}\delta})$. When X is S-strong, all the assumptions above hold with $\overline{S}$ replaced by S, and we get an algebra homomorphism $\Phi_X : W^{(r)}_{\upuparrows}(S) \to \mathrm{End}(\mathbf{V}^{\mathrm{taut}}(X)_{\beta+\mathbb{Z}\delta})$.

When X is only an S-weak Hecke pattern, regularity is not sufficient to ensure the actions of $W^{\pm}_{\downarrow}(S)$ in our setup. The reason is that $W^{\pm}_{\downarrow}(S)$ is *defined* as a subalgebra of $W^{\pm}(\overline{S})$; furthermore, it is typically not generated by degree 1 elements. Because of this, we will have to postpone the proof of an analogous statement until the end of § 7, see Corollary 7.13.

Thanks to Proposition 2.8, a result in all points analogous to Proposition 6.8 holds without any regularity assumptions if we replace $\mathbf{V}^{\mathrm{taut}}_X$ by its virtual cousin

$$\mathbf{V}^{\mathrm{vtaut}}(X_\alpha) := \Lambda(\overline{S}) \bullet [X_\alpha].$$

6.5. **Base change for Hecke patterns.** Let S be a smooth surface and let X be an S-weak two-sided Hecke pattern on S.

Proposition 6.9.

(a) *Let $\iota : S^\circ \to S$ be an open embedding. Then X is an S°-weak two-sided Hecke pattern on S° and we have $\Psi^{\pm}_X \circ \iota_! = \Psi^{\pm}_X : \mathbf{H}^c_0(S^\circ) \to \mathrm{End}(\mathbf{V}(X)_{\alpha+\mathbb{Z}\delta})$,*

(b) *Let $j : X^\circ \to X$ be an open immersion and assume that X° is also an S-weak two-sided Hecke pattern on S. Then we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{H}^c_0(S) \arrow[r] \arrow[rr, bend left, "\Psi^{\pm}_X"] \arrow[dr, "\Psi^{\pm}_{X^\circ}"'] & \mathrm{End}_{X^\circ}(\mathbf{V}(X)_{\alpha+\mathbb{Z}\delta}) \arrow[r] \arrow[d, "res"] & \mathrm{End}(\mathbf{V}(X)_{\alpha+\mathbb{Z}\delta}) \\
& \mathrm{End}(\mathbf{V}(X^\circ)_{\alpha+\mathbb{Z}\delta}) &
\end{tikzcd}$$

where $\mathrm{End}_{X^\circ}(\mathbf{V}(X)_{\alpha+\mathbb{Z}\delta})$ *is the subset of endomorphisms preserving the kernel of the map $j^* : \mathbf{V}(X)_{\alpha+\mathbb{Z}\delta} \to \mathbf{V}(X^\circ)_{\alpha+\mathbb{Z}\delta}$.*

Proof. We treat the case of Ψ^+, the other being similar. Consider the diagram

$$\begin{tikzcd}
\mathfrak{Coh}_{l\delta}(S) \times X_\alpha & \widetilde{X} \arrow[l, "\kappa_{l\delta,\alpha}"'] \arrow[r, "\pi'_{l\delta,\alpha}"] & X_{\alpha+l\delta} \times \mathrm{Sym}^l(S) \\
\mathfrak{Coh}_{l\delta}(S^\circ) \times X_\alpha \arrow[u] & \widetilde{Z} \arrow[l, "\kappa^{\#}_{l\delta,\alpha}"'] \arrow[r, "\pi^{\#}_{l\delta,\alpha}"] \arrow[u] & X_{\alpha+l\delta} \times \mathrm{Sym}^l(S^\circ) \arrow[u]
\end{tikzcd}$$

where $\widetilde{Z} = \widetilde{X} \times_{X_{\alpha+l\delta} \times \mathrm{Sym}^l(S)} (X_{\alpha+l\delta} \times \mathrm{Sym}^l(S^\circ))$ and the vertical maps are all open embeddings. Both squares are cartesian by definition of Hecke patterns. Statement (a) follows from base change properties of the morphism $\iota_{1!}$, see Proposition A.6. Now consider the diagram

$$\begin{tikzcd}
\mathfrak{Coh}_{l\delta}(S) \times X_\alpha & \widetilde{Z} \arrow[l, "\kappa^{\#}_{l\delta,\alpha}"'] \arrow[r, "\pi^{\#}_{l\delta,}"] & X_{\alpha+l\delta} \times \mathrm{Sym}^l(S)\ . \\
\mathfrak{Coh}_{l\delta}(S) \times X^\circ_\alpha \arrow[u] & \widetilde{X}^\circ \arrow[l, "\kappa^{\circ}_{l\delta,\alpha}"'] \arrow[r, "\pi'^{\circ}_{n\delta,\alpha}"] \arrow[u] & X^\circ_{\alpha+l\delta} \times \mathrm{Sym}^l(S) \arrow[u]
\end{tikzcd}$$

Again, both squares are cartesian, and the vertical arrows are all open embeddings induced by j. Statement (b) follows from proper base change in hyperbolic homology, see Lemma A.4. $\square$

7. Action on Hilbert schemes

In this section we construct actions of $\mathbf{H}_0(S)$ and $\mathbf{H}_0^c(S)$ on the homology of the Hilbert scheme of points on S, and we explicitly describe the action on tautological classes using the results of §2.3. We assume everywhere that the surface S is pure.

7.1. **The Hilbert scheme and stack.** Let Pic be the derived stack of invertible coherent sheaves on S, $B\mathbb{G}_m \to \mathrm{Pic}$ the closed immersion of the substack parametrizing trivial invertible sheaves, and $\mathfrak{Perf}$ the derived stack of perfect complexes. There is a morphism of derived stacks $\mathfrak{Coh}^{\geqslant 2}_{1,-n\delta} \to \mathfrak{Perf}_{1,-n\delta}$, which, composed with the perfect determinant map $\mathfrak{Perf}_{1,-n\delta} \to \mathrm{Pic}$ defined in [40], yields a morphism of derived stacks $\mathfrak{Coh}^{\geqslant 2}_{1,-n\delta} \to \mathrm{Pic}$. We define the *Hilbert stack* of S to be the derived fiber product

$$\mathfrak{Hilb}_n = \mathfrak{Coh}^{\geqslant 2}_{1,-n\delta} \underset{\mathrm{Pic}}{\times} B\mathbb{G}_m$$

We write $\mathfrak{Hilb} = \bigsqcup_n \mathfrak{Hilb}_n$. Let Hilb_n be the Hilbert scheme of n points on S, whose points parametrize ideal sheaves $\mathcal{I} \subset \mathcal{O}_S$ and of colength n. It is the coarse moduli space of $\mathfrak{Hilb}_n$.

Lemma 7.1. *The following hold:*

(a) $\mathfrak{Hilb}_n$ *is isomorphic to its classical truncation and* Hilb_n *is the coarse moduli space of* $\mathfrak{Hilb}_n$*,*

(b) *there is a canonical isomorphism of stacks* $\mathfrak{Hilb}_n \simeq \mathrm{Hilb}_n \times B\mathbb{G}_m$*.*

Proof. Since the morphism $\mathfrak{Coh}^{\geqslant 2}_{1,-n\delta} \to \mathrm{Pic}$ is Pic^{cl}-equivariant, we have

$$\begin{aligned}\mathrm{vdim}(\mathfrak{Hilb}_n) &= \mathrm{vdim}(\mathfrak{Hilb}_n) - \mathrm{vdim}(\mathrm{Pic}) - 1\\ &= \langle [\mathcal{O}_S] - n\delta\,,\, [\mathcal{O}_S] - n\delta\rangle - \langle [\mathcal{O}_S], [\mathcal{O}_S]\rangle - 1\\ &= 2n-1.\end{aligned}$$

The classical truncation of $\mathfrak{Hilb}_n$ parametrizes inclusions of colength n ideal sheaves $\mathcal{I} \to \mathcal{O}_S$. It is smooth and of dimension $2n-1$. The stack $\mathfrak{Hilb}_n$ is quasi-smooth. Part (a) follows using the fact that a quasi-smooth derived stack with smooth classical truncation and whose virtual dimension coincides with that of its classical truncation is underived.

We now turn to (b). The stack $\mathfrak{Hilb}_n$ parametrizes simple rank 1 sheaves. Hence it is a $\mathbb{G}_m$-gerbe over its coarse moduli space Hilb_n. Fixing a splitting of this gerbe is the same as choosing a universal sheaf $\mathcal{U}_n$ on $\mathrm{Hilb}_n \times S$. But such a canonical sheaf is given by a subsheaf of $\mathcal{O}_{\mathrm{Hilb}_n} \boxtimes \mathcal{O}_S$. □

Fix a smooth compactification $\iota : S \to \overline{S}$. We define $\mathfrak{Hilb}_n(\overline{S})$, $\mathrm{Hilb}_n(\overline{S})$ and $\mathrm{Pic}(\overline{S})$ as above, with the surface S replaced by $\overline{S}$. Let ρ be the degree one line bundle on $\mathfrak{Hilb}_n(\overline{S})$ pulled back from $B\mathbb{G}_m$. Let $\mathcal{U}_n$ and $\mathcal{I}_n \simeq \mathcal{U}_n \boxtimes \rho$ be the universal ideal sheaves in $Coh(\mathrm{Hilb}_n(\overline{S}) \times \overline{S})$ and $Coh(\mathfrak{Hilb}_n(\overline{S}) \times \overline{S})$ respectively. Let us also simply denote $\mathrm{ev}_{\mathfrak{Hilb}}$ by ev.

Lemma 7.2. *We have* $c_1(\rho) = \mathrm{ev}(p_1([\mathrm{pt}])) = \int_{\overline{S}} c_1(\mathcal{I}_n) \cdot [\mathrm{pt}]$.

Proof. Since $\mathcal{I}_n$ is of generic rank one, we have $c_1(\mathcal{I}_n) = c_1(\mathcal{U}_n) + c_1(\rho)$. We may write

$$c_1(\mathcal{I}_n) = \mathrm{ev}(p_1([\mathrm{pt}])) \otimes 1 + y \quad , \quad y \in H^*(\mathrm{Hilb}_n(\overline{S}), \mathbb{Q}) \otimes H^{>0}(\overline{S}, \mathbb{Q}).$$

Thus, given any point s in $\overline{S}$, we have

$$\mathrm{ev}(p_1([\mathrm{pt}])) = c_1(\rho) + c_1(\mathcal{U}_n|_{\mathrm{Hilb}_n(\overline{S}) \times \{s\}}),$$

It remains to prove that $c_1(\mathcal{U}_n|_{\mathrm{Hilb}_n(\overline{S}) \times \{s\}}) = 0$. There is a short exact sequence of sheaves

$$0 \to \mathcal{U}_n \to \mathcal{O}_{\mathrm{Hilb}_n(\overline{S})} \boxtimes \mathcal{O}_{\overline{S}} \to \mathcal{T}_n \to 0$$

over $\mathrm{Hilb}_n(\overline{S}) \times \overline{S}$, where $\mathcal{T}_n$ is a coherent sheaf whose restriction to $\{\xi\} \times \overline{S}$ is zero-dimensional for any $\xi \in \mathrm{Hilb}_n(\overline{S})(\mathbb{C})$. Let $\mathrm{Hilb}_n^\circ(\overline{S})$ be the open subscheme of $\mathrm{Hilb}_n(\overline{S})$ parametrizing zero-dimensional sheaves whose support does not contain s. Its complement is of codimension 2, hence the restriction map $H^i(\mathrm{Hilb}_n(\overline{S}), \mathbb{Q}) \to H^i(\mathrm{Hilb}_n^\circ(\overline{S}), \mathbb{Q})$ is an isomorphism for $i < 4$, in particular for $i = 2$. The restriction of $\mathcal{T}_n$ to $\mathrm{Hilb}_n^\circ(\overline{S}) \times \{s\}$ is zero. We deduce that

$$c_1(\mathcal{U}_n|_{\mathrm{Hilb}_n(\overline{S})\times\{s\}}) = -c_1(\mathcal{T}_n|_{\mathrm{Hilb}_n(\overline{S})\times\{s\}}) = 0. \qquad \square$$

As a consequence, we have an isomorphism

$$H_*(\mathrm{Hilb}_n(\overline{S}), \mathbb{Q}) = H_*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q}) \,/\, p_1([\mathrm{pt}]) \bullet H_*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q}). \tag{7.1}$$

We define

$$V(\overline{S}) = \bigoplus_n H_*(\mathrm{Hilb}_n(\overline{S}), \mathbb{Q}) \quad , \quad \mathbf{V}(\overline{S}) = \bigoplus_n H_*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q}).$$

We define $V(S)$ and $\mathbf{V}(S)$ similarly. Unless $S = \overline{S}$, the cohomology group $H^4(S, \mathbb{Q})$ vanishes, hence the class $p_1([\mathrm{pt}])$ vanishes in $\Lambda(S)$. Nevertheless, Lemma 7.2 and (7.1) hold if we use instead of $p_1([\mathrm{pt}])$ the restriction to $H^*(\mathfrak{Hilb}_n, \mathbb{Q})$ of the class $p_1([\mathrm{pt}])$ in $H^*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q})$. Note that this is not in conflict with Lemma 1.16, since its conditions are satisfied for Hilb_n, but not $\mathfrak{Hilb}_n$.

7.2. **Purity and generation of the cohomology by tautological classes.** In this section we collect some facts on purity and tautological classes on Hilbert schemes and Hilbert stacks.

Lemma 7.3. *The stack $\mathfrak{Hilb}_n$ is pure for all n.*

Proof. It is enough to prove that Hilb_n is pure. For any partition $\lambda = (1^{m_1}2^{m_2}\ldots)$ we set

$$\mathrm{Sym}^\lambda(S) = \mathrm{Sym}^{m_1}(S) \times \mathrm{Sym}^{m_2}(S) \times \cdots .$$

By [16, Thm. 2] there is an isomorphism of mixed Hodge structures

$$H^{i+2n}(\mathrm{Hilb}_n, \mathbb{Q}) \otimes \mathbb{Q}(n) = \bigoplus_\lambda H^{i+2l(\lambda)}(\mathrm{Sym}^\lambda(S), \mathbb{Q}) \otimes \mathbb{Q}(l(\lambda)),$$

where λ runs among all partitions of size n. Hence, it is enough to prove that each symmetric power $\mathrm{Sym}^m(S)$ is cohomologically pure. We have an isomorphism of Hodge stuctures

$$\pi^* : H^*(\mathrm{Sym}^m(S), \mathbb{Q}) \xrightarrow{\sim} H^*(S^m, \mathbb{Q})^{\mathfrak{S}_m}.$$

The lemma follows. $\square$

Corollary 7.4. *For any n, the cohomology of $\mathfrak{Hilb}_n$ is generated by tautological classes, i.e., the evaluation map $\Lambda(\overline{S}) \to H^*(\mathfrak{Hilb}_n, \mathbb{Q})$ is onto. The same holds for the evaluation map $\Lambda(S) \to H^*(\mathrm{Hilb}_n, \mathbb{Q})$.*

Proof. The evaluation map factors through $H^*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q})$. The map $\Lambda(\overline{S}) \to H^*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q})$ is surjective by [36, Thm. 7.5] and Lemma 7.1. The restriction map $H^*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q}) \to H^*(\mathfrak{Hilb}_n, \mathbb{Q})$ is surjective since the cohomology of $\mathfrak{Hilb}_n$ is pure, and $\mathfrak{Hilb}_n(\overline{S})$ is its smooth compactification. This proves the first claim. For the second claim, note that the evaluation map factors through $\Lambda(S)$ by Lemma 1.16. $\square$

By Corollary 7.4, the collection of evaluation morphisms $\Lambda(S) \to H_*(\mathrm{Hilb}_n, \mathbb{Q})$ yields a surjective linear map $\mathrm{ev}' : \mathbf{F}(S) \to V(S)$. The same holds for the surface $\overline{S}$. There is a commuting diagram

$$\begin{tikzcd} \mathbf{F}(\overline{S}) \arrow[r, "\iota^*"] \arrow[d, "\mathrm{ev}'"'] & \mathbf{F}(S) \arrow[d, "\mathrm{ev}'"'] \\ V(\overline{S}) \arrow[r, "\iota^*"] & V(S). \end{tikzcd}$$

7.3. **COHA actions on $\mathfrak{Hilb}$.** We now endow the spaces $\mathbf{V}(\overline{S})$, $\mathbf{V}(S)$ with actions of the COHAs $\mathbf{H}_0(\overline{S})$, $\mathbf{H}_0(S)$ and their compact versions. This is a reformulation of Nakajima's classical construction and of Lehn's results, see [31], [23]. Let us denote by $\mathfrak{Hilb}_{k-n,k}(\overline{S})$ the flag Hilbert stack:

$$\mathfrak{Hilb}_{k-n,k}(\overline{S}) = \{\mathcal{J} \subset \mathcal{I} \subset \mathcal{O}_S : \mathrm{lg}(\mathcal{I}) = k-n,\ \mathrm{lg}(\mathcal{J}) = k\}$$

In terms of Hecke patterns of § 6, these are precisely $\widetilde{X}_{n\delta;[\mathcal{O}_{\overline{S}}]-k\delta}$ for $X_{[\mathcal{O}_{\overline{S}}]-k\delta} = \mathfrak{Hilb}_k(\overline{S})$.

Proposition 7.5. *The stack $\mathfrak{Hilb}(\overline{S})$ is a regular two-sided Hecke pattern. The stack $\mathfrak{Hilb}$ is a regular right S-strong and left S-weak Hecke pattern.*

Proof. The proof of the fact that $\mathfrak{Hilb}$ is a two-sided Hecke pattern is the same for S and $\overline{S}$ and relies upon the following simple observation.

Lemma 7.6. *Let $0 \to \mathcal{F} \xrightarrow{a} \mathcal{G} \to \mathcal{T} \to 0$ be a short exact sequence of coherent sheaves on a smooth surface, with $\mathcal{F}, \mathcal{G}$ torsion-free and $\mathcal{T}$ zero-dimensional. Then $a^{\vee\vee} : \mathcal{F}^{\vee\vee} \simeq \mathcal{G}^{\vee\vee}$.*

Proof. It suffices to consider the case when $\mathcal{T}$ is of length one. Applying the derived duality functor $\mathbb{D}$ yields the long exact sequence

$$\to H^0(\mathbb{D}\mathcal{T}) \to H^0(\mathbb{D}\mathcal{G}) \xrightarrow{\mathbb{D}a} H^0(\mathbb{D}\mathcal{F}) \to H^1(\mathbb{D}\mathcal{T}) \to .$$

Since $\mathbb{D}\mathcal{T} \simeq \mathcal{T}[-2]$, we obtain an isomorphism $\mathcal{G}^\vee = H^0(\mathbb{D}\mathcal{G}) \simeq H^0(\mathbb{D}\mathcal{F}) = \mathcal{F}^\vee$. Note that because $\mathcal{F}, \mathcal{G}$ are torsion-free, $\mathcal{F}^\vee, \mathcal{G}^\vee$ are vector bundles. $\square$

Let $\mathcal{I} \subset \mathcal{O}_S$ be an ideal sheaf of finite colength and let $\mathcal{T}$ be a finite length sheaf. For any short exact sequence $0 \to \mathcal{J} \to \mathcal{I} \to \mathcal{T} \to 0$, $\mathcal{J}$ is obviously a finite colength ideal sheaf. For any short exact sequence $0 \to \mathcal{I} \to \mathcal{J} \to \mathcal{T} \to 0$ with $\mathcal{I} \in \mathfrak{Coh}^{\geqslant 1}$, the sheaf $\mathcal{J}$ is torsion-free (otherwise it would contain a one-dimensional subsheaf $\mathcal{E}$, whose support would intersect $S \backslash \mathrm{supp}(\mathcal{T})$, contradicting the fact that $\mathcal{I}$ is torsion free). Finally, by Lemma 7.6, there is a canonical isomorphism $\mathcal{J}^{\vee\vee} \simeq \mathcal{I}^{\vee\vee}$ hence $\mathcal{J} \in \mathfrak{Hilb}$ as wanted. The S-strongness on the right follows from Lemma 6.4.

Let us now prove the regularity of these Hecke patterns. Again, the argument is the same for S and $\overline{S}$, we will only treat the latter case. Note that $\mathfrak{Hilb}_k(\overline{S})$ is of finite type and included in $\mathcal{C}oh^{\geqslant 1}_{1,-k\delta}(\overline{S})$ for any k. It follows that we may find global resolutions for both $\mathcal{E}_{1,-k\delta}$ and $\mathrm{RHom}(\mathcal{E}_\delta, \mathcal{E}_{1,-(k+1)\delta})[1]$. We will follow the notations of § 2.3. It is well-known that $\mathrm{Hilb}_k(\overline{S})$ and $\mathrm{Hilb}_{k,k+1}(\overline{S})$ are both smooth and connected, of respective dimensions $2k$ and $2(k+1)$. Since $\mathfrak{Hilb}_k(\overline{S})$ and $\mathfrak{Hilb}_{k,k+1}(\overline{S})$ are $\mathbb{G}_m$-gerbes over $\mathrm{Hilb}_k(\overline{S})$ and $\mathrm{Hilb}_{k,k+1}(\overline{S})$ respectively, it follows that the former are smooth, irreducible and

$$\dim(\mathfrak{Hilb}_k(\overline{S})) = 2k-1, \qquad \dim(\mathfrak{Hilb}_{k+1}(\overline{S})) = \dim(\mathfrak{Hilb}_{k,k+1}(\overline{S})) = 2k+1. \tag{7.2}$$

The tautological sheaf $\mathcal{E}_{1,-k\delta}$ on $\mathfrak{Hilb}_k(\overline{S}) \times \overline{S}$ is of rank one. The section s is thus regular if and only if $\dim(\mathfrak{Hilb}_{k,k+1}(\overline{S})) = \dim(\mathfrak{Hilb}_k(\overline{S}) \times \overline{S}) + \mathrm{rk}(\mathcal{E}_{1,-k\delta}) - 1 = \dim(\mathfrak{Hilb}_k(\overline{S}) \times \overline{S})$. Likewise, s' is regular if and only if $\dim(\mathfrak{Hilb}_{k,k+1}(\overline{S})) = \dim(\mathcal{C}oh_\delta(\overline{S}) \times \mathfrak{Hilb}_{k+1}(\overline{S})) - \langle \delta, (1-(k+1)\delta) \rangle = \dim(\mathcal{C}oh_\delta(\overline{S}) \times \mathfrak{Hilb}_{k+1}(\overline{S})) - 1$. Both of these equalities follow from (7.2). $\square$

Remark 7.7. Let S be projective, and H an ample divisor. Suppose that the Assumptions A and S of [32] hold, and fix $r > 0$, $c \in H^2(S, \mathbb{Z})$. A proof similar to Proposition 7.5 yields the regularity of Hecke pattern $\mathcal{M}_{r,c}$, which is the moduli of H-stable torsion-free sheaves on S of rank r and first Chern class c.

For simplicity, we will denote the Hecke patterns $\mathfrak{Hilb}$ and $\mathfrak{Hilb}(\overline{S})$ by $\mathfrak{H}$ and $\overline{\mathfrak{H}}$ respectively. Proposition 6.6 yields two representations

$$\Psi^+_{\mathfrak{H}} : \widetilde{\mathbf{H}}^c_0(S) \to \mathrm{End}(\mathbf{V}(S)), \qquad \Psi^-_{\mathfrak{H}} : \widetilde{\mathbf{H}}_0(S) \to \mathrm{End}(\mathbf{V}(S))$$

such that the subspace $H^c_*(\mathcal{C}oh_{0,k}, \mathbb{Q})$ maps into

$$\prod_n \mathrm{Hom}\left(H_*(\mathfrak{Hilb}_n, \mathbb{Q}), H_*(\mathfrak{Hilb}_{n\mp k}, \mathbb{Q})\right).$$

We get similar representations for $\overline{S}$. Since both $\mathfrak{Hilb}$ and $\mathfrak{Hilb}(\overline{S})$ are right Hecke patterns, by Remark 6.7(i) we can lift $\Psi^-_{\mathfrak{H}}$ to an action of $\widetilde{\mathbf{H}}_0(S)$. By Proposition 6.8, regularity of the Hecke patterns above yields representations

$$\Phi^{\pm}_{\overline{\mathfrak{H}}} : W^{\pm}(\overline{S}) \to \mathrm{End}(\mathbf{V}^{\mathrm{taut}}(\overline{S})) = \mathrm{End}(\mathbf{V}(\overline{S})), \qquad \Phi^-_{\mathfrak{H}} : W^-_{\uparrow}(S) \to \mathrm{End}(\mathbf{V}(S)),$$

which for $\overline{S}$ glue to a representation $W^{(1)}(\overline{S}) \to \mathrm{End}(\mathbf{V}(\overline{S}))$.

Lemma 7.1 yields an isomorphism

$$V(\overline{S}) = \mathbf{V}(\overline{S}) \,/\, p_1([\mathrm{pt}]) \bullet \mathbf{V}(\overline{S})$$

Since the class $p_1([\mathrm{pt}]) = \psi_0([\mathrm{pt}])$ belongs to the center of $W(\overline{S})$, the representations $\Phi^{\pm}_{\overline{\mathfrak{H}}}$ descend to representations of $W^{\pm}(\overline{S})$ on $V(\overline{S})$ which we will simply denote by $\Phi^{\pm}_{\overline{S}}$. Since $\mathrm{ev}_{n\delta}(p_1([\mathrm{pt}])) = 0$ for any $n > 0$, see Example 1.17, the representations $\Psi^{\pm}_{\overline{\mathfrak{H}}}$ descend to representations of $\mathbf{H}_0(\overline{S})$ on $V(\overline{S})$ which we will simply denote by $\Psi^{\pm}_{\overline{S}}$. Similar claims hold for the representations $\Psi^{\pm}_{\mathfrak{H}}$, $\Phi^-_{\mathfrak{H}}$ associated to S.

7.4. **Nakajima operators and COHA actions.** In this section, we briefly recall the construction of the Nakajima operators, see [31] and [23] for details, and we relate them to the action of $\mathbf{H}_0(S)$, $\mathbf{H}^c_0(S)$. We begin with the case of a proper surface $\overline{S}$. For each $k \geqslant 0$ and $l \geqslant 1$, we consider the reduced subscheme

$$Z_{k+l,k}(\overline{S}) \subset \mathrm{Hilb}_{k+l}(\overline{S}) \times \mathrm{Hilb}_k(\overline{S})$$

parametrizing pairs of ideal sheaves $(\mathcal{I}, \mathcal{J})$ with $\mathcal{J} \supset \mathcal{I}$ for which the support $\mathrm{supp}(\mathcal{J}/\mathcal{I})$ consists of a single point. There is a support map

$$s : Z_{k+l,l}(\overline{S}) \to \overline{S} \quad , \quad s(\mathcal{J}, \mathcal{I}) = \mathrm{supp}(\mathcal{J}/\mathcal{I}).$$

This allows us to view $Z_{k+l,k}(\overline{S})$ as a subscheme of $\mathrm{Hilb}_{k+l}(\overline{S}) \times \overline{S} \times \mathrm{Hilb}_k(\overline{S})$. For any subset $I \subset \{1, 2, 3\}$ let p_I be the projection to the factors in I. For each $\lambda \in H^*(\overline{S}, \mathbb{Q})$ and $l \geqslant 1$, the Nakajima operator $\mathfrak{q}_l(\lambda) \in \mathrm{End}(V(\overline{S}))$ is

$$\mathfrak{q}_l(\lambda) : H_*(\mathrm{Hilb}_k(\overline{S}), \mathbb{Q}) \to H_*(\mathrm{Hilb}_{k+l}(\overline{S}), \mathbb{Q}) \quad , \quad c \mapsto p_{1*}(p_{23}^*((\lambda \cap [\overline{S}]) \otimes c) \cap [Z_{k+l,k}(\overline{S})])$$

Note that the restriction of p_1 to the support of $Z_{k+l,k}$ is proper. Exchanging the roles of $\mathrm{Hilb}_k(\overline{S})$ and $\mathrm{Hilb}_{k+l}(\overline{S})$ and using the isomorphic subscheme $Z_{k,k+l} \subset \mathrm{Hilb}_k(\overline{S}) \times \overline{S} \times \mathrm{Hilb}_{k+l}(\overline{S})$ in place of $Z_{k+l,k}$, we get the operator $\mathfrak{q}_{-l}(\lambda) \in \mathrm{End}(V(\overline{S}))$ given by

$$\mathfrak{q}_{-l}(\lambda) : H_*(\mathrm{Hilb}_{k+l}(\overline{S}), \mathbb{Q}) \to H_*(\mathrm{Hilb}_k(\overline{S}), \mathbb{Q}) \quad , \quad c \mapsto (-1)^l p_{1*}(p_{23}^*((\lambda \cap [\overline{S}]) \otimes c) \cap [Z_{k,k+l}(\overline{S})])$$

As proved by Nakajima in [31, §8], the operators $\mathfrak{q}_n(\lambda)$ with $\lambda \in H^*(\overline{S}, \mathbb{Q})$, $n \in \mathbb{Z}\backslash\{0\}$ generate an action of the Heisenberg algebra $\mathfrak{h}_{\overline{S}}$ modeled on $H^*(\overline{S}, \mathbb{Q})$, see (3.19) for relations, with central charge $C = 1$. The space $V(\overline{S})$ is isomorphic to the Fock space representation of $\mathfrak{h}_{\overline{S}}$; in particular, the action of $U(\mathfrak{h}_{\overline{S}})$ on $V(\overline{S})$ is faithful.

Now, assume that S is any quasi-projective smooth surface. We may apply the exact same construction, with the following modifications: the operators $\mathfrak{q}_{-l}(\lambda)$ decreasing the number of points are labeled by classes $\lambda \in H^*_c(S, \mathbb{Q})$, and the commutator relations are defined using the intersection pairing $H^*(S, \mathbb{Q}) \otimes H^*_c(S, \mathbb{Q}) \to \mathbb{Q}$, see [31, §8].

Proposition 7.8. *Let S be any quasi-projective smooth surface. For any $l > 0$ and any $\lambda \in H^*(S, \mathbb{Q})$, $\mu \in H^*_c(S, \mathbb{Q})$ there exist elements $E_l(\lambda) \in \mathbf{H}_0(S)$ and $E_{-l}(\mu) \in \mathbf{H}^c_0(S)$ such that $\Psi^-_S(E_l(\lambda)) = \mathfrak{q}_l(\lambda)$ and $\Psi^+_S(E_{-l}(\mu)) = \mathfrak{q}_{-l}(\mu)$.*

Proof. We begin with the case of operators $\mathfrak{q}_{-l}(\mu)$. Let $h : S \to \mathrm{Sym}^l(S)$ be the diagonal embedding. Set

$$\mathfrak{Coh}^{pt}_{l\delta} = \mathfrak{Coh}_{l\delta} \times_{\mathrm{Sym}^l(S)} S.$$

Let $s : \mathfrak{Coh}^{pt}_{l\delta} \to S$ the projection and $t : \mathfrak{Coh}^{pt}_{l\delta} \to \mathfrak{Coh}_{l\delta}$ the closed immersion. Given $k \geqslant l$, let $\widetilde{\mathfrak{Z}}_{k,k-l}$ be the derived stack parametrizing inclusions $\mathcal{I} \subset \mathcal{J}$ of ideal sheaves of colength k and $k-l$. We have the following commutative diagram with Cartesian squares

$$\begin{tikzcd}
\mathfrak{Coh}_{l\delta} \times \mathfrak{Hilb}_k & \widetilde{\mathfrak{Hilb}}_{k,k-l} \arrow[l, "\kappa"'] \arrow[r, "\pi'"] & \mathfrak{Hilb}_{k-l} \times \mathrm{Sym}^l(S) \\
\mathfrak{Coh}^{pt}_{l\delta} \times \mathfrak{Hilb}_k \arrow[u, "t\times\mathrm{Id}"] & \widetilde{\mathfrak{Z}}_{k,k-l} \arrow[l, "\kappa^{pt}"'] \arrow[r, "(\pi')^{pt}"] \arrow[u, "i'"] & \mathfrak{Hilb}_{k-l} \times S \arrow[u, "i''"]
\end{tikzcd}$$

The map κ^{pt} is quasi-smooth, because κ is quasi-smooth. The class $\mu \in H^c_*(S, \mathbb{Q})$ yields a class

$$s^*(\mu) \in H^*(\mathcal{C}oh^{pt}_{l\delta}/S, \mathbb{Q})$$

We define

$$E_{-l}(\mu) = (-1)^l t_!(s^*(\mu) \cap [\mathcal{C}oh^{pt}_{l\delta}]) \in H_*(\mathcal{C}oh_{l\delta}/\,\mathrm{Sym}^l(S), \mathbb{Q}).$$

The proper base change in Proposition A.6 implies that

$$(\pi')_!\kappa^!(t \times \mathrm{Id})_! = (\pi')_! i'_!(\kappa^{pt})^! = i''_!(\pi')^{pt}_!(\kappa^{pt})^!.$$

Composing with the projection r, we get the relation

$$\Psi^+_{\mathfrak{H}}(E_{-l}(\mu))(c) = (-1)^l r \circ (\pi')^{pt}_!(\kappa^{pt})^!((s^*(\mu) \cap [\mathcal{C}oh^{pt}_{l\delta}]) \otimes c) \quad , \quad c \in H_*(\mathfrak{Hilb}_k, \mathbb{Q}).$$

After pulling everything back from $\mathfrak{Hilb}$ to Hilb, we have

$$(\kappa^{pt})^!\,((s^*(\mu) \cap [\mathcal{C}oh_{l\delta}]) \otimes c) = ((\mu \cap [S]) \otimes c \otimes [\mathrm{Hilb}_{k-l}]) \cap [Z_{k,k-l}].$$

Applying the proper pushforward to the projection to the factor Hilb_{k-l}, we get the equality

$$\Psi^+_{\mathfrak{H}}(E_{-l}(\mu))(c) = \mathfrak{q}_{-l}(\mu)(c).$$

We now turn to the case of operators $\mathfrak{q}_l(\lambda)$. Using the cartesian diagram

$$\begin{tikzcd}
\mathfrak{Hilb}_k(\overline{S}) \times \mathfrak{Coh}_{l\delta}(\overline{S}) & \widetilde{\mathfrak{Hilb}}_{k,k+l}(\overline{S}) \arrow[l, "\overline{\kappa}"'] \arrow[r, "\overline{\pi}"] & \mathfrak{Hilb}_{k+l}(\overline{S}) \\
\mathfrak{Hilb}_k \times \mathfrak{Coh}_{l\delta} \arrow[u] & \widetilde{\mathfrak{Hilb}}_{k,k+l} \arrow[l, "\overline{\kappa}^\circ"'] \arrow[r, "\overline{\pi}^\circ"] \arrow[u] & \mathfrak{Hilb}_{k+l} \arrow[u]
\end{tikzcd}$$

in which all the vertical arrows are open embeddings, we are reduced to the case of a proper surface $S = \overline{S}$. For this we may repeat the arguments used in the case $\mathfrak{q}_{-l}(\mu)$ above. $\square$

7.5. **The Heisenberg subalgebra and Nakajima operators.** In this paragraph, we will identify the action of the Heisenberg subalgebra $\mathfrak{h}_S$ on $V(S)$ with the action of Nakajima operators. We begin with the case of a proper surface $\overline{S}$, in which case we may consider both actions $\Phi^{\pm}_{\overline{S}}$.

Proposition 7.9. *For any $\lambda \in H^*(\overline{S}, \mathbb{Q})$ and $l \geqslant 1$ we have the following formulas in* $\mathrm{End}(V(\overline{S}))$

$$\Phi^{+}_{\overline{S}}(D_{l,0}(\lambda)) = (-1)^l \mathfrak{q}_{-l}(\lambda) \quad , \quad \Phi^{-}_{\overline{S}}(D_{l,0}(\lambda)) = \mathfrak{q}_l(\lambda). \tag{7.3}$$

In particular, both representations $\Phi^{\pm}_{\overline{S}}$ of $U(\mathfrak{h}_{\overline{S}})$ are faithful.

Proof. Assume first that $l = 1$ and consider the diagram

$$\begin{tikzcd}
\mathcal{C}oh_\delta(\overline{S}) \times \mathfrak{Hilb}_{k+1}(\overline{S}) & \widetilde{\mathfrak{Hilb}}_{k+1,k}(\overline{S}) \arrow[l, "q"'] \arrow[r, "p"] & \mathfrak{Hilb}_k(\overline{S}) \\
\overline{S} \times \mathrm{Hilb}_{k+1}(\overline{S}) \arrow[u, "j_1"] & Z_{k+1,k}(\overline{S}) \arrow[l, "q'"'] \arrow[r, "p'"] \arrow[u, "j_2"] & \mathrm{Hilb}_k(\overline{S}) \arrow[u, "j_3"]
\end{tikzcd}$$

in which the vertical arrows are induced by the maps $\overline{S} \to \overline{S} \times B\mathbb{G}_m = \mathcal{C}oh_\delta(\overline{S})$ and $\mathrm{Hilb}_n(\overline{S}) \to \mathrm{Hilb}_n(\overline{S}) \times B\mathbb{G}_m = \mathfrak{Hilb}_n(\overline{S})$. Note that the right square is cartesian. Moreover, $\widetilde{\mathfrak{Hilb}}_{k+1,k}(\overline{S})$ and $Z_{k+1,k}(\overline{S})$, are smooth and the maps q, q' are lci. In this situation, for $c \in H_*(\mathrm{Hilb}_{k+1}(\overline{S}), \mathbb{Q})$ and $\lambda \in H^*(\overline{S}, \mathbb{Q})$, we have

$$\mathfrak{q}_{-1}(\lambda)(c) = -p'_*((q')^!((\lambda \cap [\overline{S}]) \otimes c)).$$

Note that by construction, the projection $H_*(\mathfrak{Hilb}_n(\overline{S}), \mathbb{Q}) \to H_*(\mathrm{Hilb}_n(\overline{S}), \mathbb{Q})$ coincides with the pullback by the morphism $\mathrm{Hilb}_n(\overline{S}) \to \mathfrak{Hilb}_n(\overline{S})$. By base change, we have

$$\begin{aligned}
\Phi^{+}_{\overline{S}}(D_{1,0}(\lambda))(c) = j_3^*(p_* q^!(\lambda \cap [\overline{S}] \otimes c)) &= p'_* j_2^* q^!(\lambda \cap [\overline{S}] \otimes c)) \\
&= p'_*(q')^! j_1^*(\lambda \cap [\overline{S}] \otimes c)) = -\mathfrak{q}_{-1}(\lambda)(j_1^*(c))
\end{aligned} \tag{7.4}$$

as wanted. Note that as the map $\widetilde{\mathfrak{Hilb}}_{k+1,k}(\overline{S}) \to \mathcal{C}oh_\delta(\overline{S}) \times \mathfrak{Hilb}_k(\overline{S})$ is lci, the refined Gysin pullback is well-defined without any need to consider derived enhancements.

In order to extend the above relation to arbitrary $l > 1$, recall Lehn's formulas [23, (2)]. Put $\mathcal{U} = \pi_*(\mathcal{T})$ where $\mathcal{T}$ is the universal subscheme on $\mathrm{Hilb}(\overline{S}) \times \overline{S}$ and π is the projection along $\overline{S}$. Setting $\mathfrak{d} = c_1(\mathcal{U})$ we have

$$\begin{aligned}
[\mathfrak{d}, \mathfrak{q}_{\pm 1}(1)] &= \mathfrak{q}'_{\pm 1}(1) \\
[\mathfrak{q}'_{\pm 1}(1), \mathfrak{q}_{\pm m}(\lambda)] &= -m \mathfrak{q}_{\pm(m+1)}(\lambda), \qquad \lambda \in H^*(\overline{S}, \mathbb{Q}),\ m \geqslant 1.
\end{aligned} \tag{7.5}$$

Observing that $\mathfrak{d} = -\frac{1}{2}\psi_2(1)$ and comparing (3.20) with (7.5) we deduce the statement by induction on m. Note that the difference in signs is due to the fact that $\Phi^{-}_{\overline{S}}$ is a *right* representation. $\square$

We now turn to the case of an arbitrary cohomologically pure[4] surface S, where (à priori) only Φ^{-}_{S} is defined. Fixing $\iota : S \to \overline{S}$, the operators $\mathfrak{q}_l(\lambda)$ for $l > 0$ and $\lambda \in H^*(\overline{S}, \mathbb{Q})$ are easily seen to

[4]When S is not pure, the same construction yields an identification of the action of the element $D_{m,0}(\alpha)$ for $\alpha \in H^*_{\mathrm{pure}}(S, \mathbb{Q})$ with the corresponding Nakajima operators $\mathfrak{q}_m(\alpha)$.

be compatible with the restriction maps from $\underline{\iota}^* : H_*(\mathrm{Hilb}_k(\overline{S}), \mathbb{Q}) \to H_*(\mathrm{Hilb}_k(S), \mathbb{Q})$ in the sense that there is a commutative diagram

$$\begin{tikzcd} H_*(\mathrm{Hilb}_k(\overline{S}), \mathbb{Q}) \arrow[r, "\underline{\iota}^*"] \arrow[d, "\mathfrak{q}_l(\lambda)"] & H_*(\mathrm{Hilb}_k, \mathbb{Q}) \arrow[d, "\mathfrak{q}_l(\lambda_{|S})"] \\ H_*(\mathrm{Hilb}_{k+l}(\overline{S}), \mathbb{Q}) \arrow[r, "\underline{\iota}^*"] & H_*(\mathrm{Hilb}_{k+l}, \mathbb{Q}) \end{tikzcd} .$$

We claim that the same base change formulas hold for the operators $\Phi^-(D_{m,0}(\lambda))$ for $m \geqslant 1$. Indeed, it follows from the cartesian diagram

$$\begin{tikzcd} \mathfrak{Hilb}_k(\overline{S}) \times \mathfrak{Coh}_\delta & \widetilde{\mathfrak{Hilb}}_{k,k+1}(\overline{S}) \arrow[l, "\overline{\kappa}"'] \arrow[r, "\overline{\pi}'"] & \mathfrak{Hilb}_{k+1}(\overline{S}) \times \overline{S} \\ \mathfrak{Hilb}_k \times \mathfrak{Coh}_\delta \arrow[u] & \widetilde{\mathfrak{Hilb}}_{k,k+1} \arrow[l, "\overline{\kappa}^\circ"'] \arrow[r, "\overline{\pi}^{\circ\prime}"] \arrow[u] & \mathfrak{Hilb}_{k+1} \times S \arrow[u] \end{tikzcd}$$

and open base change that there is a commutative diagram

$$\begin{tikzcd} H_*(\mathrm{Hilb}_k(\overline{S}), \mathbb{Q}) \arrow[r, "\underline{\iota}^*"] \arrow[d, "D_{1,n}(\lambda)"] & H_*(\mathrm{Hilb}_k, \mathbb{Q}) \arrow[d, "D_{1,n}(\lambda_{|S})"] \\ H_*(\mathrm{Hilb}_{k+1}(\overline{S}), \mathbb{Q}) \arrow[r, "\underline{\iota}^*"] & H_*(\mathrm{Hilb}_{k+1}, \mathbb{Q}) \end{tikzcd}$$

for any n. Since the collection of elements $D_{1,k}(\lambda)$ generates $W^-_\uparrow(\overline{S})$, and in particular $\mathfrak{h}^-_{\overline{S}}$, we deduce the following:

Corollary 7.10. *For any pure S and any pair (m, λ) we have*

$$\Phi^-_S(D_{m,0}(\lambda)) = \mathfrak{q}_m(\lambda) \in \mathrm{End}(V(S)).$$

In particular, the representation Φ^-_S of $U(\mathfrak{h}^-_S)$ is faithful.

We are now in position to prove the following:

Proposition 7.11. *For any pure S, $(\Phi^-_{\mathfrak{H}}, \mathbf{V}(S))$ is a faithful representation of $W^-_\uparrow(S)$. If S is proper then the same holds for $(\Phi^+_{\mathfrak{H}}, \mathbf{V}(S))$.*

Proof. Let $I = \mathrm{Ker}((\Phi^-_S)_{|W^-_\uparrow(S)})$. By definition, it is a two-sided ideal of $W^-_\uparrow(S)$. If non-zero, I must have a non-zero intersection with $U(\mathfrak{h}_S)$ by Lemma 3.17. But this would contradict Corollary 7.10. The second statement is proved in the same fashion. □

7.6. **Comparison between COHAs and W-algebras.** In this section we prove Theorem B. We will begin with the case of $\mathbf{H}_0(S)$, then deduce the case of $\mathbf{H}^c_0(S)$ by using the morphism $\underline{\iota}_! : \mathbf{H}^c_0(S) \to \mathbf{H}_0(\overline{S})$.

Proof of Theorem B for $\mathbf{H}_0(S)$. Let $\mathbf{H}'_0(S)$ be the subalgebra of $\mathbf{H}(S)$ generated by the subspace $\mathbf{H}_0(S)[1, -]$ and set $\widetilde{\mathbf{H}}'_0(S) = \Lambda(S) \ltimes \mathbf{H}'_0(S)$. Results of § 7.3 yield homomorphisms of algebras (we drop the index $\mathfrak{Hilb}$ for simplicity)

$$W^{\leqslant}(S) \xrightarrow{\Phi^-} \mathrm{End}(\mathbf{V}(S)) \xleftarrow{\Psi^-} \widetilde{\mathbf{H}}_0(S)^{\mathrm{op}} \tag{7.6}$$

Moreover, $\Phi^-_{|W^-(S)}$ is injective by Proposition 7.11 and for any $n\geqslant 0$, $m>0$ and $\lambda\in H^*(S,\mathbb{Q})$ we have

$$\Psi^-(\lambda u^n \cap [\mathcal{C}oh_\delta(S)]) = \Phi^-(T_n(\lambda)), \qquad \Psi^-(\psi_m(\lambda)) = \Phi^-(\phi_m(\lambda)) \tag{7.7}$$

(recall that $\mathfrak{Hilb}$ is a Hecke pattern of rank $r=1$). By (7.7), we have $\Phi^-(W^-(S)) = \Psi^-(\mathbf{H}'_0(S)^{\mathrm{op}})$. Hence, for any n,l, we have the chain of inequalities of graded dimensions

$$\dim(W^+(S)[n,l]) \leqslant \dim(\Psi(\mathbf{H}'_0(S)[n,l])) \leqslant \dim(\mathbf{H}'_0(S)[n,l]) \leqslant \dim(\mathbf{H}_0(S)[n,l]).$$

But by Theorems 1.5 and 3.5, we have $\dim(W^+(S)[n,l]) = \dim(\mathbf{H}_0(S)[n,l])$. This forces all the inequalities above to be equalities. As a consequence, $\mathbf{H}_0(S) = \mathbf{H}'_0(S)$, i.e. $\mathbf{H}_0(S)$ is generated in degree one, the map $\Psi^-_{|\mathbf{H}_0(S)}$ is injective, and in fact the morphism $T_n(\lambda)\mapsto(\lambda u^n)\cap[\mathcal{C}oh_\delta(S)]$ extends to the desired isomorphism of algebras $\Theta_S : W^+(S) \xrightarrow{\sim} \mathbf{H}_0(S)$. This isomorphism extends to $W^{\geqslant}(S) \xrightarrow{\sim} \widetilde{\mathbf{H}}_0(S)$. $\square$

Proof of Theorem B for $\mathbf{H}^c_0(S)$. Recall that S is assumed to be pure. Let us fix a smooth compactification $\iota : S\to\overline{S}$ and consider the algebra homomorphism $\iota_! : \mathbf{H}^c_0(S)\to\mathbf{H}_0(\overline{S})\simeq W^+(\overline{S})$. By Proposition 7.5, $\mathfrak{Hilb}(\overline{S})$ is a two-sided S-weak Hecke pattern, so in particular we have a (left) action $\Psi^{\pm}_{\mathfrak{H}} : \mathbf{H}^c_0(S)\to\mathrm{End}(\mathbf{V}(\overline{S}))$. Moreover, by Proposition 6.9, there is a commutative diagram

$$\begin{tikzcd} \mathbf{H}^c_0(S) \arrow[r, "\iota_!"] \arrow[dr, "\Psi^{\pm}_{\mathfrak{H}}"'] & \mathbf{H}^c_0(\overline{S}) \arrow[d, "\Psi^{\pm}_{\mathfrak{H}}"] & W^+(\overline{S}) \arrow[l, "\Theta_{\overline{S}}"', "\sim"] \arrow[dl, "\Phi^{\pm}_{\mathfrak{H}}"] \\ & \mathrm{End}(\mathbf{V}(\overline{S})) & \end{tikzcd}$$

Let us put $A=\Psi^{\pm}_{\mathfrak{H}}(\mathbf{H}^c_0(S))$. By Proposition 7.8, A contains the subalgebra generated by all operators $\mathfrak{q}_{-l}(\lambda)$ for $\lambda\in H^*_c(S,\mathbb{Q})$ as well as the collection of operators $\Phi^{\pm}_{\mathfrak{H}}(D_{1,m}(\lambda))$ for $m\geqslant 0$ and $\lambda\in H^*_c(S,\mathbb{Q})$ which arise as length one (compactly supported) Hecke operators. We know that $\Theta_{\overline{S}}$ is an isomorphism and by Proposition 7.11, the map $\Phi^{\pm}_{\mathfrak{H}}$ is injective. It follows from the definition of $W^+_\downarrow(S)$ that A contains $\Phi^{\pm}_{\mathfrak{H}}(W^+_\downarrow(S))$. In particular, the graded dimension of A is bounded below by that of $W^+_\downarrow(S)$. Since by Theorem 1.8 there is an equality between these graded dimensions, we deduce that we have a chain of isomorphisms

$$\begin{tikzcd} \mathbf{H}^c_0(S) \arrow[r, "\Psi^{\pm}_{\mathfrak{H}}"] & A \arrow[r, equal] & \Phi^{\pm}_{\mathfrak{H}}(W^+_\downarrow(S)) & W^+_\downarrow(S) \arrow[l, "\Phi^{\pm}_{\mathfrak{H}}"'] \end{tikzcd}$$

as desired. This also shows that the map $\iota_!$ is injective and concludes the proof of Theorem B for an arbitrary pure surface S. $\square$

Remark 7.12. The proof of Theorem B above goes through almost verbatim in the case when S is equipped with an action of an algebraic torus T. The only thing that we need in addition is equivariants counterparts of Theorems 1.5, 1.8, which are provided by remarks after said theorems.

The results above now yield:

Corollary 7.13. *Let S be a pure surface, X a two-sided Hecke pattern of rank r, and $X^\circ \subset X$ an open two-sided S-weak Hecke subpattern. Then there is a commutative diagram*

$$\begin{tikzcd} W^{\pm}_{\downarrow}(S) \arrow[r] \arrow[rr, bend left, "\Phi^{\pm}"] \arrow[dr, "\Phi^{\pm}_X"'] & \mathrm{End}_X(\mathbf{F}^{(r)}(\overline{S})) \arrow[r] \arrow[d, "\mathrm{ev}"] & \mathrm{End}(\mathbf{F}^{(r)}(\overline{S})) \\ & \mathrm{End}(\mathbf{V}^{\mathrm{taut}}_X) & \end{tikzcd}$$

which glues into an action of $W^{(r)}_{\downarrow\downarrow}(S)$ on $\mathbf{V}^{\mathrm{taut}}(X)$ provided that the conditions of Lemma 1.16 are satisfied.

Proof. We have an analogous diagram for $W^{\pm}(\overline{S})$ by Proposition 6.8. For the first claim, it suffices to show that $W^{\pm}_{\downarrow}(S)$ lands in $\mathrm{End}_{X^\circ}(\mathbf{F}^{(r)}(\overline{S}))$ under $\Phi^{\pm}$. This follows from Proposition 6.6 and the fact that $W^{+}_{\downarrow}(S) \simeq \mathbf{H}^{c}_{0}(S)$ as subalgebras of $W^{+}(\overline{S})$. Adding tautological classes in, à priori we get actions of $W^{0}(\overline{S}) \ltimes W^{\pm}_{\downarrow}(S)$. However, the map $\mathbf{F}^{(r)}(\overline{S}) \to \mathbf{V}^{\mathrm{taut}}_X$ factors through $\mathbf{F}^{(r)}(S)$ by Lemma 1.16, and so the two actions above glue to $W^{(r)}_{\downarrow\downarrow}(S)$. □

In particular, we have an action of $W^{(1)}_{\uparrow\downarrow}(S)$ on $V(S)$, which descends to a faithful action of $W^{(1)}_{\uparrow\downarrow,\mathrm{red}}$ by Lemma 3.17.

Remark 7.14. By Lemma 6.4, any two-sided S-weak Hecke pattern which is S-strong on the right is automatically S-strong on the left. In particular, the action of $W^{(r)}_{\downarrow\uparrow}(S)$ always has central charge $r = 0$ and lifts to the action of $W^{(0)}_{\uparrow\uparrow}(S)$. However, we expect that one can obtain actions of $W^{(r)}_{\downarrow\uparrow}(S)$ with non-zero central charge out of Hecke patterns living in other hearts of $D^bCoh(S)$.

8. Action on Higgs bundles

In this section we show how to apply the machinery developed in this paper to obtain explicit formulas for the action of Hecke operators on the homology of the stack of Higgs bundles on a smooth projective curve C.

8.1. **The stack of Higgs bundles.** Let us consider $S = T^*C$, where C is a smooth connected projective curve of genus g. Let $\overline{S} = \mathbb{P}(\Omega_C \oplus \mathcal{O}_C)$ be the projective completion of Ω_C, which is a smooth compactification of S, and let $p : \overline{S} \to C$ be the projection. We consider the (derived) stack $\mathfrak{Higgs}_{r,d}$ classifying Higgs sheaves on C of rank r and degree d, or equivalently via the BNR correspondence, coherent sheaves $\mathcal{E}$ on $\overline{S}$ whose support does not intersect $D_\infty = \overline{S}\backslash S$ and for which $p_*(\mathcal{E}) \in Coh(C)$ is of rank r and degree d. When $r = 0$ we recover the stacks $\mathfrak{Coh}_{d\delta}(S)$. We will sometimes view a Higgs sheaf as a pair $(\mathcal{F}, \theta)$ where $\mathcal{F} \in Coh(C)$ and $\theta \in \mathrm{Hom}(\mathcal{F}, \mathcal{F} \otimes \Omega_C)$. The correspondence is given by $\mathcal{E} \mapsto \mathcal{F} := p_*(\mathcal{E})$ and reads as follows on the Chern classes:

$$c_0(\mathcal{E}) = 0, \qquad c_1(\mathcal{E}) = r, \qquad c_2(\mathcal{E}) = r(r+1)(1-g) - d.$$

Note that a Higgs sheaf $\mathcal{E}$ is of dimension $\geqslant 1$ is and only if it is pure of dimension 1 if and only if the associated sheaf $\mathcal{F}$ on C is a vector bundle. We will denote by $\mathcal{H}iggs_{r,d}$ the classical truncation of $\mathfrak{Higgs}_{r,d}$.

As opposed to the case of the moduli stack of coherent sheaves on a curve, the stack $\mathfrak{Higgs}_{r,d}$ is not irreducible. Luckily, as soon as $g > 1$, the stack $\mathfrak{Higgs}^{\circ}_{r,d}$ parametrizing Higgs *bundles* of rank

r and degree d is irreducible. More precisely, denote by $\mathfrak{Higgs}_{r,d}^{tor=l}$ the locally closed substack of $\mathfrak{Higgs}_{r,d}$ parametrizing Higgs sheaves whose maximal torsion subsheaf is of degree l.

Proposition 8.1. *Assume that $g > 1$. Then the Zariski closures of $\mathfrak{Higgs}_{r,d}^{tor=l}$ for $l \geqslant 0$ form a complete set of irreducible components of $\mathfrak{Higgs}_{r,d}$. Moreover the stack $\mathcal{H}iggs_{r,d}^{tor=l}$ is of dimension $2r^2(g-1)+l+1$. In particular, we have* $\dim \mathcal{H}iggs_{r,d}^{\circ} = 2r^2(g-1)+1$.

Proof. Let us first show that the stack $\mathfrak{Higgs}_{r,d}^{tor=0}$ of Higgs *bundles* of rank r and degree d is irreducible. This can be easily deduced from [3, Section 2], we sketch the argument for the sake of completeness. Let $\mathfrak{Bun}_{r,d}$ be the stack classifying vector bundles on C of rank r and degree d. It is a smooth, irreducible stack of dimension $(g-1)r^2$, with a non-empty open substack $\mathfrak{Bun}_{r,d}^{st}$ parametrizing stable vector bundles. The canonical morphism $\pi : \mathfrak{Higgs}_{r,d}^{tor=0} \to \mathfrak{Bun}_{r,d}$ identifies $\mathfrak{Higgs}_{r,d}$ with the cotangent stack of $\mathfrak{Bun}_{r,d}$. It follows that any irreducible component of $\mathcal{H}iggs_{r,d}^{\circ}$ is of dimension at least $2r^2(g-1)+1$ (one can see this, for instance, by locally realizing $\mathcal{H}iggs_{r,d}^{\circ}$ as a symplectic quotient). On the other hand, the morphism π is representable and $\pi^{-1}(\mathcal{F}) \simeq \mathrm{Ext}^1(\mathcal{F},\mathcal{F})^*$. By Serre duality, $\dim\ \pi^{-1}(\mathcal{F}) = (g-1)r^2 + \dim\ \mathrm{End}(\mathcal{F})$. In particular, $\pi^{-1}(\mathcal{H}iggs_{r,d}^{\circ})$ is irreducible, of dimension $2(g-1)r^2+1$. To see that there is no other irreducible component, it is enough to check that for any $n>0$, the constructible substack $\mathfrak{X}_n \subset \mathfrak{Bun}_{r,d}$ parametrizing objects whose automorphism group is of dimension n, is of codimension at least n, i.e. of dimension at most $(g-1)r^2-n$. In other words, one must check that $\mathfrak{Bun}_{r,d}$ is very good in the sense of [3, Section 1]. This is done in *loc.cit.* (2.10.5) for the stack $\mathfrak{Bun}_{SL_r}$ (over a curve of genus $g>1$); the proof is easily adapted to our case.

Next, let us fix $l > 0$. Projecting a Higgs sheaf to its vector bundle quotient and torsion subsheaf yields a morphism $\mathfrak{Higgs}_{r,d}^{tor=l} \to \mathfrak{Higgs}_{r,d}^{\circ} \times \mathfrak{Higgs}_{0,l}$ which is a stack vector bundle of rank 0 (indeed, the fiber over a pair of Higgs sheaves $(\mathcal{V},\mathcal{T})$ is equal to $\mathbf{R}\,\mathrm{Hom}(\mathcal{V},\mathcal{T})[1]$ which is of perfect amplitude $[-1,0]$ and of virtual rank $\langle (r,d), l\delta\rangle = 0$). Hence $\mathcal{H}iggs_{r,d}^{tor=l}$ is irreducible, of dimension $2(g-1)r^2+l+1$. Finally, it is easy to see that the union $\mathfrak{Higgs}_{r,d}^{tor\leqslant l} = \bigsqcup_{n \leqslant l} \mathfrak{Higgs}_{r,d}^{tor=n}$ is an open substack for any n. Since the dimensions of the irreducible locally closed substacks $\mathfrak{Higgs}_{r,d}^{tor=l}$ increase with l, we deduce that the Zariski closures of each $\mathfrak{Higgs}_{r,d}^{tor=l}$ is an irreducible component of $\mathfrak{Higgs}_{r,d}$. □

Remark 8.2.

(i) For $g=0$ or 1 the situation is quite different. When $g=0$, the stack $\mathfrak{Higgs}_{r,d}^{\circ}$ has infinitely many irreducible components whose classical truncations are all of dimension $-r^2$. In that case $\mathfrak{Higgs}_{r,d}^{\circ}$ coincides with the global nilpotent cone. When $g=1$, the classical stack $\mathcal{H}iggs_{r,d}^{\circ}$ is also not irreducible, but the dimensions of the irreducible components may vary between 1 and r. Similar results hold for $\mathfrak{Higgs}_{r,d}^{tor=l}$ for any positive l.

(ii) For any g and l, the stack $\mathfrak{Higgs}_{r,d}^{tor=l}$ is of (virtual) dimension $2(g-1)r^2$. In particular, this dimension is independent of l.

8.2. **Regularity of the Hecke pattern.** For any $r \geqslant 1$, let us put $\mathfrak{Higgs}_r^{\circ} = \bigsqcup_d \mathfrak{Higgs}_{r,d}^{\circ}$. The aim of this paragraph is to prove the following result:

Proposition 8.3. *Assume that $g>1$. Then the substack $\mathfrak{Higgs}_r^{\circ}$ is a regular S-strong two-sided Hecke pattern on $S = T^*C$.*

Proof. The fact that $\mathfrak{Higgs}_r^\circ$ is an S-strong two-sided Hecke pattern follows from Example 6.3. Since C is of genus $g>1$, $\mathcal{H}iggs_{r,d}^\circ$ is irreducible and of dimension $2(g-1)r^2+1$ for any d. Let us fix $d\in\mathbb{Z}$ and set $\alpha=(r,d),\gamma=\alpha+\delta=(r,d+1)$. We write $\mathcal{E}_\eta$ for the tautological sheaf on $\mathfrak{Higgs}_\eta\times S$. Let $\mathfrak{U}$ be any finite type open substack of $\mathfrak{Higgs}_\gamma^\circ\times S$, let $\mathcal{E}_{-1}\to\mathcal{E}_0$ be a presentation of $\mathcal{E}_\gamma$ as a perfect complex, and let s be the associated section. The (virtual) rank of $\mathcal{E}_\gamma$ being equal to zero, the (virtual) dimension of the map $\pi_{\delta,\alpha}:\widetilde{\mathfrak{Higgs}}_{\delta;\alpha}\to\mathfrak{Higgs}_\gamma$ is equal to 1. It follows that $\dim(\widetilde{\mathcal{H}iggs}^\circ_{\delta;\alpha})\geqslant\dim(\mathcal{H}iggs_\gamma^\circ)+1$. We will show that s is regular by proving that $\dim(\widetilde{\mathcal{H}iggs}^\circ_{\delta;\alpha})\leqslant\dim(\mathcal{H}iggs_\gamma^\circ)+1$, which will in fact imply equality. This will also give the regularity of the section s', see (2.10). For this, let us denote by

$$\mu:\mathcal{H}iggs_\gamma^\circ\to\mathcal{A}:=\bigoplus_{i=1}^r H^0(C,\Omega_C^{\otimes i})$$

the Hitchin map. If $a\in\mathcal{A}$ we write C_a for the corresponding spectral curve, and we denote by $\mathcal{C}\subset\mathcal{A}\times S$ the universal spectral curve. For $i\geqslant 1$, we set

$$\mathcal{R}_i:=\{(a,(x,\xi))\in\mathcal{C}\mid p_{|C_a}:C_a\to C\text{ is ramified of order } i \text{ at } (x,\xi)\}\subset\mathcal{A}\times S.$$

We denote by $\mathcal{A}_i^{(x,\xi)}\subset\mathcal{A}$ the fiber over (x,ξ) of the projection $\mathcal{R}_i\to S$. Note that $\mathcal{R}_{\geqslant i}=\bigsqcup_{j\geqslant i}\mathcal{R}_j$ is closed and $\mathcal{R}_{\geqslant 1}=\mathcal{C}$. We set $\overline{\mathcal{R}}_i=(\mu\times\mathrm{Id})^{-1}(\mathcal{R}_i)$.

Lemma 8.4. *For any $i\geqslant 1$ and $(x,\xi)\in S$ we have* $\operatorname{codim}_{\mathcal{A}}(\mathcal{A}_i^{(x,\xi)})=i$.

Proof. Since Ω_C is base-point free, the evaluation at $x\in C$ yields a surjective linear map $i_x^*:\mathcal{A}\to\bigoplus_{i=1}^r T_x^*(C)^{\otimes i}$. The locally closed subset $\mathcal{A}_i^{(x,\xi)}$ is the inverse image of the subset of degree $r-1$ polynomials in one variable $y\in T_x^*C$ vanishing with order i at ξ. This condition defines a subset of codimension i. □

Since $g>1$ the map μ is flat by [14, Cor. 9]. It follows that $\operatorname{codim}_{\mathcal{H}iggs_\gamma^\circ}(\mu^{-1}(\mathcal{A}_i^{(x,\xi)}))=i$ hence $\operatorname{codim}_{\mathcal{H}iggs_\gamma^\circ\times S}(\mathcal{R}_i))=i$. Recall the morphism $\pi:\widetilde{\mathcal{H}iggs}^\circ_{\delta,\alpha}\to\mathcal{H}iggs_\gamma^\circ\times S$ which sends a filtration $\mathcal{H}\subset\mathcal{E}$ to $(\mathcal{G},\operatorname{supp}(\mathcal{E}/\mathcal{H}))$. By construction, π lands in the closed substack $\overline{\mathcal{R}}_{\geqslant 1}$. For any Higgs bundle $\mathcal{F}$ whose support is ramified at (x,ξ) of order i we have $\dim(\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_{(x,\xi)}))\leqslant i$. It follows that $\dim(\pi^{-1}(\mathcal{E},(x,\xi)))\leqslant i-1$ if $(\mathcal{E},(x,\xi)))\in\overline{\mathcal{R}}_i$, hence

$$\dim(\pi^{-1}(\overline{\mathcal{R}}_i))\leqslant\dim(\overline{\mathcal{R}}_i)+i-1=\dim(\mathcal{H}iggs_\gamma^\circ\times S)-i+i-1=\dim(\mathcal{H}iggs_\gamma^\circ)+1.$$

Since this is true for all $i=1,\ldots,r$, we get the desired dimension estimate. □

Remark 8.5.

(i) Refining the above dimension estimates, one can show that for $g>1$ the Hecke correspondence $\widetilde{\mathcal{H}iggs}^\circ_{\delta,\alpha}$ is irreducible.

(ii) When $g=0,1$, $\mathfrak{Higgs}_r^\circ$ is *not* a regular Hecke pattern. For instance, if $gcd(r,d)=1$ and $r>1$ then a generic (stable) Higgs sheaf $\mathcal{E}=(\mathcal{F},\theta)$ has a scalar Higgs field, hence the fiber of $p_{\delta,\alpha}$ over $\mathcal{E}$ is of dimension $r-1$ (and thus $\widetilde{\mathcal{H}iggs}^\circ_{\delta,\alpha}$ has a component of dimension $r+\dim(\mathcal{H}iggs_{r,d}^{st})$ lying over $\mathcal{H}iggs_{r,d}^{st}$).

(iii) A similar argument also shows that the stacks of $\mathfrak{Higgs}^{\mathcal{L},\circ}$ of $\mathcal{L}$-twisted Higgs sheaves with $\deg(\mathcal{L})>2(g-1)$ form a regular two-sided Hecke pattern on $\operatorname{Tot}(\mathcal{L})$.

8.3. **Action on tautological classes.** Unlike the case of the stack of coherent sheaves on C [17], the homology of the stack of Higgs sheaves on C is *not* generated by tautological classes. Note, however that by Markman's theorem [27] this is the case after restriction to the stack of *stable* Higgs sheaves in the case $(r,d)=1$. This motivates the study of Hecke operators on the subspace of tautological classes of $H_*(\mathfrak{Higgs}^\circ,\mathbb{Q})$. By Propostion 6.8 there is an action of $W^{(0)}_{\uparrow\uparrow}(T^*C)$ on $\mathbf{V}^{\mathrm{taut}}_{\mathfrak{Higgs}^\circ}$. Moreover since $H^*(S,\mathbb{Q})=H^*(C,\mathbb{Q})$, by degree reasons we have $c_1^2=c_2=0$ and $c_1\Delta_S=0$. Analogously to Corollary 5.9 we conclude that $W^{(0)}_{\uparrow\uparrow}(T^*C)\simeq U(\mathfrak{w}_{T^*C})$ where

$$\mathfrak{w}_{T^*C}=\bigoplus_{\substack{m,n\geqslant 0\\(m,n)\neq 0}}\bigoplus_{\gamma\in\Pi}\mathbb{Q}D_{m,n}(\gamma),\qquad \Pi=\{1,\gamma_1,\dots,\gamma_{2g},\omega\}$$

with $\gamma_1,\dots,\gamma_{2g}$ a symplectic basis of $H^1(C,\mathbb{Q})$, with relations

$$[D_{m,n}(\gamma),D_{m',n'}(\gamma')]=(mn'-m'n)D_{m+m',n+n'-1}(\gamma\gamma')$$

for all tuples $(m,m',n,n',\gamma,\gamma')$.

We should stress that Corollary 4.8 does not apply in this case, because Higgs bundles have zero rank as sheaves on T^*C. This is also evidenced by the fact that, contrary to the case of Hilbert scheme, the action of both W^0 and $W^+_\uparrow$ on $\mathbf{V}^{\mathrm{taut}}_{\mathfrak{Higgs}^\circ}$ is not faithful. However, this non-faithfulness is a feature; it is related to the existence of a certain "rational" degeneration of $W^{\geqslant}_{\uparrow}(T^*C)$, which was used in [18] to prove the $P=W$ conjecture of de Cataldo-Hausel-Migliorini; see also § 8.4 below.

Example 8.6. Consider Higgs bundles of rank 1 on a curve C of genus $g>1$. In this case $\mathfrak{Higgs}^\circ_{1,d}\simeq \mathrm{Jac}^dC\times B\mathbb{G}_m\times H^0(C,\Omega_C)$. We have the Poincaré line bundle $\mathcal{P}$ on $(\mathrm{Jac}^d\times B\mathbb{G}_m)\times C$, and it is known that

$$c_1(\mathcal{P})=u\otimes 1+\sum_\gamma b_\gamma\otimes\overline{\gamma}+1\otimes[\mathrm{pt}],$$

where u is the generator of $H^*(B\mathbb{G}_m,\mathbb{Q})$, and b_γ form the basis of $H^1(C,\mathbb{Q})\subset\Lambda^\bullet H^1(C,\mathbb{Q})\simeq H^*(\mathrm{Jac}_d,\mathbb{Q})$. Starting from this observation, an easy but tedious computation shows that

$$\psi_1([\mathrm{pt}])=a,\quad \psi_1(\gamma)=b_\gamma,\quad \psi_0([\mathrm{pt}])=1,$$

where the classes ψ_i are obtained from the universal sheaf on a natural compactification of $\mathfrak{Higgs}^\circ_{1,d}\times T^*C$ in accordance with § 1.6. This means that the cohomology of $\mathfrak{Higgs}^\circ_{1,d}$ is generated by the classes ψ_1. In particular, $\psi_2(1)$ can be expressed in terms of these classes for all d at once, so that the action of $W^0(T^*C)$ on $\mathbf{V}^{\mathrm{taut}}_{\mathfrak{Higgs}^\circ_1}$ is not faithful. Applying $\mathrm{Ad}_{D_{10}(1)}$ twice to such an expression, we see that $\mathfrak{q}_2(1)$ expresses in terms of $\mathfrak{q}_1$'s, and so the action of $W^+_\uparrow(T^*C)$ is not faithful either.

8.4. **One-dimensional sheaves on K3 surfaces.** Let us now briefly consider another example. Let S be a K3 surface, and fix a smooth curve $C\subset S$ of genus $g>1$ which is a very ample divisor. We denote by $\mathfrak{Mukai}_r$ the moduli stack of coherent sheaves $\mathcal{E}$ of purely 1-dimensional support on S, such that $c_1(\mathcal{E})=rC$. It is well known [30, 7] that $\mathfrak{Mukai}_r$ behaves similarly to $\mathfrak{Higgs}^\circ_r$; in particular, it admits a morphism $\mathfrak{Mukai}_r\to\mathbb{P}^{r^2(g-1)+1}$ with properties analogous to the Hitchin map $\mathfrak{Higgs}^\circ_r\to\mathcal{A}$.

It follows again from Example 6.3 that $\mathfrak{Mukai}_r$ is a two-sided Hecke pattern. While we expect that $\mathfrak{Mukai}_r$ is in fact regular, let us for simplicity's sake restrict our attention to the subspace $\mathbf{V}^{\mathrm{vtaut}}_{\mathfrak{Mukai}_r}\subset H_*(\mathfrak{Mukai}_r,\mathbb{Q})$ of classes obtained by capping the virtual fundamental class $[\mathfrak{Mukai}_r]_{\mathrm{vir}}$ with tautological classes. Invoking Proposition 2.8, we can apply an analogue of Proposition 6.8 obtain an action of $W^{(0)}(S)$ on $\mathbf{V}^{\mathrm{vtaut}}_{\mathfrak{Mukai}_r}$. While $c_1=0$, it is known that $s_2=-24[\mathrm{pt}]$. Thus we are placed in the semi-deformed situation of § 3.8, where we can take $q = (-\int_S C^2/24)^{-1/2}C$ up to passing to a finite extension of $\mathbb{Q}$ in the coefficients.

Similarly to Example 8.6, it is easy to check that the action of $W^{(0)}(S)$, or indeed of $W^+(S)$ on $\mathbf{V}^{\mathrm{vtaut}}_{\mathfrak{Mukai}_r}$ is not faithful. However, with an argument analogous to the one found in [18, Section 6], one can show that the action of $W^{\geqslant}(S)$ degenerates to the action of the algebra $U(\mathfrak{w}^{\log}_S)$, where $\mathfrak{w}^{\log}_S$ has basis $x^m\partial^n\lambda$, $m,n\in\mathbb{N}$, $\lambda\in H^*(S,\mathbb{Q})$, and the Lie bracket is given by

$$[x^m\partial^a\lambda, x^n\partial^b\mu] = \sum_{i\geqslant 1} i!\left(\binom{a}{i}\binom{n}{i} - \binom{b}{i}\binom{m}{i}\right) x^{m+n-i}\partial^{n+b-i}q^{i-1}\lambda\mu.$$

One way to think of this is that $\mathfrak{w}_S$ looks like the Lie algebra of differential operators on $\mathbb{C}^*$ with coefficients in $H^*(S,\mathbb{Q})$, and so rational degeneration should look like the Lie algebra of differential operators on $\mathbb{C}$.

Note that the defining relations of $\mathfrak{w}^{\log}_S$ imply that $\{x^2/2, x\partial + 2q, \partial^2/2\}$ is an $\mathfrak{sl}_2$-triple, which should control the perverse filtration on the cohomology of the stable locus of $\mathfrak{Mukai}_r$. We plan to return to this in the future work.

9. Some conjectures

9.1. **Action beyond tautological classes.** Let us informally summarize what we proved, omitting most adjectives. First, given any two-sided Hecke pattern X, we have two actions of $W^+(S)$ on the Borel-Moore homology of X. Second, for a regular Hecke pattern these two actions glue to an action of $W^{(\mathbf{c})}(S)$ *on the subspace of tautological classes.* Somewhat frustratingly, our methods do not prove the relation (4.8) for non-tautological classes. This reflects the fact that $W^{(\mathbf{c})}(S)$ is supposed to be related to a Drinfeld double of $\mathbf{H}_0(S)$, which has not yet been defined. In view of Proposition 2.8, it also seems natural drop the regularity condition.

Conjecture 9.1. *For any two-sided Hecke pattern X of rank r, there is an action of $W_{\star,\star}(S)$ on $H_*(X)$ for appropriate $\star\in\{\uparrow,\downarrow\}$.*

Our main theorems were proved under the assumption that S has pure cohomology, but we conjecture that they hold without this assumption. The reason for this assumption is that we need $\mathbf{V}^{\mathrm{taut}}(\overline{S})$ to be a faithful representation of $U(\mathfrak{h}_{\overline{S}})$ in order to compare the deformed W-algebra with the COHA as subalgebras of $\mathrm{End}(\mathbf{V}^{\mathrm{taut}}(\overline{S}))$. We plan to address this problem in future work by considering another family of tautological classes on $\mathrm{Hilb}(S)$, obtained from the correspondence

$$S \longleftarrow Q_n \longrightarrow \mathrm{Hilb}_n(S)$$

where Q_n is the universal subscheme.

9.2. **W-algebras for 3-dimensional varieties.** Let S be a smooth surface. The Borel-Moore homology of the stack $\mathcal{C}oh^0(S)$ is isomorphic to the critical cohomology $H^*_{\mathrm{crit}}(\mathcal{C}oh^0(M))$ of the stack of finite length sheaves $\mathcal{C}oh^0(M)$ on the Calabi-Yau *threefold* $M=\mathrm{Tot}_S(K_S)$ by dimensional reduction [21]. For any 3-Calabi-Yau M, the space $\mathbf{H}^{\mathrm{crit}}_0(M) = H^*_{\mathrm{crit}}(\mathcal{C}oh^0(M))$ is an associative algebra, called critical COHA, at least admitting Joyce's conjectures [1, Conjecture 5.22]. We expect that an isomorphism similar to the one of Theorem B should hold for critical CoHAs as well:

Conjecture 9.2. *Let $W^+(M)$ be an algebra generated by $T_n(\lambda)$, $n\geqslant 0$, $\lambda\in H^*(M)$, modulo the relations* (b), (e)-(g) *of Definition 3.1, where we replace s_2 by $c_2(TM)$, and $c_1\Delta_S$ by Δ_M. Then $\mathbf{H}^{\mathrm{crit}}_0(M)\simeq W^+(M)$.*

One may wonder if a similar statement can be made without Calabi-Yau condition. Our preliminary computations suggest that the analogues of relations (f), (g) become significantly more complicated. We hope to return to this in future work.

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Appendix A. Borel-Moore homology for derived stacks

A derived stack X is a functor $R \mapsto X(R)$ assigning an ∞-groupoid to every simplicial commutative ring, that satisfies étale hyperdescent. We say that X is 1-Artin if its diagonal is representable by an algebraic space and if there is a scheme Y with a smooth and surjective morphism $Y \to X$. This morphism is called a smooth atlas for X. Unless specified otherwise, in this work all derived stacks are supposed to be 1-Artin, and locally quotient stacks of finite type. We will work over the ground field $\mathbb{C}$.

A.1. **Dualizing complexes and virtual classes.** We will use the formalism introduced in [20], to which we refer for details (see also [35]). In particular, for any derived stack X there is an ∞-category $\mathrm{Sh}_{\mathbb{Q}}(X)$ of constructible $\mathbb{Q}$-sheaves on X, and these satisfy a six-functor formalism (see [20, Thm. A.5]). The dualizing complex is defined as $\mathbb{D}_X = p^!\mathbb{Q}$ where $p : X \to \mathrm{Spec}(\mathbb{C})$ is the map to the point. The sheaf of Borel-Moore chains on X is

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X, \mathbb{Q}) = p_* p^! \mathbb{Q} = p_* \mathbb{D}_X \in \mathrm{Sh}_{\mathbb{Q}}(\mathrm{Spec}(\mathbb{C})) = D(\mathbb{Q}\text{-mod}).$$

The Borel-Moore homology is obtained as usual by taking derived global sections $H_i(X, \mathbb{Q}) = H_i^{\mathrm{BM}}(X, \mathbb{Q}) = H^{-i}(\mathrm{C}^{\mathrm{BM}}_\bullet(X, \mathbb{Q}))$. Likewise, the sheaf of cochains and the cohomology groups are defined as

$$\mathrm{C}^\bullet(X, \mathbb{Q}) = p_* p^* \mathbb{Q} = p_* \mathbb{Q}_X, \qquad H^i(X, \mathbb{Q}) = H^i(\mathrm{C}^\bullet(X, \mathbb{Q})).$$

These satisfy all the usual properties, see [20, Section 2]. In addition, Borel-Moore homology is insensitive to the derived structure, in the sense that the direct image map $H_i(X^{cl}, \mathbb{Q}) \to H_i(X, \mathbb{Q})$ is an isomorphism for any X and i. Of crucial importance for us are the notions of Gysin pullback and virtual fundamental classes for quasi-smooth morphisms. Let $f : X \to Y$ be a quasi-smooth morphism of dimension d. There is a map $f^! : H_i(Y, \mathbb{Q}) \to H_{i+2d}(X, \mathbb{Q})$ which is in fact induced by a morphism

$$[f]_{vir} : f^* \mathbb{D}_Y \to f^! \mathbb{D}_Y[-2d] = \mathbb{D}_X[-2d].$$

In particular, when X is itself quasi-smooth then the (virtual) fundamental class of X is defined as $[X] = p^!(1) \in H_{2\dim_X}(X, \mathbb{Q})$. Note that if X is smooth and classical then $[X]$ is just the usual fundamental class; in general, $[X]$ and $[X^{cl}]$ differ –in fact they typically live in different homological degrees of $H_\bullet(X, \mathbb{Q}) = H_\bullet(X^{cl}, \mathbb{Q})$.

We collect here some of the basic properties of derived pullbacks which we will use.

Proposition A.1. *Let $f : X \to Y$ be a quasi-smooth morphism of derived stacks. The following hold:*

(a) *If f is an open embedding then $f^! = f^*$,*

(b) *If $g : Y \to Z$ is quasi-smooth then $[g]_{vir}[f]_{vir} = [gf]_{vir} : g^*f^*\mathbb{D}_Z \to \mathbb{D}_X[-2\dim(f)-2\dim(g)]$; in particular, $(gf)^! = g^!f^! : H_*(Z,\mathbb{Q}) \to H_{*+2\dim(f)+2\dim(g)}(X,\mathbb{Q})$, and hence $f^!([Y]) = [X]$,*

(c) *Compatibility with cap product: for any $c \in H^*(Y,\mathbb{Q}), c' \in H_*(Y,\mathbb{Q})$ we have $f^!(c \cap c') = f^*(c) \cap f^!(c')$,*

(d) *Assume furthermore that f is proper representable, of finite Tor-amplitude, so that $f_* : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ is well-defined. Then the following projection formula holds: for any class $c \in H^*(X,\mathbb{Q})$ and any $\alpha \in H_*(Y,\mathbb{Q})$, we have $f_!(c \cap f^!(\alpha)) = f_*(c) \cap \alpha$.*

(e) *Let*

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"] & Y \arrow[d, "g"] \\ X' \arrow[r, "f"] & Y' \end{tikzcd}$$

be a cartesian diagram of derived stacks with f quasi-smooth and g proper. Then the proper base change formula holds: $\overline{g}_!\overline{f}^! = f^!g_! : H_*(Y,\mathbb{Q}) \to H_*(X',\mathbb{Q})$.

A.2. **Relative and hyperbolic homology.** Let X be a derived stack, S a scheme and let $\pi : X \to S$ be a morphism. Let us also denote by $p : S \to \mathrm{pt}$ the projection. We define the space of relative Borel-Moore chains on X/S as

$$C^{BM}_*(X/S,\mathbb{Q}) = \pi_*\mathbb{D}_X \in D^b(S),$$

and we define the S-hyperbolic, or simply hyperbolic homology, as

$$H_i(X/S,\mathbb{Q}) = H^{-i}(p_!\pi_*\mathbb{D}_X).$$

Example A.2. Taking $X = S$ we get that $H_*(S/S,\mathbb{Q})$ is the usual homology of S. On the other hand, if S is a point then $H_*(X/S,\mathbb{Q})$ is the Borel-Moore homology of X.

When S is understood from the context, we might abbreviate $H^c_i(X,\mathbb{Q}) = H_i(X/S,\mathbb{Q})$. There is a canonical isomorphism $H^c_*(X^{cl},\mathbb{Q}) \xrightarrow{\sim} H^c_*(X,\mathbb{Q})$. Dually, we define the space of relative cochains on X/S as

$$C^*(X/S,\mathbb{Q}) = \pi_*\mathbb{Q}_X \in D^b(S),$$

and we define the S-hyperbolic, or simply hyperbolic cohomology as

$$H^i(X/S,\mathbb{Q}) = H^i(p_!\pi_*\mathbb{Q}_X).$$

There is a natural morphism $\pi^* : H^*_c(S,\mathbb{Q}) \to H^*(X/S,\mathbb{Q})$.

Lemma A.3. *There is a natural action $\cap$ of $H^*(X,\mathbb{Q})$ on $H_*(X/S,\mathbb{Q})$:*

$$\cap : H^i(X,\mathbb{Q}) \otimes H_j(X/S,\mathbb{Q}) \to H_{j-i}(X/S,\mathbb{Q})$$

for all i, j. Likewise, there is a natural cap product

$$\cap : H^i(X/S,\mathbb{Q}) \otimes H_j(X,\mathbb{Q}) \to H_{j-i}(X/S,\mathbb{Q})$$

for all i, j.

Proof. Consider the following diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\Delta"] & S \arrow[d, "\Delta'"] \arrow[dr, "\mathrm{Id}"] & & \\ X \times X \arrow[r, "\pi\times\pi"] & S \times S \arrow[r, "pr_2"] & S \arrow[r, "p"] & \{pt\} \end{tikzcd}$$

where $pr_2 : S \times S$ is the second projection. We have $\Delta^*(\mathbb{Q}_X \boxtimes \mathbb{D}_X) = \mathbb{D}_X$. The adjunction $\mathrm{Id} \to \Delta_*\Delta^*$ yields a map $\mathbb{Q}_X \boxtimes \mathbb{D}_X \to \Delta_*\mathbb{D}_X$. Applying $(\pi\times\pi)_*$ we get a morphism $(\pi\times\pi)_*(\mathbb{Q}_X \boxtimes \mathbb{D}_X) \to \Delta'_*\pi_*\mathbb{D}_X$, hence a map $p_! pr_{2*}(\pi\times\pi)_*(\mathbb{Q}_X \boxtimes \mathbb{D}_X) \to p_!\pi_*\mathbb{D}_X$. The construction of the second cap product is similar. □

Lemma A.4. *Let X, Y be derived stacks, S a scheme. Let $\pi_X : X \to S, \pi_Y : Y \to S$ and $f : X \to Y$ be morphisms such that $\pi_Y \circ f = \pi_X$. Then*

(a) *Assume that f is quasi-smooth of relative dimension d. Then there is a canonical morphism $f^* : H_i(Y/S, \mathbb{Q}) \to H_{i+2d}(X/S, \mathbb{Q})$,*

(b) *Assume that f is proper. Then there is a canonical morphism $f_* : H_i(X/S, \mathbb{Q}) \to H_i(Y/S, \mathbb{Q})$. Moreover, the projection formula holds, i.e. for any $c \in H^*(Y, \mathbb{Q})$ and any $x \in H_*(X/S)$ we have $f_*(f^*(c) \cap x) = c \cap f_*(x)$,*

(c) *For any cartesian diagram of S-stacks*

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"] & Y \arrow[d, "g"] \\ X' \arrow[r, "f"] & Y' \end{tikzcd}$$

with f smooth and g proper we have $\overline{g}_!\overline{f}^! = f^! g_! : H_*(Y/S, \mathbb{Q}) \to H_*(X'/S, \mathbb{Q})$.

Proof. Let $p : S \to \mathrm{pt}$ be the projection. Assume that f is quasi-smooth of dimension d. The virtual fundamental class gives a morphism $f^*\mathbb{D}_Y \to \mathbb{D}_X[-2d]$. Applying $p_!\pi_{X*}$ and using the adjunction $\mathrm{Id} \to f_*f^*$ yields a canonical morphism $p_!\pi_{Y*}\mathbb{D}_Y \to p_!\pi_{X*}\mathbb{D}_X[-2d]$, proving (a). The construction of the direct image morphism follows directly from the adjunction $f_!f^! \to \mathrm{Id}$. We leave the proof of the projection formula to the reader. It boils down to the commutativity of the following diagram

$$\begin{tikzcd} f_*\mathbb{D}_X \boxtimes \mathbb{Q}_Y \arrow[r, "\mathbb{Q}_Y \to f_*\mathbb{Q}_X"] \arrow[d, "f_*\mathbb{D}_X \to \mathbb{D}_Y"'] & f_*\mathbb{D}_X \boxtimes f_*\mathbb{Q}_X \arrow[r, "1\to\Delta_*\Delta^*"] & (f\times f)_*\Delta_*\mathbb{D}_X = \Delta_* f_*\mathbb{D}_X \arrow[d, "f_*\mathbb{D}_X \to \mathbb{D}_Y"] \\ \mathbb{D}_Y \boxtimes \mathbb{Q}_Y \arrow[rr, "1\to\Delta_*\Delta^*"] & & \Delta_*\mathbb{D}_Y \end{tikzcd}$$

The proper base change statement (c) is obtained by taking compactly supported cohomology of the proper base change over the base S. □

We will also need to consider base change operations associated to an open immersion $\iota : S^\circ \to S$. For $\pi_X : X \to S$ a derived S-stack we set $X^\circ = X \times_S S^\circ$ and denote by $\iota_X : X^\circ \to X$ and $\pi_X^\circ : X^\circ \to S^\circ$ the induced maps. We define a pushforward map $i_{X!} : H_*(X^\circ/S^\circ, \mathbb{Q}) \to H_*(X/S, \mathbb{Q})$ as the following composition of identifications and morphisms:

$$p_!^\circ \pi_{X*}^\circ \mathbb{D}_{X^\circ} = p_!^\circ \pi_{X*}^\circ i_X^! \mathbb{D}_X = p_!^\circ i_S^! \pi_{X*}\mathbb{D}_X = p_! i_{S!} i_S^! \pi_{X*}\mathbb{D}_X \xrightarrow{i_{S!}i_S^! \to 1} p_!\pi_{X*}\mathbb{D}_X.$$

Example A.5. Assume that S is proper. Then the composition $\iota^*\iota_! : H^c_*(S^\circ, \mathbb{Q}) = H_*(S^\circ/S^\circ, \mathbb{Q}) \to H_*(S/S, \mathbb{Q}) = H_*(S, \mathbb{Q}) \to H_*(S^\circ, \mathbb{Q})$ is the canonical morphism from usual homology to Borel-Moore homology.

Proposition A.6. *Let S°, S be as above. Let $\pi_X : X \to S, \pi_Y : Y \to S$ be two derived S-stacks, $f : X \to Y$ a morphism of S-stacks and let $f^\circ : X^\circ \to Y^\circ$ be the base change of f. The following hold:*

(a) *If f is quasi-smooth then $f^! i_{Y!} = i_{X!} f^{\circ !} : H_*(Y^\circ/S^\circ, \mathbb{Q}) \to H_{*+2\dim(f)}(X/S, \mathbb{Q})$,*

(b) *If f is proper then $f_! i_{X!} = i_{Y!} f^\circ_! : H_*(X^\circ/S^\circ, \mathbb{Q}) \to H_*(Y/S, \mathbb{Q})$.*

Proof. This follows from some tedious but unimaginative diagram chasing (see also [35, §3, 4] where similar results are proven in a dual context). □

References

[1] Amorim, L., Ben-Bassat, O., *Perversely categorified Lagrangian correspondences*, Adv. Theor. Math. Phys. **21**, No. 2, 289–381 (2017).

[2] Arbesfeld, N., Schiffmann, O., *A presentation of the deformed $W_{1+\infty}$ algebra*, Symmetries, integrable systems and representations, 1–13. Springer Proc. Math. Stat., **40**, (2013).

[3] Beilinson, A., Drinfeld, V., *Quantization of Hitchin's integrable system and Hecke eigensheaves*, available at https://math.uchicago.edu/∼drinfeld/langlands/QuantizationHitchin.pdf

[4] Davison, B., *The critical CoHA of a quiver with potential*, Q. J. Math. **68** (2017), no.2, 635–703.

[5] Davison, B., *Purity in 2CY categories*, arXiv:2106.07692.

[6] Davison, B., *Affine BPS algebras, W algebras, and the cohomological Hall algebra of $\mathbb{A}^2$*, arXiv:2209.05971.

[7] Donagi, R., Ein, L., Lazarsfeld, R., *Nilpotent cones and sheaves on K3 surfaces*, Contemp. Math. **207** (1997), 51–61.

[8] Davison, B., Hennecart, L., Schlegel Mejia, S., *BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks*, arXiv:2212.07668.

[9] Davison, B., Hennecart, L., Schlegel Mejia, S., *BPS Lie algebras and generalised Kac–Moody algebras from 2-Calabi-Yau categories*, arXiv:2303.12592.

[10] Deligne, P., Théorie de Hodge II, Publ. Math. IHES **40** (1971), 5–57.

[11] Diaconescu, E., Porta, M., Sala, F., *Cohomological Hall algebras and their representations via torsion pairs*, arXiv:2207.08926.

[12] Davison, B., Kinjo, T., *in preparation.*

[13] Feigin, B., Tsymbaliuk, A., *Heisenberg action in the equivariant K-theory of Hilbert schemes via Shuffle Algebra*, Kyoto J. Math. **51** (2011), no. 4, 831–854.

[14] Ginzburg, V., *The global nilpotent variety is lagrangian*, Duke Math. J. **109** (2001), no. 3, 511–519.

[15] Goresky, M., Kottwitz, R., MacPherson R., *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83.

[16] Gottsche, L., Soergel, W., *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. **296** (1993), 235–245.

[17] Heinloth, J., *Cohomology of the moduli stack of coherent sheaves on a curve*, Geometry and arithmetic, EMS Series of Congress Reports (2012), 165–171.

[18] Hausel, T., Mellit, A., Minets, A., Schiffmann, O., *$P = W$ via $\mathcal{H}_2$*, arXiv:2209.05429.

[19] Jiang, Q., *Derived projectivization of complexes*, arXiv:2202.11636.

[20] Khan, A.A., *Virtual fundamental classes of derived stacks I*, arXiv:1909.01332.

[21] Kinjo, T., *Dimensional reduction in cohomological Donaldson-Thomas theory*, Compos. Math. **158**, No. 1, 123–167 (2022).

[22] Kapranov, M., Vasserot, E., *The cohomological Hall algebra of a surface and factorization cohomology*, J. Eur. Math. Soc. **25** (2023), no. 11, pp. 4221–4289.

[23] Lehn, M., *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. **136** (1999), no. 1, 157–207.

[24] Lehn, M., Sorger, C., *Symmetric groups and the cup product on the cohomology of Hilbert schemes*, Duke Math. J. **110** (2001), no. 2, 345–357.

[25] Li, W.-P., Qin, Z., Wang, W., *Hilbert schemes and W-algebras*, Intern. Math. Res. Notices **27** (2002), 1427–1456.

[26] Macdonald, I., *Symmetric functions and Hall polynomials*, Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[27] Markman, E., *Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces*, J. Reine Angew. Math. **544** (2002), 61–82.
[28] Maulik, D., Okounkov, A., *Quantum Groups and Quantum Cohomology*, Astérisque(2019), no. **408**, ix+209 pp.
[29] Minets, A., *Cohomological Hall algebras for Higgs torsion sheaves, moduli of triples and sheaves on surfaces*, Sel. Math., New Ser. **26**, No. 2, Paper No. 30, 67 p. (2020).
[30] Mukai, S., *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101–116.
[31] Nakajima, H., *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.
[32] Negut, A., *Hecke correspondences for smooth moduli spaces of sheaves*, Publ. Math. Inst. Hautes Études Sci. **135** (2022), 337–418.
[33] Negut, A., *Shuffle algebras associated to surfaces*, Sel. Math. New Ser. **25**, 36 (2019).
[34] Porta, M., Sala, F., *Two-dimensional categorified Hall algebras*, J. Eur. Math. Soc. **25** (2023), no. 3, pp. 1113–1205.
[35] Porta, M., Yue Yu, T., *Non-archimedean Gromov-Witten invariants*, arXiv:2209.13176.
[36] Qin, Z., *Hilbert Schemes of Points and Infinite Dimensional Lie Algebras*, Mathematical Surveys and Monographs, 228. American Mathematical Society, Providence, RI, 2018.
[37] Sala, F., Schiffmann, O., *Cohomological Hall algebra of Higgs sheaves on a curve*, Algebr. Geom. **7** (2020), no. 3, 346–376.
[38] Schiffmann, O., Vasserot, E., *The elliptic Hall algebra and the K-theory of the Hilbert scheme of* $\mathbb{A}^2$, Duke Math. J. **162** (2013), no. 2, 279–366.
[39] Schiffmann, O., Vasserot, E., *Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on* $\mathbb{A}^2$, Publ. Math. Inst. Hautes Études Sci. **118** (2013), 213–342.
[40] Schürg, T, Toën B., Vezzosi G., *Derived algebra geomety, detrminants of perfect complexes, and applications to obstruction theories for maps and complexes*, J. reine angew. Math. **702** (2015), 1–40.
[41] Shan, P., Varagnolo, M., Vasserot, E., *On the center of quiver Hecke algebras*, Duke Math. J. **166** (2017), no.6, 1005–1101.
[42] Tsymbaliuk, A., *The affine Yangian of* $\mathfrak{gl}_1$ *revisited*, Adv. Math. **304** (2017), 583–645.
[43] Vasserot, E., *Sur l'anneau de cohomologie du schéma de Hilbert de* $\mathbb{C}^2$, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 1, 7–12.
[44] Yang, Y., Zhao, G., *The cohomological Hall algebra of a preprojective algebra*, Proc. Lond. Math. Soc. (3) **116**, No. 5, 1029–1074 (2018).
[45] Zhao, Y., *On the K-theoretic Hall algebra of a surface*, Int. Math. Res. Not. IMRN 2021, no. **6**, 4445–4486.

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