SCHURIFICATION OF POLYNOMIAL QUANTUM WREATH PRODUCTS

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ABSTRACT. We study the Schur algebra counterpart of a vast class of quantum wreath products. This is achieved by developing a theory of twisted convolution algebras, inspired by geometric intuition. In parallel, we provide an algebraic Schurification via a Kashiwara–Miwa–Stern-type action on a tensor space. We give a uniform proof of Schur duality, and construct explicit bases of the new Schur algebras. This provides new results for, among other examples, Vignéras' pro-p Iwahori Hecke algebras of type A, degenerate affine Hecke algebras, Kleshchev–Muth's affine zigzag algebras, and Rosso–Savage's affine Frobenius Hecke algebras.

0. INTRODUCTION

0.1. **Background.** Consider algebras over a field $\Bbbk$. Following [KM22], by *Schurification* we mean a procedure that, given an algebra $\mathbf{A}$, produces a new algebra $\mathbf{S}(\mathbf{A})$ which enjoys favorable properties similar to the classical Schur algebra, e.g., the double centralizer property, and the existence of functors that relate the representation theory of $\mathbf{A}$ and of $\mathbf{S}(\mathbf{A})$.

Let $\mathcal{H}_q(\mathfrak{S}_d)$ be the Hecke algebra of the symmetric group $\mathfrak{S}_d$ with $q \in \Bbbk^\times$. One instance of Schurification is the well-known Dipper–James' construction of the q-Schur algebras [DJ89]

$$\mathbf{S}^{\mathrm{DJ}}(\mathcal{H}_q(\mathfrak{S}_d)) \equiv S_q(n,d) := \mathsf{End}_{\mathcal{H}_q(\mathfrak{S}_d)}\left(\bigoplus_{\lambda\in\Lambda_{n,d}} x_\lambda \mathcal{H}_q(\mathfrak{S}_d)\right),$$

in terms of permutation modules. Another instance is Beilinson–Lusztig–MacPherson's realization of q-Schur algebra via convolution algebras [BLM90], later generalized by Pouchin [Pou09]:

$$\mathbf{S}^{\mathrm{BLM}}(\mathcal{H}_q(\mathfrak{S}_d)) \equiv \Bbbk_{\mathrm{GL}_d}(Y_{n,d}\times Y_{n,d}),$$

where $Y_{n,d}$ is the (finite) set of partial flags of length n in $\mathbb{F}_q^d$, and $\Bbbk_{\mathrm{GL}_d}(-)$ is the space of GL_d-invariant $\Bbbk$-valued functions. The two constructions can be identified [DJ91, Gro92]:

$$\bigoplus_{\lambda\in\Lambda_{n,d}} x_\lambda \mathcal{H}_q(\mathfrak{S}_d) \equiv (\Bbbk^n)^{\otimes d} \equiv \Bbbk_{\mathrm{GL}_d}(Y_{n,d}).$$

Each construction has its own merits. The convolution algebra approach usually accounts for positivity behaviors; while Dipper–James' approach involves Coxeter group combinatorics, and allows potential generalizations to the case of unequal parameters.

Schurification (and further development) for these flavors of Hecke algebras of various types has been studied intensively; see e.g. [BW18, BKLW18, LL21, FL15, DLZ24] for type B/C/D, [GRV94, Lus99, DF15] for affine type A, [BWW18, FLL⁺20a, FLL⁺23, CFW24] for affine type B/C/D. Note that the works on affine types use Coxeter presentation instead of Bernstein–Lusztig presentation, and hence do not generalize in an obvious way to certain interesting variants, e.g. the quantum wreath products $B \wr \mathcal{H}(d)$ introduced in [LNX24].

To our knowledge, only partial results are obtained regarding Schurification for algebras defined via the Bernstein–Lusztig presentation (see [KMS95, FLL⁺20b]). It is an interesting question whether one can extend the theory of Schurification to these algebras. If such an algebraic theory exists, does it admit a geometric counterpart in terms of the convolution algebras? In this paper, we provide affirmative answers to both questions, based on a new construction of convolution algebras with a twist, and a Demazure-type operator twisted by weak Frobenius elements.

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0.2. **An overview.** In this paper, we construct Schurification for algebras which admit the Bernstein–Lusztig presentation. Such algebras include the affine Hecke algebras for GL_d, their degenerate, 0-Hecke, and nil-Hecke variants, Kleshchev–Muth's affine zigzag algebras [KM19], Vignéras' pro-p Iwahori Hecke algebras $\mathcal{H}(q_s, c_s)$ [Vig16] for $\mathrm{GL}_d(\mathbb{Q}_p)$ (which are isomorphic to the affine Yokonuma algebras introduced by Chlouveraki-d'Andecy [Cd15]), certain Rosso–Savage's affine Frobenius Hecke algebras [Sav20, RS20], and Rees affine Frobenius Hecke algebras considered in an ongoing work by Mathas–Stroppel [MS].

Precisely speaking, we consider a family of quantum wreath products $B \wr \mathcal{H}(d)$ (which we call of *polynomial type*, or PQWP), in which the base algebra $B = F[x]$ (or $F[x^{\pm 1}]$) is the ring of (Laurent) polynomials over a $\Bbbk$-algebra F. Typically, F is either the ground field $\Bbbk$, the group algebra $\Bbbk[t]/(t^m - 1)$ of a cyclic group, or the cohomology ring of a smooth variety, e.g. $H^*(\mathbb{P}^n) = \Bbbk[c]/(c^{n+1})$. The parameters (S, R, σ, ρ) for such PQWPs are of the form

$$S = \Delta^{10} - \Delta^{01}, \quad R \in (Z(F \otimes F))^{\mathfrak{S}_2}, \quad \sigma = \mathrm{flip}, \quad \rho = \partial^{\beta},$$

where $\beta = \sum_{0 \le i,j \le 1} \Delta^{ij}(x^i \otimes x^j)$ for some weak Frobenius elements $\Delta^{ij} \in (F \otimes F)^{\mathfrak{S}_2}$, and ∂^{β} is the Demazure operator twisted by β (see (2.3)).

On the other hand, we consider *twisted*, $\mathfrak{S}_d$-equivariant convolution algebras of functions valued in $\mathcal{R} := B^{\otimes d}$, where the product in given by

$$(f * g)(x, y) := \sum_{z \in X} f(x, z) e(z)^{-1} g(z, y).$$

for some $e : X \to \mathcal{R}$.

Recall that a uniform proof of Schur duality for quantum wreath products was provided in [LNX24, §7.1–5] under strict assumptions, including finite-dimensionality of the base algebra B. In the present paper, our Schurification allows a uniform proof for the aforementioned quantum wreath products with infinite-dimensional base algebras.

Main results. *Let $B \wr \mathcal{H}(d)$ be a quantum wreath product of polynomial type (see Theorem 2.7) satisfying conditions (C1)–(C3) of Section 4.1.*

(A) [Theorem 4.2] There is an embedding $B \wr \mathcal{H}(d) \hookrightarrow \mathcal{R}^{(e)}_{\mathfrak{S}_d}(\mathfrak{S}_d \times \mathfrak{S}_d)$ into a twisted convolution algebra, with the twist given by

$$e(z) = z \left(\prod_{1 \le i < j \le d} (\sigma(\beta)_{ij}(x_i - x_j) - \sigma(\alpha)_{ij}(x_i - x_j)^2) \right).$$

In particular, $H_i \mapsto \xi_{1, \beta_i/(x_i - x_{i+1})} + \xi_{s_i, \alpha_i + \beta_i/(x_i - x_{i+1})}$.

(B) [Theorems 3.11, 4.6 and 5.9] There is a Schurification of $B \wr \mathcal{H}(d)$ via twisted convolution algebras such that the Schur counterpart, i.e., the **coil*** *Schur algebra*

$$\mathrm{S}^{\mathrm{BLM}}(B \wr \mathcal{H}(d)) := \mathcal{R}^{T}_{G}(Y \times Y)$$

has a double centralizer property (DCP) with $B \wr \mathcal{H}(d)$, when the elements (4.15) are invertible. When invertibility fails, DCP continues to hold for a slightly bigger **laurel*** *Schur algebra $\overline{\mathrm{S}}^{\mathrm{BLM}}$.*

(C) [Theorems 6.7 and 6.8] There is another Schurification of $B \wr \mathcal{H}(d)$ via permutation modules such that DCP holds for the corresponding **wreath*** *Schur algebra*

$$\mathrm{S}^{\mathrm{DJ}}(B \wr \mathcal{H}(d)) := \mathrm{End}_{B \wr \mathcal{H}(d)}\Big(\bigoplus_{\lambda \in \Lambda_{n,d}} M^{\lambda}\Big) \equiv \mathrm{End}_{B \wr \mathcal{H}(d)}(V_n^{\otimes d}),$$

provided that $n \ge d$. The algebras $\overline{\mathrm{S}}^{\mathrm{BLM}}$ and S^{DJ} are Morita-equivalent.

(D) [Theorem 6.5] There exists an explicit basis $\{\theta_{A,P}\}$ of these Schur algebras, where A lies in the set $\Theta_{n,d}$ of n-by-n matrices with non-negative integer entries summing up to d, and P is a partially symmetric polynomial in $B^{\otimes d}$.

While the conditions (C1) and (C3) are quite restrictive, we only expect to be able to remove (C2); see the discussion in Section 5.4.

* "Wreath" is reserved for the centralizer algebra. Both "coil" and "laurel" evoke the skeletal shape of wreath, with laurel being slightly thicker.

The basis $\{\theta_{A,P}\}$ in (D) is a generalization of the Dipper–James basis $\{\theta^{g}_{\lambda,\mu}\}$ of the q-Schur algebra, and is related to the "chicken-foot" basis in [SW24a, SW24b]. A similar basis also appears in an ongoing work [DKMZ] by Davidson-Kujawa-Muth-Zhu.

0.3. **Applications.** Let us highlight some applications that we find exciting; see Section 7 for details.

Imaginary Strata of Affine KLR Algebras. Recall that quantum groups are categorified by quiver Hecke algebras. In particular, the study of PBW bases of affine quantum groups categorifies to the study of stratification of quiver Hecke algebras of affine type. While this was carried out in characteristic 0 in [KM19], the general case remains mysterious, the hard being the computation of the so-called *imaginary* strata. We propose that (idempotent truncations of) the coil Schur algebras $\mathrm{S}^{\mathrm{BLM}}$ describe the imaginary strata in any characteristic (Section 7.4).

Representation Theory of p-adic Groups. The pro-p Iwahori Hecke algebra (Section 7.3) and its representation theory plays an important role in the representation theory of p-adic groups, especially when one considers representations in local characteristic [Vig16, Oll10, Abe19], or metaplectic covers of p-adic groups [GGK24]. In the latter case, the pro-p Iwahori Hecke algebra and its Gelfand–Graev modules encode information about certain metaplectic Whittaker functions, see also [BP22]. We expect our theory for $\overline{\mathrm{S}}^{\mathrm{BLM}}$ and S^{DJ} to be useful to understand Schurification arising from [GGK24].

0.4. **Organization.** In Section 1, we recall the definition of convolution algebras, as well as the combinatorics used in the Dipper–James construction. We also remind readers the definition of QWP and the conditions for it to have a PBW basis. In Section 2, we introduce twisted Demazure operators, and use them to define the class of quantum wreath products of polynomial type. We show PQWPs afford a PBW basis. In Section 3, we introduce twisted convolution algebras, and prove the double centralizer property for their certain sublattices. In Section 4, we realize PQWPs as subalgebras of twisted convolution algebras, and hence deduce the Schur duality under an invertibility assumption for the coil Schur algebras. The assumption is removed in Section 5 for the price of replacing the Schur algebra with a larger laurel Schur algebra. In Section 6, we prove the double centralizer property for the wreath Schur algebra in the sense of Dipper–James, and then compare these approaches. Finally, we summarize particular cases of new results in Section 7.

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1. Prerequisites

1.1. **Schurification via convolution algebras.** Let G be a finite group acting on a finite set X, and $\mathcal{R}$ a unital ring equipped with a G-action and free as a $\Bbbk$-module. Denote by $\mathcal{R}_G(X)$ the set of G-equivariant $\mathcal{R}$-valued functions on X. Then, the set $\mathcal{R}_G(X \times X)$ of G-equivariant $\mathcal{R}$-valued functions on $X \times X$ is a unital associative algebra, with multiplication given by convolution:

$$(f * g)(x, y) = \sum\nolimits_{z\in X} f(x, z)g(z, y). \tag{1.1}$$

Such convolution algebras and the corresponding double centralizer property have been systematically studied in [Pou09].

Let $\mathcal{R} = \Bbbk$ with the trivial G-action. In the case $G = \mathrm{GL}_d(\Bbbk)$ acting on the set $X = Y_d$ of complete flags in $\Bbbk^d$, the convolution algebra $\mathcal{R}_G(X \times X)$ realizes the Hecke algebra $\mathcal{H}_q(\mathfrak{S}_d)$. A well-known Schurification of $\Bbbk_G(X \times X) \equiv \mathcal{H}_q(\mathfrak{S}_d)$, due to Beilinson-Lusztig-MacPherson [BLM90], proceeds by replacing Y_d with the set of n-step partial flags in $\Bbbk^d$. This produces an algebra $\Bbbk_G(Y_{n,d} \times Y_{n,d})$ with monomial and canonical bases, which are indexed by the set of G-orbits in $Y_{n,d} \times Y_{n,d}$. Note that this set is naturally identified with the set $\Theta_{n,d}$ of n-by-n matrices with non-negative integer coefficients that add up to d.

The aforementioned bases can be constructed from the basis consisting of the following characteristic functions:

$$\xi_\pi \in \mathcal{R}(Y_{n,d} \times Y_{n,d}), \qquad \xi_\pi(x,y) = \sum_{g \in G/\operatorname{Stab}_G(\pi)} \delta_{g\pi,(x,y)} = \begin{cases} 1 & \text{if } (x,y) \in G\cdot\pi; \\ 0 & \text{otherwise}, \end{cases} \tag{1.2}$$

where π runs over a fixed choice of representatives of $\Pi_{n,d}$.

The convolution algebra $\mathcal{R}_G(Y_{n,d} \times Y_{n,d})$ is isomorphic to the q-Schur algebra $S_q(n,d)$ of Dipper–James [DJ89].

1.2. **Schurification via permutation modules.** Let us recall the combinatorics used in [DJ89]. Denote the simple transpositions by $s_i = (i, i+1) \in \mathfrak{S}_d$. For the Hecke algebra $\mathcal{H}_q(\mathfrak{S}_d)$, denote by $\{T_w\}_{w\in\mathfrak{S}_d}$ its standard basis with multiplication rules determined by the quadratic relation $T_i^2 = (q-1)T_i + q$. Let $\Lambda_{n,d}$ be the set of (weak) compositions $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_i \geq 0$ of d into n parts. Denote by $\mathfrak{S}_\lambda$ the corresponding Young subgroup $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n} \subseteq \mathfrak{S}_d$, and let $\mathfrak{S}^\lambda$ and ${}^\lambda\mathfrak{S}$ be the sets of shortest left and right coset representatives of $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_d$, respectively. When it is convenient, we identify each representative with the coset:

$$\mathfrak{S}^\lambda \equiv \mathfrak{S}_d/\mathfrak{S}_\lambda, \qquad {}^\lambda\mathfrak{S} \equiv \mathfrak{S}_\lambda\backslash\mathfrak{S}_d.$$

The set $\Theta_{n,d}$ can be identified with the set of triples (λ, g, μ) where $\lambda, \mu \in \Lambda_{n,d}$ are the column/row sum vectors of A, respectively, and $g \in {}^\lambda\mathfrak{S}^\mu := {}^\lambda\mathfrak{S} \cap \mathfrak{S}^\mu$ is the shortest representative in the double coset $\mathfrak{S}_\lambda g \mathfrak{S}_\mu$ such that $a_{ij} = {}^\#(\mathfrak{I}_i^\lambda \cap g\mathfrak{I}_j^\mu)$ for all i, j, where

$$\mathfrak{I}_i^\lambda := \{\lambda_1 + \cdots + \lambda_{i-1} + 1,\ \lambda_1 + \cdots + \lambda_{i-1} + 2,\ \dots,\ \lambda_1 + \cdots + \lambda_i\}.$$

Let $G \subseteq \mathfrak{S}_d$ be a subset with the unique longest element $w_\circ^G$. In particular, write $w_\circ^\lambda := w_\circ^{\mathfrak{S}_\lambda}$ and $w_\circ^A := w_\circ^{\mathfrak{S}_\lambda g \mathfrak{S}_\mu}$, where $A \equiv (\lambda, g, \mu)$. The following facts on symmetric groups are well-known, see e.g. [DDPW08]:

Lemma 1.1. *Suppose that $A \equiv (\lambda, g, \mu)$. Then,*

(a) There is a unique strong composition $\delta^c = \delta^c(A) \in \Lambda_{n',d}$ for some n' such that $\mathfrak{S}_{\delta^c} = g^{-1}\mathfrak{S}_\lambda g \cap \mathfrak{S}_\mu$. Moreover, δ^c is obtained by column reading of nonzero entries of A.

(b) There is a unique strong composition $\delta^r = \delta^r(A) \in \Lambda_{n',d}$ for some n' such that $\mathfrak{S}_{\delta^r} = g\mathfrak{S}_\mu g^{-1} \cap \mathfrak{S}_\lambda$. Moreover, $\delta^r = \delta^c({}^tA)$, and is obtained from row reading of nonzero entries of A.

(c) Write $\delta = \delta^c(A)$ and $G = {}^\delta\mathfrak{S}_\mu$. Then, $\mathfrak{S}_\lambda g \mathfrak{S}_\mu = \{w \mid g \leq w \leq w_\circ^A\}$, in which the longest element is $w_\circ^A = w_\circ^\lambda g w_\circ^G$, where $w_\circ^G = w_\circ^\delta w_\circ^\mu$ with $\ell(w_\circ^G) = \ell(w_\circ^\mu) - \ell(w_\circ^\delta)$. In other words, the map $\kappa : \mathfrak{S}_\lambda \times ({}^{\delta^c}\mathfrak{S}_\mu) \to \mathfrak{S}_\lambda g \mathfrak{S}_\mu$, $(x,y) \mapsto xgy$ is a bijection satisfying $\ell(xgy) = \ell(x) + \ell(g) + \ell(y)$.

Example 1.2. If $A := \left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)$, then $\delta^c(A)$ is obtained from $(a_{11}, a_{21}, a_{12}, a_{22})$ by removing the zeroes, and hence $\delta^c(A) = (1,2,1)$. Similarly, $\delta^r(A) = (1,1,2)$. The row sum and column sum vectors of A are $(2,2)$ and $(3,1)$, respectively. Then, $A \equiv ((2,2), g, (3,1))$ with $g = |1342| = s_2 s_3$, since $\mathfrak{I}_1^\lambda = \{1,2\}, \mathfrak{I}_2^\lambda = \{3,4\}$, $g\mathfrak{I}_2^\mu = \{1,3,4\}$ and $g\mathfrak{I}_2^\mu = \{2\}$. The longest element is $w_\circ^A = (s_1 s_3)(s_2 s_3)(s_2)(s_2 s_1 s_2) = s_1 s_3 s_2 s_3 s_1 s_2$.

Recall that $\mathcal{H}_q(\mathfrak{S}_d)$ acts on the d-fold tensor product of $\Bbbk^n \equiv \bigoplus_{1\leq i\leq n} \Bbbk v_i$ by

$$v_f \cdot T_i = \begin{cases} v_{f\cdot s_i} & \text{if } f_i < f_{i+1}; \\ q v_f & \text{if } f_i = f_{i+1}; \\ q v_{f\cdot s_i} + (q-1) m_f & \text{if } f_i > f_{i+1}, \end{cases} \tag{1.3}$$

where $v_f = v_{f_1} \otimes \cdots \otimes v_{f_d}$, $f = (f_i)_i \in \{1, \dots, n\}^d$, on which $\mathfrak{S}_d$ acts by place permutation. As a $\mathcal{H}_q(\mathfrak{S}_d)$-module, $(\Bbbk^n)^{\otimes d}$ can be decomposed into the sum of q-permutation modules $x_\lambda \mathcal{H}_q(\mathfrak{S}_d)$ over $\lambda \in \Lambda_{n,d}$, where $x_\lambda := \sum_{w\in\mathfrak{S}_\lambda} T_w$. The T_i-action on x_λ is explicit, because one can rewrite the quadratic relation as $T_i(T_i + 1) = q(T_i + 1)$, i.e., $(T_i + 1)$ is a q-eigenvector of T_i. For $A = (\lambda, g, \mu)$ we write $G(A) := {}^{\delta^c(A)}\mathfrak{S}_\mu$.

The basis of the q-Schur algebra $S_q(n,d)$ is given by $\{\theta_A\}_{A\in\Theta_{n,d}}$, where the basis elements are right $\mathcal{H}_q(\mathfrak{S}_d)$-linear maps given by

$$\theta_A : x_\mu \mathfrak{H}_q(\mathfrak{S}_d) \to x_\lambda \mathfrak{H}_q(\mathfrak{S}_d), \quad x_\mu \mapsto x_A, \text{ where } x_A := \sum_{w\in\mathfrak{S}_\lambda g \mathfrak{S}_\mu} T_w = x_\lambda T_g \sum_{w\in G(A)} T_w. \tag{1.4}$$

The map is well-defined thanks to Theorem 1.1. It is immediate from this construction that θ_A's are $\mathcal{H}_q(\mathfrak{S}_d)$-module homomorphisms.

To sum up, there is an identification $\Bbbk_G(Y_{n,d}) \equiv \bigoplus_{\lambda\in\Lambda_{n,d}} x_\lambda \mathcal{H}_q(\mathfrak{S}_d) \equiv (\Bbbk^n)^{\otimes d}$ that leads to the identification $\Bbbk_G(Y_{n,d}\times Y_{n,d}) \equiv S_q(n,d)$. Moreover, the map $\theta_A \in S_q(n,d)$ is identified with the characteristic function $\xi_\pi \in \Bbbk_G(Y_{n,d}\times Y_{n,d})$ where π is the representative in the orbit corresponding to $A \in \Theta_{n,d}$.

1.3. **Quantum wreath products.** Let B be a unital associative $\Bbbk$-algebra, free over $\Bbbk$ with basis $\{b_i\}_{i\in I}$. Let $d \in \mathbb{Z}_{\geq 2}$. By Q we mean a quadruple $(R,S,\rho,\sigma) \in (B\otimes B)^2 \times \mathrm{End}_\Bbbk(B\otimes B)$. We use the following abbreviations, for each $1 \le i \le d$:

$$\begin{aligned} &Y_i := 1^{\otimes i-1} \otimes Y \otimes 1^{\otimes d-i-k} \in B^{\otimes d}, && Y \in B^{\otimes k+1};\\ &\varphi_i : B^{\otimes d} \to B^{\otimes d}, \ \textstyle\bigotimes_j b_j \mapsto (\bigotimes_{j=1}^{i-1} b_j)\otimes \varphi(b_i\otimes b_{i+1}) \otimes (\bigotimes_{j=i+2}^{d} b_j), && \varphi \in \mathrm{End}_\Bbbk(B^{\otimes 2}). \end{aligned} \tag{1.5}$$

For $Y = \sum_k a^{(k)} \otimes b^{(k)} \in B\otimes B$, we also write, for $1 \le i < j \le d$:

$$Y_{i,j} := \sum\nolimits_k a_i^{(k)} b_j^{(k)}, \quad Y_{j,i} := \sum\nolimits_k b_j^{(k)} a_i^{(k)} \in B^{\otimes d}. \tag{1.6}$$

In particular, $Y_i := Y_{i,i+1}$ and $Y_{i+1,i} = \sigma_i(Y_i)$ if $\sigma : a\otimes b \mapsto b\otimes a$ is the flip map.

Definition 1.3. The *quantum wreath product (QWP)* is the associative $\Bbbk$-algebra, denoted by $B\wr\mathcal{H}(d) = B\wr_Q \mathcal{H}(d)$, generated by the algebra $B^{\otimes d}$ and Hecke-like generators $H_1,\dots,H_{d-1}$ modulo the following relations, for $1\le k\le d-2$, $1 \le i \le d-1$, $|j-i|\ge 2$, $b\in B^{\otimes d}$:

(braid relations)	$H_kH_{k+1}H_k = H_{k+1}H_kH_{k+1}, \quad H_iH_j = H_jH_i,$
(quadratic relations)	$H_i^2 = S_iH_i + R_i,$
(wreath relations)	$H_ib = \sigma_i(b)H_i + \rho_i(b).$

For any $w\in\mathfrak{S}_d$ with a reduced expression $w = s_{i_1}\dots s_{i_N}$ we can define an element $H_w := H_{i_1}\dots H_{i_N} \in B\wr\mathcal{H}(d)$. Note that H_w is independent of the choice of a reduced expression due to the braid relations above. We say that $B\wr\mathcal{H}(d)$ *has a PBW basis* if the natural spanning sets $\{(\bigotimes_{j=1}^d b_{i_j})H_w \mid i_j\in I, w\in\mathfrak{S}_d\}$ and $\{H_w(\bigotimes_{j=1}^d b_{i_j}) \mid i_j \in I, w\in\mathfrak{S}_d\}$ are linearly independent.

Proposition 1.4 ([LNX24, Theorem 3.3.1]). *$B\wr\mathcal{H}(d)$ has a PBW basis if and only if*

Conditions (P1) – (P4) hold, and (P5) – (P9) hold additionally if $d\ge 3$.

Here, the conditions are:

$$\sigma(1\otimes 1) = 1\otimes 1, \quad \rho(1\otimes 1) = 0, \tag{P1}$$

$$\sigma(ab) = \sigma(a)\sigma(b), \quad \rho(ab) = \sigma(a)\rho(b) + \rho(a)b, \tag{P2}$$

$$\sigma(S)S + \rho(S) + \sigma(R) = S^2 + R, \quad \rho(R) + \sigma(S)R = SR, \tag{P3}$$

$$r_S\sigma^2 + \rho\sigma + \sigma\rho = l_S\sigma, \quad r_R\sigma^2 + \rho^2 = l_S\rho + l_R, \tag{P4}$$

where l_X, r_X for $X\in B\otimes B$ are $\Bbbk$-endomorphisms defined by left and right multiplication in $B\otimes B$ by X, respectively,

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \rho_i\sigma_j\sigma_i = \sigma_j\sigma_i\rho_j, \tag{P5}$$

$$\rho_i\sigma_j\rho_i = r_{S_j}\sigma_j\rho_i\sigma_j + \rho_j\rho_i\sigma_j + \sigma_j\rho_i\rho_j, \tag{P6}$$

$$\rho_i\rho_j\rho_i + r_{R_i}\sigma_i\rho_j\sigma_i = \rho_j\rho_i\rho_j + r_{R_j}\sigma_j\rho_i\sigma_j, \tag{P7}$$

where $\{i,j\}=\{1,2\}$, r_X *for* $X\in B^{\otimes 3}$ *is understood as right multiplication in* $B^{\otimes 3}$ *by* X,

(P8) $$S_i=\sigma_j\sigma_i(S_j),\quad R_i=\sigma_j\sigma_i(R_j),\quad \rho_j\sigma_i(S_j)=0=\rho_j\sigma_i(R_j),$$

(P9) $$\sigma_j\rho_i(S_j)S_j+\rho_j\rho_i(S_j)+\sigma_j\rho_i(R_j)=0=\rho_j\rho_i(R_j)+\sigma_j\rho_i(S_j)R_j,$$

where $\{i,j\}=\{1,2\}$.

2. Quantum wreath products of polynomial type

Quantum wreath products cover various examples of deformations of wreath products appearing in literature. Unfortunately, this notion is rather unwieldy, since it is in some sense the most general definition one can come up with. In this paper we will only consider a certain class of quantum wreath products, which has the flavor of affine Hecke algebras of type A.

2.1. Twisted Demazure operators.

Let F be a unital finite-dimensional algebra over $\Bbbk$.

Definition 2.1. A *weak Frobenius* element of F is $\Delta\in F\otimes F$ satisfying $(a\otimes b)\Delta=\Delta(b\otimes a)$ for any $a,b\in F$.

It is clear that weak Frobenius elements form a vector space, which we denote by $W(F)\subseteq F\otimes F$. Such elements are sometimes called *intertwiners* or *teleporters*. We call them weak Frobenius since they are the evaluation at identity of the comultiplication of weak Frobenius algebras (see [CG03]). While usual Frobenius elements (which satisfy an additional non-degeneracy condition) are essentially unique, there can be many linearly independent weak Frobenius elements.

Example 2.2. Let $F=\Bbbk[c]/(c^{n+1})$. Then for every $k\geq 0$ the element $\sum_{i+j=n+k}c^i\otimes c^j$ is weak Frobenius.

Definition 2.3. Let $\beta=\sum_{i,j\geq 0}\Delta^{i,j}(x^i\otimes x^j)$ be an element of $W(F)^{\mathfrak{S}_2}[x_1,x_2]$, i.e., a polynomial in two variables with coefficients in symmetric weak Frobenius elements in F. The β*-twisted Demazure operator* $\partial^\beta:F^{\otimes 2}[x_1,x_2]\to F^{\otimes 2}[x_1,x_2]$ is given by

$$\partial^\beta(a\otimes b)=\frac{\beta(a\otimes b)-(b\otimes a)\beta}{x\otimes 1-1\otimes x}.$$

Remark 2.4. A similar Demazure operator appears in [Sav20, Lemma 4.3].

Note that ∂^β is well defined since, writing $a=f'x^k$, $b=f''x^l$ for some $f',f''\in F$:

$$\begin{aligned}\beta(a\otimes b)-(b\otimes a)\beta&=\sum_{i,j}\Delta^{i,j}(x^i\otimes x^j)(f'\otimes f'')(x^k\otimes x^l)-\sum_{i,j}(f''\otimes f')(x^l\otimes x^k)\Delta^{i,j}(x^i\otimes x^j)\\&=\beta(f'\otimes f'')(x^k\otimes x^l-x^l\otimes x^k)=(f''\otimes f')(x^k\otimes x^l-x^l\otimes x^k)\beta,\end{aligned}$$

and hence $\partial^\beta(fx^k\otimes gx^l)=\beta(f\otimes g)\partial(x^k\otimes x^l)=(g\otimes f)\partial(x^k\otimes x^l)\beta$, or

(2.1) $$\partial^\beta(fP)=\sigma(f)\partial(P)\beta,\quad \text{for all}\quad f\in F\otimes F,\ P\in\Bbbk[x_1,x_2],$$

where $\sigma: a\otimes b\mapsto b\otimes a$ is the flip map, and $\partial:\Bbbk[x_1,x_2]\to\Bbbk[x_1,x_2]$ is the usual Demazure operator.

Let us collect some useful properties of ∂^β.

Lemma 2.5. *Suppose that* β *is an element as in Theorem 2.3. Then,*

(a) *If* $P\in\Bbbk[x_1,x_2]$, *then* $\partial^\beta(P)=\beta\partial(P)=\partial(P)\beta$. *Moreover,* $\partial^\beta\sigma(P)=-\partial^\beta(P)$.
(b) *If* $f\in F\otimes F$, *then* $\partial^\beta(f)=0$.
(c) *For any* $a,b\in(F\otimes F)[x_1,x_2]$ *we have* $\partial^\beta(ab)=\sigma(a)\partial^\beta(b)+\partial^\beta(a)b$. *In other words,* ∂^β *is a* σ*-twisted left derivation.*

Proof. The first two claims are direct consequences of (2.1). The last claim follows from a quick computation:

$$\begin{aligned}\partial^\beta(ab)&=\frac{\beta ab-\sigma(ab)\beta}{x\otimes 1-1\otimes x}=\frac{\sigma(a)\beta b-\sigma(a)\sigma(b)\beta+\beta ab-\sigma(a)\beta b}{x\otimes 1-1\otimes x}\\&=\sigma(a)\partial^\beta(b)+\partial^\beta(a)b.\end{aligned}$$

□

Remark 2.6. Note that when F is not commutative, ∂^β has no reason to be a right derivation.

2.2. **Quantum wreath products of polynomial type.** Let F be a unital finite-dimensional algebra as before, and let B be either $F[x]$ or $F[x^{\pm 1}]$.

Definition 2.7. A quantum wreath product $B \wr \mathcal{H}(d)$, $Q = (R, S, \rho, \sigma)$ is said to be of *polynomial type (PQWP)* with respect to the pair $(R, \beta) \in (F \otimes F)^{\mathfrak{S}_2} \times (B \otimes B)$ if

(A1) R is central in $F \otimes F$;

(A2) $\beta = \sum_{0 \le i,j \le 1} \Delta^{ij}(x^i \otimes x^j) \in W(F)^{\mathfrak{S}_2}[x_1, x_2]$, and $\Delta_1^{00}\Delta_2^{11} = \Delta_1^{01}\Delta_2^{10}$;

(A3) σ is the flip map, $\rho = \partial^\beta$, and $S = \Delta^{10} - \Delta^{01}$.

Remark 2.8. If $\Delta^{ij} = a^{ij}\Delta$, $a^{ij} \in \Bbbk$, then the relation $\Delta_1^{00}\Delta_2^{11} = \Delta_1^{01}\Delta_2^{10}$ in (A2) is equivalent to β factoring as $\beta = \Delta\beta_1\beta_2$, where $\beta_i \in \Bbbk[x_i]$. See the proof of Theorem 2.11.

Lemma 2.9. *The following identities hold in a PQWP:*

$$\begin{aligned} &HS = SH, \quad HR = RH, \\ &PH = H\sigma(P) + \partial(P)\beta \quad \text{for all} \quad P \in \Bbbk[x]^{\otimes 2}, \\ &\beta H = H\sigma(\beta) + S\beta, \quad H\beta = \sigma(\beta)H + S\beta. \end{aligned}$$

Proof. The first two lines are direct consequences of Theorem 2.5 since both S and R lie in $(F \otimes F)^{\mathfrak{S}_2}$.
Since $\sigma(\beta) = \Delta^{00} + \Delta^{01}x_1 + \Delta^{10}x_2 + \Delta^{11}x_1x_2$, we have $\beta - \sigma(\beta) = S(x_1 - x_2)$, and hence

$$\rho(\beta) = \frac{\beta^2 - \sigma(\beta)\beta}{x_1 - x_2} = \frac{\beta - \sigma(\beta)}{x_1 - x_2}\beta = S\beta.$$

Similarly, $\rho\sigma(\beta) = -S\beta$. The last line follows from the wreath relations. ☐

B	R	β	S	α	PQWP
$F[x]$	1	0	0	1	usual wreath product
$\Bbbk[\hbar][x]$	1	$\hbar$	0	1	graded affine Hecke
$\Bbbk[x]$	0	1	0	0	nil-Hecke algebra
$\Bbbk[x^{\pm 1}]$	q	$(q-1)x_1$ / $(1-q)x_2$	$q-1$	1 or $-q$	affine Hecke algebra
$\Bbbk[x]$	0	x_1x_2	0	0	opposite nil-Hecke algebra $NH_d^{\downarrow}$
$F[x^{\pm 1}]$	1	$q\Delta x_2$	$q\Delta$	may not exist	affine Frobenius Hecke algebra [RS20]
$F[\hbar, t][x^{\pm 1}]$	$\hbar^2$	$-S(x_2 + \hbar t)$	$\eta\tau$	may not exist	Rees affine Frobenius Hecke algebra [MS]
$\frac{\Bbbk[c]}{(c^2)}[x^{\pm 1}]$	1	$c_1 + c_2$	0	1	affine zigzag algebra of type A_1 [MM22]
$\frac{\Bbbk[t]}{(t^{p-1}-1)}[x^{\pm 1}]$	1	Sx_1 / $-Sx_2$	$(q-q^{-1})e$	$(1+q^{-1})e - 1 \otimes 1$	pro-p Iwahori Hecke algebra

TABLE 1. Examples of quantum wreath products of polynomial type

Example 2.10. (a) When $(R, \beta) = (1, 0)$, we recover the usual wreath product $B \wr \mathfrak{S}_d$.
(b) Let $F = \Bbbk$. The following choices of parameters recover various flavors of affine Hecke algebras of type A: The degenerate affine Hecke algebra (resp. its graded version) are PQWP for $(R, \beta) = (1, 1)$ (resp. $(1, \hbar)$) with $B = \Bbbk[x]$ (resp. $B = \Bbbk[\hbar][x]$). The nil-Hecke algebra is a PQWP for $(0, 1)$ with $B = \Bbbk[x]$. For $B = \Bbbk[x^{\pm 1}]$, the type A affine Hecke algebra is a PQWP for $(q, (q-1)x_1)$ or $(q, (1-q)x_2)$, and hence the affine 0-Hecke algebra is a PQWP for $(0, -x_1)$ or $(0, x_2)$.

(c) Let us highlight another curious example. Let $B = \Bbbk[x]$, $R = 0$, and $\beta = x_1x_2$. The usual Demazure operator satisfies the following relation after extension to Laurent polynomials:

$$\partial x_1^{-1} = x_2^{-1}\partial - (x_1x_2)^{-1}.$$

This tells us that the PQWP for $(R, \beta) = (0, x_1x_2)$ is isomorphic to the subalgebra $NH_d^{\downarrow}$ of difference operators on $\Bbbk[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ generated by Demazure operators and multiplications by x_i^{-1}, $1 \leq i \leq d$. It can be viewed as the "opposite" of the usual nil-Hecke algebra NH_d. One can easily check that $NH_d^{\downarrow} \not\simeq NH_d$ for $d \geq 2$.

(d) Let F be a Frobenuis algebra with Frobenius form $\Delta \in F \otimes F$. Setting $B = F[x]$, $R = 1$, $\beta = \Delta$, our PQWP $B \wr \mathcal{H}_d$ is Savage's affine wreath product algebra [Sav20]. If we set $B = \Bbbk[x^{\pm 1}]$, $R = 1$, $\beta = -qx_2\Delta$, the algebra $B \wr \mathcal{H}_d$ is isomorphic to Rosso-Savage's affine Frobenius Hecke algebra [RS20].

These examples, as well as further examples from Section 7, are summarized in Table 1 (see Section 4.1 for the meaning of column α).

Lemma 2.11. *The following equalities hold:*

$$\rho_1(\beta_{13}) = \rho_2(\beta_1) + \beta_{13}S_2 = -\sigma_2\rho_1(\beta_2).$$

Proof. We will only prove the first equality, since the other reduces to the same computation. First of all, we have

$$\rho_1(\beta_{13}) = \beta_{12}(\Delta_{13}^{10} + \Delta_{13}^{11}x_3) = \Delta_{12}^{00}\Delta_{13}^{10} + \Delta_{12}^{10}\Delta_{13}^{10}x_1 + \Delta_{12}^{01}\Delta_{13}^{10}x_2 + \Delta_{12}^{00}\Delta_{13}^{11}x_3 + \Delta_{12}^{11}\Delta_{13}^{10}x_1x_2 + \Delta_{12}^{10}\Delta_{13}^{11}x_1x_3 + \Delta_{12}^{01}\Delta_{13}^{11}x_2x_3 + \Delta_{12}^{11}\Delta_{13}^{11}x_1x_2x_3.$$

On the other hand,

$$\rho_2(\beta_1) = \beta_{23}(\Delta_{12}^{01} + \Delta_{12}^{11}x_1) = \Delta_{23}^{00}\Delta_{12}^{01} + \Delta_{23}^{00}\Delta_{12}^{11}x_1 + \Delta_{23}^{10}\Delta_{12}^{01}x_2 + \Delta_{23}^{01}\Delta_{12}^{01}x_3 + \Delta_{23}^{10}\Delta_{12}^{11}x_1x_2 + \Delta_{23}^{01}\Delta_{12}^{11}x_1x_3 + \Delta_{23}^{11}\Delta_{12}^{01}x_2x_3 + \Delta_{23}^{11}\Delta_{12}^{11}x_1x_2x_3,$$

and

$$\beta_{13}S_2 = \Delta_{13}^{00}(\Delta_{23}^{10} - \Delta_{23}^{01}) + \Delta_{13}^{10}(\Delta_{23}^{10} - \Delta_{23}^{01})x_1 + \Delta_{13}^{01}(\Delta_{23}^{10} - \Delta_{23}^{01})x_3 + \Delta_{13}^{11}(\Delta_{23}^{10} - \Delta_{23}^{01})x_1x_3.$$

Note that for any $\Delta, \Delta' \in W(F)^{\mathfrak{S}_2}$ we have

$$\Delta_{12}\Delta'_{13} = \Delta'_{23}\Delta_{12} = \Delta_{13}\Delta'_{23} = \Delta'_{12}\Delta_{13} = \Delta_{23}\Delta'_{12} = \Delta'_{13}\Delta_{23}. \tag{2.2}$$

This simple observation takes care of comparing all coefficients except the two coefficients at x_1 and at x_3. For the remaining two coefficients to coincide, we need $\Delta_{23}^{00}\Delta_{12}^{11} = \Delta_{13}^{10}\Delta_{23}^{01}$ and $\Delta_{12}^{00}\Delta_{13}^{11} = \Delta_{13}^{01}\Delta_{23}^{10}$. This is precisely the condition (A2) we imposed on β in Theorem 2.7. □

Proposition 2.12. *The quantum wreath product of polynomial type with respect to (R, β) has a PBW basis.*

Proof. Applying Theorem 1.4, we need to check the relations (P1)–(P9). Relations (P1)–(P3), (P5), (P8), (P9) follow immediately from Theorem 2.5. Let us check the relations (P4):

$$(\rho\sigma + \sigma\rho)(a) = \frac{\beta\sigma(a) - a\beta}{x_1 - x_2} + \frac{\sigma(\beta)\sigma(a) - a\sigma(\beta)}{x_2 - x_1} = \frac{(\beta - \sigma(\beta))\sigma(a) - a(\beta - \sigma(\beta))}{x_1 - x_2}$$
$$= S\sigma(a) - aS = (l_S\sigma - r_S\sigma^2)(a),$$

$$\rho^2(a) = \rho\left(\frac{\beta a - \sigma(a)\beta}{x_1 - x_2}\right) = \frac{\beta^2 a - \beta\sigma(a)\beta + \sigma(\beta)\sigma(a)\beta - a\sigma(\beta)\beta}{(x_1 - x_2)^2}$$
$$= \frac{\beta - \sigma(\beta)}{x_1 - x_2}\frac{\beta a - \sigma(a)\beta}{x_1 - x_2} = S\frac{\beta a - \sigma(a)\beta}{x_1 - x_2} = l_S\rho(a) = (l_S\rho + l_R - r_R\sigma^2)(a),$$

where we used the fact that R is central in the last equality.

Before checking (P6)–(P7), let us make the following observation. It follows from Theorem 2.5 that for any $P \in \Bbbk[x_1, x_2]$, $f \in F^{\otimes 2}$ we have $\rho(fP) = \sigma(f)\rho(P)$. In particular, when applying either (P6) or

(P7) to fP, the element $\sigma_1\sigma_2\sigma_1(f)$ will factor out, and the rest would only depend on P. Thus it suffices to check (P6)–(P7) only on polynomials $P \in \Bbbk[x_1, x_2, x_3]$. Since the relations are manifestly linear, we can further restrict to P being monomials. We will show that (P6)–(P7) hold when evaluated at P if and only if they hold when evaluated at x_iP, $1 \le i \le 3$.

Checking the equivalence above for all three relations and all three x_i's would take too much space; we will therefore only consider x_1, and leave the other two variables for the interested reader. First, let us look at the relation (P6) for $i = 1$, $j = 2$:

$$\begin{aligned}
\rho_1\sigma_2\rho_1(x_1P) &= \rho_1\sigma_2(x_2\rho_1(P) + \beta_1 P) = x_3\rho_1\sigma_2\rho_1(P) + \beta_2\rho_1\sigma_2(P) + \rho_1(\beta_{13})\sigma_2(P),\\
\rho_2\rho_1\sigma_2(x_1P) &= \rho_2(\beta_1\sigma_2(P) + x_2\rho_1\sigma_2(P)) = x_3\rho_2\rho_1\sigma_2(P) + \rho_2(\beta_1)\sigma_2(P) + \beta_2\rho_1\sigma_2(P) + \varphi,\\
r_{S_2}\sigma_2\rho_1\sigma_2(x_1P) &= r_{S_2}\sigma_2(\beta_1\sigma_2(P) + x_2\rho_1\sigma_2(P)) = x_3 r_{S_2}\sigma_2\rho_1\sigma_2(P) + \varphi',\\
\sigma_2\rho_1\rho_2(x_1P) &= \sigma_2(\beta_1\rho_2(P) + x_2\rho_1\rho_2(P)) = x_3\sigma_2\rho_1\rho_2(P) + \varphi'',
\end{aligned}\tag{2.3}$$

where the terms $\varphi := \beta_{13}\rho_2\sigma_2(P)$, $\varphi' := r_{S_2}\beta_{13}P$, and $\varphi'' := \beta_{13}\sigma_2\rho_2(P)$ sum up to $\beta_{13}S_2\sigma_2(P)$, thanks to (P4). Moreover, the first terms on the right hand sides of (2.3) sum up to the evaluation of (P6) at P multiplied by x_3. Therefore, it remains to check that $\rho_1(\beta_{13}) = \rho_2(\beta_1) + \beta_{13}S_2$, which follows from Theorem 2.11. The relation (P6) with $i = 2$, $j = 1$ is proved in an analogous fashion.

Let us finally consider the relation (P7). First, consider the two simpler terms:

$$\begin{aligned}
r_{R_1}\sigma_1\rho_2\sigma_1(x_1P) &= r_{R_1}\sigma_1(\beta_2\sigma_1(P) + x_3\rho_2\sigma_1(P)) = \beta_{13}PR_1 + x_3 r_{R_1}\sigma_1\rho_2\sigma_1(P),\\
r_{R_2}\sigma_2\rho_1\sigma_2(x_1P) &= r_{R_2}\sigma_2(\beta_1\sigma_2(P) + x_2\rho_1\sigma_2(P)) = \beta_{13}PR_2 + x_3 r_{R_2}\sigma_2\rho_1\sigma_2(P).
\end{aligned}$$

Since R is central and symmetric, and the coefficients of β are weak Frobenius, we have

$$\beta_{13}PR_1 = R_{32}\beta_{13}P = \beta_{13}PR_{32} = \beta_{13}PR_2.$$

Now, for the other two terms:

$$\begin{aligned}
\rho_1\rho_2\rho_1(x_1P) &= \rho_1\rho_2(\beta_1P + x_2\rho_1(P)) = \rho_1(\rho_2(\beta_1)P + \beta_{13}\rho_2(P) + \beta_2\rho_1(P) + x_3\rho_2\rho_1(P))\\
&= \rho_1\rho_2(\beta_1)P + (\sigma_1\rho_2(\beta_1) + \rho_1(\beta_2) + \beta_{13}S_1)\rho_1(P) + \rho_1(\beta_{13})\rho_2(P) + \beta_2\rho_1\rho_2(P) + x_3\rho_1\rho_2\rho_1(P),\\
\rho_2\rho_1\rho_2(x_1P) &= \rho_2(\beta_1\rho_2(P) + x_2\rho_1\rho_2(P)) = (\rho_2(\beta_1) + \beta_{13}S_2)\rho_2(P) + \beta_2\rho_1\rho_2(P) + x_3\rho_2\rho_1\rho_2(P).
\end{aligned}$$

Note that the S_i's appear from using the second equation of (P4). Comparing the coefficients at P, $\rho_1(P)$ and $\rho_2(P)$, it remains to show that

$$\rho_1(\beta_{13}) = \rho_2(\beta_1) + \beta_{13}S_2, \qquad \rho_1\rho_2(\beta_1) = 0, \qquad \sigma_1\rho_2(\beta_1) + \rho_1(\beta_2) + \beta_{13}S_1 = 0.$$

The first relation follows directly from Theorem 2.11. The second relation is obtained from the first one by applying ρ_1 and using (P4). For the last one, we have

$$\begin{aligned}
\sigma_1\rho_2(\beta_1) + \rho_1(\beta_2) + \beta_{13}S_1 &= \sigma_1(\rho_2(\beta_1) + \sigma_1\rho_1(\beta_2) + \beta_2S_1) = \sigma_1(\rho_2(\beta_1) + S_1\beta_{13} - \rho_1(\beta_{13}))\\
&= \sigma_1(S_1\beta_{13} - \beta_{13}S_2),
\end{aligned}$$

where we used (P4) and Theorem 2.11. Finally, since β has weak Frobenius components, we have $S_{12}\beta_{13} = \beta_{13}S_{23}$ by (2.2), and so we may conclude. □

3. Double centralizer property

In this section we extend the main theorem of [Pou09] to the setting of "twisted" convolution algebras. Such algebras arise naturally after applying equivariant localization to convolution algebras in Borel-Moore homology; see discussion in Section 7.4.

3.1. Twisted convolution algebras.

Recall the setup from Section 1.1.

Definition 3.1. By a *twist*, we mean a function $e \in \mathcal{R}_G(X)$ such that $e(x)$ is invertible for any $x \in X$. Given a twist e, the corresponding *twisted convolution algebra* is the associative $\Bbbk$-algebra $(\mathcal{R}_G(X \times X), *)$

whose multiplication is given by

$$(f * g)(x,y) = \sum\nolimits_z f(x,z)e(z)^{-1}g(z,y). \tag{3.1}$$

Lemma 3.2. *The twisted convolution algebra $\mathcal{R}_G(X \times X)$ with respect to a given twist e is a unital associative algebra.*

Proof. Let $f, g, h \in \mathcal{R}_G(X \times X)$. The chain of equialities below follows directly from the formula (3.1):

$$((f * g) * h)(x,y) = \sum_{x',x''} f(x,x')e(x')^{-1}g(x',x'')e(x'')^{-1}h(x'',y) = (f * (g * h))(x,y).$$

This proves the associativity. We conclude by observing that the element

$$1_X \in \mathcal{R}_G(X \times X), \qquad 1_X(x,y) = \delta_{x,y}e(x)$$

is a unit of $\mathcal{R}_G(X \times X)$. □

Applying Theorem 3.2 to the disjoint union of two G-sets X, Y, we obtain a left $\mathcal{R}_G(X \times X)$-action and a right $\mathcal{R}_G(Y \times Y)$-action on $\mathcal{R}_G(X \times Y)$. These two actions obviously commute. In particular, setting $Y = \mathrm{pt}$ each $\mathcal{R}_G(X \times X)$ acquires a natural representation $\mathcal{R}_G(X)$.

The following lemma is immediate.

Lemma 3.3. *For each $\pi \in \Pi$ and $r \in \mathcal{R}$, consider*

$$\xi_{\pi,r} \in \mathcal{R}_G(X \times X), \qquad \xi_{\pi,r}(x,x') = \sum\nolimits_{g \in G/\operatorname{Stab}_G(\pi)} \delta_{g\pi,(x,x')}e(x)g(r). \tag{3.2}$$

Given a basis $B^\pi_{\mathcal{R}}$ of $\mathcal{R}^{\operatorname{Stab}_G(\pi)}$ for each π, the collection $\{\xi_{\pi,r} : \pi \in \Pi, r \in B^\pi_{\mathcal{R}}\}$ is a basis of $\mathcal{R}_G(X \times X)$.

We will add a superscript to $\mathcal{R}_G(X \times X)$ when the twist e needs to be specified. Observe that the map $f \mapsto e \cdot f$, $(e \cdot f)(x,y) = e(x)f(x,y)$ establishes an isomorphism of algebras $\mathcal{R}^{(1)}_G(X \times X) \mapsto \mathcal{R}^{(e)}_G(X \times X)$. While this renders our definition of $\mathcal{R}^{(e)}_G(X \times X)$ somewhat superfluous at the first glance, its usefulness will become clear in Section 3.3.

Remark 3.4. The product in $\mathcal{R}_G(X \times X)$ is typically only $\Bbbk$-linear, and not $\mathcal{R}$-linear.

3.2. **Generators.** Let (Λ, ω) be a pointed finite set. For each $\lambda \in \Lambda$, fix a finite G-set Y_λ, and denote $Y = \bigsqcup_{\lambda \in \Lambda} Y_\lambda$, $X = Y_\omega$. We further assume that for each $\lambda \in \Lambda$ we have a fixed G-equivariant surjection $p_\lambda : X \to Y_\lambda$. Fix a twist $e \in \mathcal{R}_G(Y)$, and denote

$$\mathbf{A} := \bigoplus\nolimits_{\lambda,\mu \in \Lambda} \mathbf{A}_{\lambda\mu}, \quad \mathbf{C} := \bigoplus\nolimits_{\lambda \in \Lambda} \mathbf{C}_\lambda, \quad \mathbf{B} := \mathcal{R}_G(X \times X), \tag{3.3}$$

where $\mathbf{A}_{\lambda\mu} := \mathcal{R}_G(Y_\lambda \times Y_\mu)$, $\mathbf{C}_\lambda := \mathcal{R}_G(Y_\lambda \times X)$. Both $\mathbf{A}$ and $\mathbf{B}$ are twisted convolution algebras (with twist e), and $\mathbf{C}$ is an $(\mathbf{A},\mathbf{B})$-bimodule. We will identify both $\mathbf{A}_{\omega\omega}$ and $\mathbf{C}_\omega$ as right $\mathbf{B}$-modules with $\mathbf{B}$ by means of p_λ.

Definition 3.5. Let $\lambda \in \Lambda$, $y, y' \in Y_\lambda$, and $x \in X$. Define elements which we call (full) *splits* and *merges*, respectively, by

$$S_\lambda \in \mathbf{A}_{\omega\lambda},\ S_\lambda(x,y) := \delta_{p_\lambda(x),y}e(y); \quad M_\lambda \in \mathbf{A}_{\lambda\omega},\ M_\lambda(y,x) := \delta_{y,p_\lambda(x)}e(y). \tag{3.4}$$

Next, define elements $1_\lambda, K_\lambda \in \mathbf{A}_{\lambda\lambda}$, and $m \in \mathcal{R}_G(Y)$ via

$$1_\lambda(y,y') := \delta_{y,y'}e(y), \quad K_\lambda(x,x') := \delta_{p_\lambda(x),p_\lambda(x')}e(p_\lambda(x)), \quad m(y) := \sum\nolimits_{x \in p_\lambda^{-1}(y)} e(x)^{-1}e(y). \tag{3.5}$$

Finally, for any $r \in \mathcal{R}_G(Y)$, define $r_\lambda \in \mathbf{A}_{\lambda\lambda}$ and $\widetilde{r}_\lambda \in \mathcal{R}_G(Y_\lambda)$ by

$$r_\lambda(y,y') := \delta_{y,y'}e(y)r(y), \quad \widetilde{r}_\lambda(y) = r(p_\lambda(y)). \tag{3.6}$$

Remark 3.6. (a) The set $\{1_\lambda : \lambda \in \Lambda\}$ is a complete set of orthogonal (but not necessarily primitive) idempotents in $\mathbf{A}$.

(b) If we equip $\mathcal{R}_G(Y)$ with pointwise multiplication, the map $\mathcal{R}_G(Y) \to \mathbf{A}_{\lambda\lambda}$, $r \mapsto r_\lambda$ is an algebra monomorphism.

Let us compute the compositions of M_λ and S_λ:

Lemma 3.7. *For any $\lambda \in \Lambda$, we have*

$$S_\lambda * M_\lambda = K_\lambda, \quad M_\lambda * S_\lambda = m_\lambda.$$

Proof. It follows by a direct computation that

$$\begin{aligned} S_\lambda * M_\lambda(x,x') &= \sum\nolimits_{y\in Y_\lambda} \delta_{p_\lambda(x),y}\delta_{y,p_\lambda(x')}e(y) = \delta_{p_\lambda(x),p_\lambda(x')}e(p_\lambda(x)),\\ M_\lambda * S_\lambda(y,y') &= \sum\nolimits_{x\in X} \delta_{y,p_\lambda(x)}\delta_{p_\lambda(x),y'}e(y)e(y')e(x)^{-1} = \delta_{y,y'}\sum\nolimits_{x\in p_\lambda^{-1}(y)} e(y)e(x)^{-1}e(y)\\ &= \delta_{y,y'}e(y)\sum\nolimits_{x\in p_\lambda^{-1}(y)} e(x)^{-1}e(y), \end{aligned}$$

where we used the notations of Theorem 3.5 in the second equality. □

The following lemmas are elementary, we leave their proofs to the interested reader.

Lemma 3.8. *Let $\lambda,\mu \in \Lambda$. Consider the natural injective maps $\psi_\lambda^R : \mathbf{A}_{\mu\lambda} \to \mathbf{A}_{\mu\omega}$, $\psi_\lambda^L : \mathbf{A}_{\lambda\mu} \to \mathbf{A}_{\omega\mu}$, given by pulling back along p_λ. We have $\psi_\lambda^R(f) = f * M_\lambda$, $\psi_\lambda^L(f) = S_\lambda * f$. Furthermore, left multiplication by 1_λ is identified with the projection $\mathbf{C} \twoheadrightarrow \mathbf{C}_\lambda$.*

Lemma 3.9. *Let $r \in \mathcal{R}_G(Y)$. Then, $r_\lambda * M_\lambda = M_\lambda * \widetilde{r}_\lambda$, $S_\lambda * r_\lambda = \widetilde{r}_\lambda * S_\lambda$.*

3.3. **Double centralizer property.** We will be mostly interested not in the convolution algebras per se, but in their interesting subalgebras, which should be thought of as "integral forms". Let us fix a G-invariant subring $T \subseteq \mathcal{R}$, and make the following assumption:

(3.7) $$m(y) \in \mathcal{R} \text{ defined in (3.5) is invertible for any } y \in Y, \text{ and } m(y)^{\pm 1} \in T.$$

Definition 3.10. Consider the following subalgebras:

$$\mathbf{B}^T = \langle K_\lambda, t_\omega \,:\, \lambda \in \Lambda, t \in T\rangle \subseteq \mathbf{B}, \quad \mathbf{A}^T = \langle \mathbf{B}^T, S_\lambda, M_\lambda, \,:\, \lambda \in \Lambda\rangle \subseteq \mathbf{A}.$$

We also define $\mathbf{C}^T = \mathbf{A}^T * 1_\omega$; it is an $(\mathbf{A}^T, \mathbf{B}^T)$-bimodule.

The following equalities immediately follow from the definition:

$$\mathbf{A}^T_{\mu\lambda} = M_\mu * \mathbf{B}^T * S_\lambda, \quad \mathbf{C}^T_\lambda = M_\lambda * \mathbf{B}^T.$$

Following closely the proof of [Pou09, Theorem 2.1], we have the following result.

Theorem 3.11. *Assume that the condition (3.7) holds. Then, we have the following double centralizer property:*

$$\mathrm{End}_{\mathbf{A}^T}(\mathbf{C}^T) = \mathbf{B}^T, \qquad \mathrm{End}_{\mathbf{B}^T}(\mathbf{C}^T) = \mathbf{A}^T.$$

In particular, $\mathbf{B}^T = 1_\omega * \mathbf{A}^T * 1_\omega$, and the Schur functor is given by $\mathbf{A}^T$-mod $\to \mathbf{B}^T$-mod, $M \mapsto 1_\omega * M$.

Proof. The inclusions $\mathbf{B}^T \subseteq \mathrm{End}_{\mathbf{A}^T}(\mathbf{C}^T)$, $\mathbf{A}^T \subseteq \mathrm{End}_{\mathbf{B}^T}(\mathbf{C}^T)$ are obvious. Let us begin by showing the inclusion $\mathrm{End}_{\mathbf{A}^T}(\mathbf{C}^T) \subseteq \mathbf{B}^T$. To this end, let $P \in \mathrm{End}_{\mathbf{A}^T}(\mathbf{C}^T)$. Since P commutes with 1_λ, the last statement of Theorem 3.8 implies that the direct sum decomposition $\mathbf{C}^T = \bigoplus_\lambda \mathbf{C}^T_\lambda$ is preserved by P. Furthermore, $\mathbf{C}^T$ is a cyclic $\mathbf{A}^T$-module generated by $1_\omega \in \mathbf{C}^T_\omega$. Thus P is completely determined by the element $P(1_\omega) \in \mathbf{C}^T_\omega \simeq \mathbf{B}^T$, and so P lies in $\mathbf{B}^T$.

It remains to show that $\mathrm{End}_{\mathbf{B}^T}(\mathbf{C}^T) \subseteq \mathbf{A}^T$. Let $P \in \mathrm{End}_{\mathbf{B}^T}(\mathbf{C}^T)$. We can rewrite P as a sum of maps $P_{\mu\lambda}$, $\lambda,\mu \in \Lambda$, where each $P_{\mu\lambda}$ belongs to $\mathrm{Hom}_{\mathbf{B}^T}(\mathbf{C}^T_\lambda, \mathbf{C}^T_\mu)$. It suffices to show that each $P_{\mu\lambda}$ belongs to $\mathbf{A}^T_{\mu\lambda}$. From now on, we fix $\lambda,\mu \in \Lambda$ and write $P' = P_{\mu\lambda}$. First of all, for any $f \in \mathbf{C}^T_\lambda$ we have

$$P'(f) = P'(m_\lambda^{-1} * M_\lambda * S_\lambda * f) = P'(M_\lambda) * (\widetilde{m}_\lambda^{-1} * S_\lambda * f).$$

Thus P' is determined by a single element $P'(M_\lambda) \in \mathbf{C}^T_\mu$. Observe that

$$P'(M_\lambda) * K_\lambda = P'(M_\lambda * S_\lambda * M_\lambda) = P'(m_\lambda * M_\lambda) = P'(M_\lambda) * \widetilde{m}_\lambda.$$

We claim that

(3.8) $$\{h \in \mathbf{C}^T_\mu \,:\, h * K_\lambda = h * \widetilde{m}_\lambda\} = \mathbf{A}^T_{\mu\lambda} * M_\lambda.$$

It will follow from (3.8) that $P'(M_\lambda) = g * M_\lambda$ for some $g \in \mathbf{A}^T_{\mu\lambda}$, and so we may conclude.

The inclusion $\supseteq$ in (3.8) is clear:

$$(f * M_\lambda) * K_\lambda = f * M_\lambda * S_\lambda * M_\lambda = f * m_\lambda * M_\lambda = (f * M_\lambda) * \widetilde{m}_\lambda.$$

For the opposite inclusion $\subseteq$ in (3.8), let $h \in \mathbf{C}^T_\mu$ satisfying $h * K_\lambda = h * \widetilde{m}_\mu$. Then,

$$h = h * K_\lambda * \widetilde{m}_\lambda^{-1} = (h * \widetilde{m}_\lambda^{-1} * S_\lambda) * M_\lambda,$$

where we used Theorem 3.9. The claim is proved. ☐

We can slightly relax the condition (3.7).

Corollary 3.12. *Let T' be a ring, and let $e \in T'_G(Y)$. Assume that $e(y), m(y)$ are not zero divisors for all $y \in Y$. Consider the localization $\mathcal{R} := T'[e(y)^{-1}, m(y)^{-1}; y \in Y]$, and its subring $T = T'[m(y)^{-1}]$. Define $\mathbf{A}^T, \mathbf{B}^T, \mathbf{C}^T$ as in Theorem 3.10. Then the double centralizer property of Theorem 3.11 holds.* ☐

While the double centralizer theorem above is very general, it has two problems. First, neither of algebras $\mathbf{A}^T$, $\mathbf{B}^T$ is very explicit; even their size is not obvious. Second, there is no reason for the condition (3.7) to be satisfied, and it indeed fails in some situations of interest (see Theorem 4.7). The rest of the paper is dedicated to the detailed study of a particular setup where both of these issues can be controlled.

4. Coil Schur algebras

4.1. PQWP as twisted convolutions. From now on, we impose the following conditions PQWPs we consider:

(C1) There exists an element $\alpha \in F \otimes F$ satisfying $\alpha(\sigma(\alpha) + S) = R$;

(C2) Both α and β are central in $F \otimes F$;

(C3) The element $P := \alpha(x_1 - x_2) + \beta$ is not a zero divisor.

We expect that the condition (C2) is an artifact of our approach, see Section 5.4 for a discussion on how one might tackle removing it, and the importance of the other two conditions. Since R is expressed in terms of α and β, we will say that such PQWP depends on $(\alpha, \beta) \in (F \otimes F) \times (B \otimes B)$.

Example 4.1. The element α for certain PQWP can be found in Table 1. For general affine Frobenius Hecke algebra, we need to solve the following equation in $F \otimes F$:

$$\alpha(\sigma(\alpha) + \Delta) = 1.$$

It is not guaranteed such a solution in $F \otimes F$ exists. However, when either Δ is nilpotent, or F is graded and Δ has positive degree, we can express α as a formal series in Δ; e.g. when $\Delta^2 = 0$, we can take $\alpha = (\pm 1) - \Delta/2$. Under the same assumptions, the element P is not a zero divisor both for Savage and Rosso-Savage algebra. Solution for pro-p Iwahori Hecke algebras for $\mathrm{GL}_d(\mathbb{Q}_p)$ is obtained in an ad hoc manner.

We will realize PQWPs satifying the conditions above as twisted convolution algebras. In the notations of Section 3.1, let $X = G = \mathfrak{S}_d$, and let $\mathcal{R} = F^{\otimes d}(x_1, \dots, x_d)$ be field of fractions. Here, G acts on X by left multiplication, and on $\mathcal{R}$ by place permutation. Thanks to (C3), we can consider the following twist:

$$e(g) = g\left(\textstyle\prod_{1 \le i < j \le d} P_{ji}(x_i - x_j)\right). \tag{4.1}$$

This gives rise to a twisted convolution algebra $\mathbf{H}_d := \mathcal{R}_G(X \times X)$.

Let us recall the basis of Theorem 3.3. Since the action of G on X is transitive, we can choose the representatives π to be $(1, g)$, $g \in \mathfrak{S}_d$. Denote $\xi_{g,r} := \xi_{(1,g),r}$. By definition, $\xi_{g,r}(x, y) = \delta_{y,xg} e(x) x(r)$.

Note that

$$\xi_{g,r} * \xi_{g',r'}(x,y) = \sum\nolimits_{x'\in\mathfrak{S}_n} \delta_{x',xg} e(x)x(r)e(x')^{-1}\delta_{y,x'g'} e(x')x'(r') = \delta_{y,xgg'} e(x)x(rg(r')) \\ = \xi_{gg',rg(r')}(x,y). \tag{4.2}$$

Recall that we have an embedding of algebras $\mathcal{R} = \mathcal{R}_{\mathfrak{S}_d}(\mathfrak{S}_d) \to \mathbf{H}_d$, $r \mapsto \xi_{1,r}$ by Theorem 3.6. For $i = 1, \dots, d-1$, consider the following elements in $\mathbf{H}_d$:

$$H_i := K_i - \xi_{1,\alpha_i}, \quad K_i := \xi_{1,\frac{P_{i,i+1}}{x_i - x_{i+1}}} + \xi_{s_i,\frac{P_{i,i+1}}{x_i-x_{i+1}}}. \tag{4.3}$$

Proposition 4.2. *Let H_i be the element defined in (4.3). Then,*

(a) The H_i's satisfy the relations in Theorem 1.3. In particular, we obtain an algebra homomorphism $\Phi : B \wr \mathcal{H}_d \to \mathbf{H}_d$.
(b) The map Φ is injective.

Proof. Part (a) requires careful bookkeeping while applying (4.2). For the wreath and quadratic relations, it suffices to consider the $d = 2$ case. The wreath relation follows since

$$Hb - \sigma(b)H = \left(\xi_{1,\frac{\beta}{x_1-x_2}} + \xi_{\sigma,\frac{P}{x_1-x_2}}\right) * \xi_{1,b} - \xi_{1,\sigma(b)} * \left(\xi_{1,\frac{\beta}{x_1-x_2}} + \xi_{\sigma,\frac{P}{x_1-x_2}}\right) \\ = \xi_{1,\frac{\beta b - \sigma(b)\beta}{x_1-x_2}} + \xi_{\sigma,\frac{P\sigma(b)-\sigma(b)P}{x_1-x_2}} = \xi_{1,\partial^\beta(b)} = \rho(b), \tag{4.4}$$

while the quadratic relation follows since

$$H^2 = \left(\xi_{1,\frac{\beta}{x_1-x_2}} + \xi_{\sigma,\frac{P}{x_1-x_2}}\right)^2 = \xi_{1,\frac{\beta^2 - P\sigma(P)}{(x_1-x_2)^2}} + \xi_{\sigma,\frac{\beta P - P\sigma(\beta)}{(x_1-x_2)^2}} \\ = \xi_{1,\alpha\sigma(\alpha)+\frac{(\alpha(x_1-x_2)+\beta)(\beta-\sigma(\beta))}{(x_1-x_2)^2}} + \xi_{\sigma,\frac{P(\beta - s(\beta))}{(x_1-x_2)^2}} = \xi_{1,\alpha(\sigma(\alpha)+S)+\frac{S\beta}{(x_1-x_2)}} + \xi_{\sigma,\frac{SP}{x_1-x_2}} \\ = \xi_{1,R} + \xi_{1,S} * H = R + SH. \tag{4.5}$$

Next, we verify the braid relations. Suppose that $|i - j| > 1$. Then, $H_iH_j = H_jH_i$ follows from the fact that $\xi_{g,r}$ commutes with $\xi_{g',r'}$ provided $gg' = g'g$, $g(r') = r'$ and $g'(r) = r$.

Suppose that $|i - j| = 1$. It suffices to consider the $d = 3$ case. When we compute both sides of the relation, we get six terms corresponding to six elements of $\mathfrak{S}_3$. The terms corresponding to s_1s_2, s_2s_1, $s_1s_2s_1$ immediately coincide. For the rest, we have the following:

$$H_1H_2H_1 - H_2H_1H_2 = \xi_{1,\frac{\beta_1^2\beta_2(x_2-x_3)(x_1-x_3) - \beta_1\beta_2^2(x_1-x_2)(x_1-x_3) - \beta_{13}\beta_1\beta_{21}(x_2-x_3)^2 + \beta_{13}\beta_2\beta_{32}(x_1-x_2)^2}{(x_1-x_2)^2(x_1-x_3)(x_2-x_3)^2}} \\ + \xi_{s_1,P_1\frac{\beta_1\beta_2(x_1-x_3)-\beta_{13}\beta_{21}(x_2-x_3)-\beta_{13}\beta_2(x_1-x_2)}{(x_1-x_2)^2(x_1-x_3)(x_2-x_3)}} + \xi_{s_2,P_2\frac{\beta_1\beta_{13}(x_2-x_3)-\beta_1\beta_2(x_1-x_3)+\beta_{13}\beta_{32}(x_1-x_2)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)^2}}.$$

We need to check that the nominators vanish. After substituting $\beta = \Delta^{00} + \Delta^{10}x_1 + \Delta^{01}x_2 + \Delta^{11}x_1x_2$, we can use the relations (2.2) and centrality of Δ^{ij} to drop the subscripts and pretend that Δ^{ij}'s are commuting variables. By direct computation, we can check that all the nominators become divisible by $\Delta^{00}\Delta^{11} - \Delta^{10}\Delta^{01} = 0$, thanks to (A2).

For part (b), let us compute the image of the basis provided by Theorem 2.12. Thanks to the formula (4.2),

$$\Phi(H_w) = \xi_{w,P_w} + \sum\nolimits_{w'<w} \xi_{w',P_{w,w'}}$$

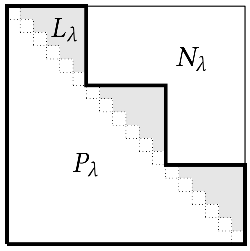
for some $P_{w,w'} \in \mathcal{R}$. In particular, the PBW monomial bH_w gets sent to ξ_{w,bP_w} modulo lower terms. We can conclude by applying Theorem 3.3 once we show that P_w is not a zero divisor for all $w \in \mathfrak{S}_n$. However, by formula (4.2) it has the form $\prod_k \frac{P_{i_k,j_k}}{(x_{i_k} - x_{j_k})}$, and each P_{i_k,j_k} is not a zero divisor by (C3). □

4.2. **The coil Schur algebras.** Let Λ be the set of compositions of d, $\omega = (1^d)$, $Y_\lambda = \mathfrak{S}_d/\mathfrak{S}_\lambda$ for any $\lambda \in \Lambda$, and $Y = \bigsqcup_\lambda Y_\lambda$. Note that for any λ,

$$\mathcal{R}_G(Y_\lambda) = \mathcal{R}_{\mathfrak{S}_d}(\mathfrak{S}_d/\mathfrak{S}_\lambda) = \mathcal{R}_{\mathfrak{S}_\lambda}(\mathrm{pt}) = \mathcal{R}^{\mathfrak{S}_\lambda}.$$

Given a composition λ of r parts, define

$$N_\lambda = \bigcup_{k=1}^{r}\{(i,j) \,:\, 1 \le i \le \lambda_1 + \cdots + \lambda_k < j \le d\},$$
$$P_\lambda = \{(i,j) \,:\, 1 \le i \ne j \le d\} \setminus N_\lambda,$$
$$L_\lambda = \{(i,j) \,:\, 1 \le i < j \le d\} \setminus N_\lambda,$$
$$e_\lambda = \prod_{(i,j)\in N_\lambda}(x_i - x_j)\prod_{(i,j)\in P_\lambda} P_{ij}.$$



Consider the twisted convolution algebra $\mathbf{S}_d := \mathcal{R}_G(Y \times Y)$ for the twist $e \in \mathcal{R}_G(Y)$ given by $e([g]) = g(e_\lambda)$ for $[g] \in \mathfrak{S}_d/\mathfrak{S}_\lambda$; note that $1_\omega \mathbf{S}_d 1_\omega = \mathbf{H}_d$ by definition. By our setup in Section 3.1, $\mathbf{S}_d$ acts via twisted convolutions on $\mathbf{T}_d := \mathcal{R}_G(Y) = \bigoplus_{\lambda\in\Lambda}\mathcal{R}^{\mathfrak{S}_\lambda}$, which we call the *polynomial representation*.

Proposition 4.3. *The action of* $\mathbf{S}_d$ *on the polynomial representation* $\mathbf{T}_d$ *is faithful.*

Proof. Assume we have a nonzero element $\varphi \in \mathbf{S}_d$, such that $\varphi v = 0$ for any $v \in \mathbf{T}_d$. Truncating by idempotents, we can assume that $\varphi \in \mathcal{R}_G(Y_\lambda \times Y_\mu)$. Recall that for $v \in \mathcal{R}_G(Y_\mu)$ and $[g] \in \mathfrak{S}_d/\mathfrak{S}_\lambda$, we have

$$\varphi v([g]) = \sum_{h\in\mathfrak{S}_d/\mathfrak{S}_\mu} \varphi([g],[h])e([h])^{-1}v([h]) = \sum_{h\in\mathfrak{S}_d/\mathfrak{S}_\mu} \varphi([g],[h])h(e_\mu)^{-1}h(v([1])). \tag{4.6}$$

Let us assume that $\mu = \omega$ for simplicity; the general case is analogous. Consider the monomial basis $B = \{x_1^{i_1} \dots x_{d-1}^{i_{d-1}} \,:\, 0 \le i_j \le d - j\}$ of the ring of coinvariants

$$\mathrm{Co}_d = \Bbbk[x_1,\dots,x_d]/(\Bbbk[x_1,\dots,x_d]^{\mathfrak{S}_d}).$$

It is well known that Co_d is isomorphic to the regular $\mathfrak{S}_d$-module. In particular, the $d! \times d!$ matrix $(h(b))_{h\in\mathfrak{S}_d, b\in B}$ is invertible. According to the formula (4.6) and the assumption $\varphi v = 0$, this implies that the vector $(\varphi([1],[h])h(e)^{-1})_{h\in\mathfrak{S}_d}$ vanishes. Since φ is $\mathfrak{S}_d$-equivariant, this means that $\varphi = 0$, and so we arrived at a contradiction. □

Let us write out the action of our favorite elements of $\mathbf{S}_d$ on $\mathbf{T}_d \equiv \bigoplus_\lambda \mathcal{R}^{\mathfrak{S}_\lambda}$. Each split S_λ fixes $b \in \mathcal{R}^{\mathfrak{S}_\lambda}$ since $S_\lambda b = \sum_{g\in\mathfrak{S}_d/\mathfrak{S}_\lambda} \delta_{[g],[1]} e(g)g(b)e(g)^{-1} = b$. For $b \in \mathcal{R}$,

$$\begin{aligned} M_\lambda b &= \sum_{g\in\mathfrak{S}_d} \delta_{[1],[g]} e_\lambda g(b) g(e)^{-1} = \sum_{g\in\mathfrak{S}_\lambda} g(b\, e_\lambda/e) = \sum_{g\in\mathfrak{S}_\lambda} g\Big(b\prod_{(i,j)\in L_\lambda}\tfrac{P_{ij}}{x_i - x_j}\Big) \\ &= \partial_\lambda\Big(b\prod_{(i,j)\in L_\lambda} P_{ij}\Big), \end{aligned} \tag{4.7}$$

where ∂_λ is the Demazure operator associated to the longest element in $\mathfrak{S}_\lambda$, and the last equality is standard (see e.g. [MM22, Lemma 5.5]).

Using the map Φ from Theorem 4.2, let us define $H_w := \Phi(H_w)$ for all $w \in \mathfrak{S}_d$ by abuse of notation. Note that for an elementary transposition s_i, $1 \le i \le d-1$ we have

$$H_i = \Big(\xi_{1,\frac{P_{i,i+1}}{x_i - x_{i+1}}} + \xi_{s_i,\frac{P_{i,i+1}}{x_i - x_{i+1}}}\Big) - \xi_{1,\alpha_{i,i+1}} = S_{\lambda(s_i)} * M_{\lambda(s_i)} - \xi_{1,\alpha_i}, \tag{4.8}$$

where $\lambda(s_i) = (1^{i-1}, 2, 1^{d-i-1})$ is a strict composition of length $d-1$ with 2 at i-th place. In particular, let $T = B^{\otimes d}$. Then all elements H_w belong to $\mathbf{H}_d^T = \mathbf{B}^T$, as introduced in Theorem 3.10.

Denote the set of inversions of $w \in \mathfrak{S}_d$ by

$$\mathrm{Inv}(w) = \{1 \le i < j \le d \mid w(i) > w(j)\}.$$

The following statement is the computational heart of the paper.

Proposition 4.4. *Recall* K_λ *from* (3.5)*. Then,*

$$K_\lambda = \sum_{w\in\mathfrak{S}_\lambda} \prod_{(i,j)\in L_\lambda\setminus \mathrm{Inv}(w)} \alpha_{ij} H_w. \tag{4.9}$$

Consequently, $\mathbf{H}_d^T \simeq B \wr \mathcal{H}_d$.

Proof. Both sides clearly factor into a product over the components of λ, therefore it suffices to prove the claim for $\lambda = (d)$. Thanks to Theorem 4.3, it suffices to prove it on the polynomial representation. We thus identify $K_{(d)}$ with the difference operator $\partial_{w_0} \prod_{1 \le i < j \le n} P_{ij}$. This operator can be further factorized as $M_{d-1}M_{d-2} \dots M_1$, where

$$M_k = \partial_{(k-1,1)} \prod_{i=1}^{k-1} P_{ik}, \qquad \partial_{(k-1,1)} := \partial_1 \dots \partial_{k-1}. \tag{4.10}$$

Analogously, we can factor the right hand side:

$$\sum_{w \in \mathfrak{S}_\lambda} H_w = H'_{d-1} H'_{d-2} \dots H'_1, \qquad H'_k = H_{(1\,2\,\dots\,k)} + \alpha_{12} H_{(2\,\dots\,k)} + \dots + \prod_{i=1}^{k-1} \alpha_{i,k}. \tag{4.11}$$

Therefore, it suffices to show that $M_k = H'_k$ for all k. For $k = 1$, this follows from the definition of H_i in Theorem 4.2. Furthermore, note that $H'_{k+1} = H'_k H_{(k\,k+1)} + \prod_{i=1}^{k} \alpha_{i,k+1}$. Therefore, reasoning by recurrence, we are reduced to proving that

$$M_{k+1} = \prod_{i=1}^{k} \alpha_{i,k+1} + M_k H_k \tag{4.12}$$

as operators on $\mathcal{R}^{\mathfrak{S}_k} = (F^{\otimes k}[x_1, \dots, x_k])^{\mathfrak{S}_k} \otimes F^{\otimes(d-k)}[x_{k+1}, \dots, x_d]$. For $j < k$, write

$$\Pi_{j,k} = \prod_{i=1}^{j} P_{j,k}. \tag{4.13}$$

We have:

$$\begin{aligned}
M_{k+1} - M_k H_k &= \partial_{(k,1)} \Pi_{k,k+1} - \partial_{(k-1,1)} \Pi_{k-1,k} (\partial_k P_{k,k+1} - \alpha_k) \\
&= \partial_{(k-1,1)} \left(\Pi_{k-1,k} \partial_k P_{k,k+1} + \partial_k(\Pi_{k-1,k+1}) P_{k,k+1} - \Pi_{k-1,k} \partial_k P_{k,k+1} + \alpha_k \Pi_{k-1,k} \right) \\
&= \partial_{(k-1,1)} \left(\alpha_k (\Pi_{k-1,k+1} - \Pi_{k-1,k}) + \beta_k \partial_k (\Pi_{k-1,k+1}) + \alpha_k \Pi_{k-1,k} \right) \\
&= \partial_{(k-1,1)} \left(\alpha_k \Pi_{k-1,k+1} + \rho_k (\Pi_{k-1,k+1}) \right).
\end{aligned}$$

Note that this operator commutes with multiplication by any element in $\mathcal{R}^{\mathfrak{S}_k}$. Thus it remains to prove the following equality of elements in $\mathcal{R}$:

$$\partial_{(k-1,1)} \left(\alpha_k \Pi_{k-1,k+1} + \rho_k (\Pi_{k-1,k+1}) \right) = \prod_{i=1}^{k} \alpha_{i,k+1}. \tag{4.14}$$

We proceed by induction on k, with the base case $k = 1$ being trivial. Using the condition (A2) on β, we get the following:

$$\begin{aligned}
&\partial_{k-1} \left(\alpha_k P_{k-1,k+1} + \rho_k (P_{k-1,k+1}) \right) \\
&= \partial_{k-1} \left(\alpha_k \alpha_{k-1,k+1} (x_{k-1} - x_{k+1}) + \alpha_k \beta_{k-1,k+1} + \alpha_{k-1,k+1} \beta_k + \beta_k (\Delta^{01}_{k-1,k+1} + \Delta^{11}_{k-1,k+1} x_{k-1}) \right) \\
&= \alpha_k \alpha_{k-1,k+1} + \partial_{k-1} \left(\beta_k (\Delta^{01}_{k-1,k+1} + \Delta^{11}_{k-1,k+1} x_{k-1}) \right) \\
&= \alpha_k \alpha_{k-1,k+1} + \left(\Delta^{00}_k \Delta^{11}_{k-1,k+1} - \Delta^{10}_k \Delta^{01}_{k-1,k+1} \right) + \left(\Delta^{01}_k \Delta^{11}_{k-1,k+1} - \Delta^{11}_k \Delta^{01}_{k-1,k+1} \right) x_{k+1} \\
&= \alpha_k \alpha_{k-1,k+1}.
\end{aligned}$$

As a consequence, using Leibniz rule for ∂_{k-1} and ρ_k (see (P2)), we get

$$\begin{aligned}
&\partial_{(k-1,1)} \left(\alpha_k \Pi_{k-1,k+1} + \rho_k (\Pi_{k-1,k+1}) \right) \\
&= \partial_{(k-2,1)} \left(\Pi_{k-2,k+1} \partial_{k-1} (\alpha_k P_{k-1,k+1}) + \partial_{k-1} (\Pi_{k-2,k+1} \rho_k (P_{k-1,k+1}) + \rho_k (\Pi_{k-2,k+1}) P_{k-1,k}) \right) \\
&= \partial_{(k-2,1)} \left(\Pi_{k-2,k+1} \partial_{k-1} (\alpha_k P_{k-1,k+1} + \rho_k (P_{k-1,k+1})) + \partial_{k-1} (\rho_k (\Pi_{k-2,k+1}) P_{k-1,k}) \right) \\
&= \partial_{(k-2,1)} (\alpha_k \alpha_{k-1,k+1} + \alpha_{k-1} \rho_k + (\alpha_{k-1} + S_{k-1}) \sigma_{k-1} \rho_k + \rho_{k-1} \rho_k)(\Pi_{k-2,k+1}) \\
&= \partial_{(k-2,1)} \left((\alpha_{k-1} \rho_k + (\alpha_{k-1} + S_{k-1}) \sigma_{k-1} \rho_k + \rho_{k-1} \rho_k)(\Pi_{k-2,k+1}) - \alpha_k \sigma_k \rho_{k-1} (\Pi_{k-2,k}) \right) \\
&\quad + \prod_{i=1}^{k} \alpha_{i,k+1},
\end{aligned}$$

where we applied the inductive assumption to $\partial_{(k-2,1)} (\alpha_{k-1,k+1} \Pi_{k-2,k+1})$ in the last line. Since $\Pi_{k-2,k}$ is independent of $(k-1)$-st factor in $\mathcal{R}$, and $\rho_{k-1} = \beta_{k-1} \partial_{k-1}$, we have

$$\alpha_k \sigma_k \rho_{k-1} (\Pi_{k-2,k}) = \alpha_{k-1} \sigma_k \rho_{k-1} (\Pi_{k-2,k}) = \alpha_{k-1} \sigma_{k-1} \rho_k (\Pi_{k-2,k+1}).$$

Continuing our chain of equalities, we have

$$\begin{aligned}
&\partial_{(k-1,1)}\left(\alpha_k \Pi_{k-1,k+1} + \rho_k(\Pi_{k-1,k+1})\right)\\
&= \prod_{i=1}^{k} \alpha_{i,k+1} + \partial_{(k-2,1)}(\alpha_{k-1}\rho_k + S_{k-1}\sigma_{k-1}\rho_k + \rho_{k-1}\rho_k)(\Pi_{k-2,k+1})\\
&= \prod_{i=1}^{k} \alpha_{i,k+1} + \partial_{(k-2,1)}(\alpha_{k-1}\rho_k + S_{k-1}\sigma_{k-1}\rho_k\sigma_{k-1} + \rho_{k-1}\rho_k\sigma_{k-1})(\Pi_{k-2,k+1})\\
&= \prod_{i=1}^{k} \alpha_{i,k+1} + \partial_{(k-2,1)}\alpha_{k-1}\rho_k(\Pi_{k-2,k+1}) + \partial_{(k-2,1)}\rho_k\sigma_{k-1}\rho_k(\Pi_{k-2,k+1}),
\end{aligned}$$

where we used (P6) and the fact that $\rho_{k-1}(\Pi_{k-2,k+1}) = 0$.

Finally, consider the last term in the expression above, and use the inductive assumption:

$$\begin{aligned}
&\partial_{(k-2,1)}\rho_k\sigma_{k-1}\rho_k(\Pi_{k-2,k+1}) = \partial_{(k-2,1)}\rho_k\sigma_k\rho_{k-1}(\Pi_{k-2,k}) = \rho_k\sigma_k\partial_{(k-2,1)}\rho_{k-1}(\Pi_{k-2,k})\\
&= \rho_k\sigma_k\left(-\partial_{(k-2,1)}(\alpha_{k-1}\Pi_{k-2,k}) + \prod_{i=1}^{k-1}\alpha_{i,k}\right) = -\partial_{(k-2,1)}\rho_k\sigma_k(\alpha_{k-1}\Pi_{k-2,k})\\
&= -\partial_{(k-2,1)}\rho_k(\alpha_{k-1,k+1}\Pi_{k-2,k+1}) = -\partial_{(k-2,1)}\alpha_{k-1}\rho_k(\Pi_{k-2,k+1}).
\end{aligned}$$

This concludes the proof of (4.14), which implies (4.12), and thus the proposition is proved. □

Corollary 4.5. *We have the following expression for* m_λ*:*

$$m_\lambda = \sum_{w\in\mathfrak{S}_\lambda} \prod_{(i,j)\in L_\lambda\setminus \mathrm{Inv}(w)} \alpha_{ij} \prod_{(i,j)\in L_\lambda\cap \mathrm{Inv}(w)} (\sigma(\alpha)+S)_{ij}\,. \tag{4.15}$$

In particular, if $\alpha \in F\otimes F$ *is invertible and* $S\in F\otimes F$ *is nilpotent,* m_λ *is invertible as long as* $\operatorname{char}\Bbbk > d$.

Proof. Recall that $m_\lambda = M_\lambda S_\lambda(1)$. Since the action of S_λ on V_d is by inclusion $\mathcal{R}^{\mathfrak{S}_\lambda} \hookrightarrow \mathcal{R}$,

$$m_\lambda = M_\lambda S_\lambda(1) = M_\lambda(1) = K_\lambda(1) = \sum_{w\in\mathfrak{S}_\lambda}\prod_{(i,j)\in L_\lambda\setminus\mathrm{Inv}(w)} \alpha_{ij}H_w(1).$$

It remains to show that $H_w(1) = \prod_{(i,j)\in L_\lambda\cap\mathrm{Inv}(w)}(\alpha_{ij}+S_{ij})$. We know that

$$H_i(1) = S_{\lambda(s_i)}M_{\lambda(s_i)}(1) - \alpha_i = \alpha_i + (\sigma_i(\alpha_i)+S_i) - \alpha_i = \sigma_i(\alpha_i) + S_i,$$

and moreover, $H_i t = s_i(t)H_i$ for any $t\in F^{\otimes d}$. Writing out a reduced expression for H_w, we arrive at the desired formula. □

Recall the subalgebra $\mathbf{A}^T$ and the module $\mathbf{C}^T$ from Theorem 3.10. We call the subalgebra $\mathbf{S}_d^T := \langle \mathbf{H}_d^T, S_\lambda, M_\lambda \mid \lambda\in\Lambda\rangle$ the *coil Schur algebra*, i.e., $\mathbf{S}^{\mathrm{BLM}} = \mathbf{S}_d^T$ as in the introduction. Next, $\mathbf{C}^T = \bigoplus_\lambda M_\lambda \mathbf{H}_d^T$ is the $(\mathbf{S}_d^T, \mathbf{H}_d^T)$-bimodule. The following double centralizer property is an immediate consequence of Theorem 3.11:

Corollary 4.6. *Assume that* $m_{(i)}$ *is invertible for all* $i\leq d$*. Then,* $\mathrm{End}_{\mathbf{S}_d^T}(\mathbf{C}^T) = B\wr\mathcal{H}(d)$, $\mathrm{End}_{B\wr\mathcal{H}(d)}(\mathbf{C}^T) = \mathbf{S}_d^T$.

Example 4.7. Let us come back to Theorem 2.10.

- For nil-Hecke algebras, we have $\alpha = S = 0$. Therefore, $m_\lambda = 0$ for all λ, and so Theorem 4.6 does not apply;
- For degenerate affine Hecke algebras, we have $\alpha = 1$, $S = 0$, so that $m_\lambda = \prod_i \lambda_i!$. In particular, $m_{(i)} = i!$ for all i, and hence Theorem 4.6 applies when $\operatorname{char}\Bbbk > d$;
- For affine Hecke algebras, $\alpha = 1$, $S = q-1$, therefore $m_\lambda = \prod_i [\lambda_i]_q!$ are q-factorials. In particular, $m_{(i)} = \prod_{t=1}^{i}\frac{q^t-1}{q-1}$ for all i, and hence Theorem 4.6 applies when q is not a root of unity of order $\leq d$; note that $m_\lambda = 1$ for 0-Hecke algebra, and so Theorem 4.6 always applies;
- Finally, in the case of affine Frobenius Hecke algebras, Theorem 4.5 applies when the quadratic relation splits (see Theorem 4.1). In this case, Theorem 4.6 applies when $\operatorname{char}\Bbbk > d$.

4.3. (α,S)-**multinomial coefficients.** For any $\gamma\in Z(F\otimes F)$, let us define

$$\gamma_w := \prod_{(i,j)\in\mathrm{Inv}(w)} \gamma_{i,j} \in B^{\otimes d}, \qquad \gamma^*_w := \gamma_{w^{-1}}.$$

Lemma 4.8. *We have*

$$\gamma_w = \gamma_{i_N}\sigma_{i_N}(\gamma_{i_{N-1}})\dots(\sigma_{i_N}\dots\sigma_{i_2})(\gamma_{i_1}),\qquad \gamma^*_w := \gamma_{i_1}\sigma_{i_1}(\gamma_{i_2})\dots(\sigma_{i_1}\dots\sigma_{i_{N-1}})(\gamma_{i_N}),$$

where $w = s_{i_1}\dots s_{i_N} \in \mathfrak{S}_d$ is any reduced expression. It is understood that $\gamma_e = \gamma^*_e = 1^{\otimes d}$.

In particular, if $\gamma = q(1\otimes 1)$ for some $q \in \Bbbk$, then $\gamma_w = q^{\ell(w)}(1\otimes 1)$.

Proof. The proof for γ_w follows from an induction on the length of w. The initial case is trivial. The inductive case follows from the fact that $\mathrm{Inv}(s_i w) = \mathrm{Inv}(s_i w) \sqcup \{w^{-1}(i) < w^{-1}(i+1)\}$ if $s_i w > w$. ☐

As a corollary, (4.9) and (4.15) become, respectively,

$$K_\lambda = \sum\nolimits_{w\in\mathfrak{S}_\lambda} \alpha_{w^\lambda_\circ w} H_w,\quad m_\lambda = \sum\nolimits_{w\in\mathfrak{S}_\lambda} \alpha_{w^\lambda_\circ w}(\sigma(\alpha)+S)_w. \tag{4.16}$$

Let F be a unital ring, $S \in F\otimes F$ a weak Frobenius element, and let $\alpha \in Z(F\otimes F)$ be such that $\overline{\alpha} := \sigma(\alpha) + S$ is also central.

In view of Theorems 4.5 and 4.7, we may define a notion of (α, S)-*multinomial coefficients*, for $\lambda \vDash d$, by

$$\begin{bmatrix} d \\ \lambda \end{bmatrix}_{(\alpha,S)} := \sum_{w\in\mathfrak{S}^\lambda}\ \prod_{(i,j)\in N_\lambda\setminus \mathrm{Inv}(w)} \alpha_{ij} \prod_{(i,j)\in N_\lambda\cap \mathrm{Inv}(w)} \overline{\alpha}_{ij}. \tag{4.17}$$

Such multinomial coefficients appear when one tries to describe relations in the laurel Schur algebras, see Theorem 5.7. In particular, $m_{(d)} = \begin{bmatrix} d \\ 1^d \end{bmatrix}_{(\alpha,S)}$, and it specializes to the q-factorial $[d]_q! = \sum_{w\in\mathfrak{S}_d} q^{\ell(w)}$ when $\alpha = 1\otimes 1$ and $S = (q-1)(1\otimes 1)$. When $\lambda = (k, d-k)$, the set $\mathfrak{S}^\lambda$ is identified with the set of size k subsets of $\{1,\dots,d\}$. In the case $F = \Bbbk$, $\alpha = q(1\otimes 1) \equiv q$, $S = t - q$, we recover the (q,t)-binomial coefficients

$$\begin{bmatrix} d \\ k, d-k \end{bmatrix}_{(q,t-q)} = \sum_{I\subseteq[d],\ {}^\#I=k} q^{c(I,<)} t^{c(I,>)},\qquad c(I,\gtrless) := {}^\#\{(i,j)\in I\times([d]\setminus I) \mid i \gtrless j\}.$$

Note that $\begin{bmatrix} d \\ k,d-k \end{bmatrix}_{(\alpha,S)} \neq \begin{bmatrix} d \\ d-k,k \end{bmatrix}_{(\alpha,S)}$ in general. It is an interesting combinatorial question to see which classical formula for (q,t)-binomial coefficients extend to this setting. We would also like to know the geometric meaning of these elements when F is the cohomology ring of a smooth manifold X, and $S \in H^{\dim X}(X\times X)$ is the class of the diagonal.

4.4. **Bases of coil Schur algebras.** The basis of $\mathbf{S}^T_d$ can be expressed in a straightforward way from the basis of $B \wr \mathcal{H}(d)$. Recall that by Theorem 3.10, we have the following direct sum decomposition:

$$\mathbf{S}^T_d \simeq \bigoplus\nolimits_{\lambda,\mu} \mathbf{S}^T_{\lambda,\mu},\quad \text{where}\quad \mathbf{S}^T_{\lambda,\mu} := M_\lambda \mathbf{S}^T_d S_\mu.$$

Let us fix two compositions $\lambda,\mu \in \Lambda$ until the end of this section. By Theorem 2.12, $B\wr\mathcal{H}_d$ has a PBW basis in which a PBW monomial is of the form bH_w where $b \in B^{\otimes d}$ and $w \in \mathfrak{S}_d$, and hence any element in $B\wr\mathcal{H}(d)$ can be spanned by PBW monomials in a different order as below:

$$H_{w_1} b H_g H_{w_2},\quad b \in B^{\otimes d},\ w_1 \in \mathfrak{S}_\lambda,\ w_2 \in \mathfrak{S}_\mu,\ g \in {}^\lambda\mathfrak{S}^\mu. \tag{4.18}$$

Lemma 4.9. *For any $w \in \mathfrak{S}_\lambda$, we have $H_w S_\lambda = \overline{\alpha}_w S_\lambda$, $M_\lambda H_w = M_\lambda \overline{\alpha}_w$.*

Proof. Interpreting $\mathbf{S}^T_d$ as a subalgebra of the convolution algebra $\mathbf{S}_d$ from Section 4.2, we see that the two equations are completely symmetric. It therefore suffices to check the first one. By Theorem 4.3 we can check it on the polynomial representation $\mathbf{T}_d$. Let $f \in R^{\mathfrak{S}_\lambda}$, and let $w = s_{i_1}\dots s_{i_l}$ be a reduced expression. Then,

$$\begin{aligned} H_w S_\lambda(f) = H_w(f) &= H_{s_{i_1}}\dots H_{s_{i_l}}(f) = H_{s_{i_1}}\dots H_{s_{i_{l-1}}}((\sigma_{i_l}(\alpha_{i_l}) + S_{i_l})\sigma_{i_l}(f) + \rho_{i_l}(f)) \\ &= H_{s_{i_1}}\dots H_{s_{i_{l-1}}}(\overline{\alpha}_{i_l} f) = \dots = \prod\nolimits_{(i,j)\in L_\lambda, w(i)>w(j)} \overline{\alpha}_{ij} f = \overline{\alpha}_w S_\lambda, \end{aligned}$$

where we have repeatedly used that f is $\mathfrak{S}_\lambda$-symmetric. ☐

Let $\nu = \delta^r(\lambda, g, \mu)$, $\delta = \delta^c(\lambda, g, \mu)$ (see Theorem 1.1).

Proposition 4.10. *Pick a $\Bbbk$-basis $\mathbb{B}$ of $M_\nu T S_\nu \subseteq T^{\mathfrak{S}_\nu}$. Then the set below is a $\Bbbk$-basis of $\mathbf{S}^T_{\lambda,\mu}$:*

$$\{M_\lambda b H_g S_\mu \mid g \in {}^\lambda\mathfrak{S}^\mu, b \in \mathbb{B}\}. \tag{4.19}$$

Proof. Theorem 4.9 together with (4.18) imply that $\mathbf{S}^T_{\lambda,\mu}$ has the following spanning set:

$$\{M_\lambda(\textstyle\bigotimes_{j=1}^d b_{i_j})H_w S_\mu \mid i_j \in I, w \in {}^\lambda\mathfrak{S}^\mu\}.$$

Furthermore, combining Theorems 4.4, 4.5 and 4.9 we obtain

$$M_\lambda K_\nu = M_\lambda m_\nu, \qquad K_\nu S_\lambda = m_\nu S_\lambda. \tag{4.20}$$

Let $f \in B^{\otimes d}$. Using (4.20) and the braid relations in $B \wr \mathcal{H}_d$, we obtain

$$\begin{aligned} M_\lambda f H_w S_\mu &= M_\lambda K_\nu m_\nu^{-1} f H_w m_\delta^{-1} K_\delta S_\mu \doteq M_\lambda K_\nu m_\nu^{-2} f H_w K_\delta S_\mu \\ &= M_\lambda K_\nu m_\nu^{-2} f K_\nu H_w S_\mu = M_\lambda S_\nu (M_\nu m_\nu^{-2} f S_\nu) M_\nu H_w S_\mu \\ &= M_\lambda K_\nu m_\nu^{-1}(M_\nu m_\nu^{-1} f S_\nu) H_w S_\mu = M_\lambda (M_\nu m_\nu^{-1} f S_\nu) H_w S_\mu, \end{aligned}$$

where the dot over an equality $\doteq$ means that it holds up to lower terms in w. We deduce that the set (4.19) spans $\mathbf{S}^T_{\lambda,\mu}$ over $\Bbbk$.

In order to check linear independence, recall the basis of $\mathbf{S}_d$ from Theorem 3.3:

$$\xi_{w,f}(y,y') = \sum_{\sigma \in \mathfrak{S}_d/\mathfrak{S}_\nu} \delta_{[\sigma],y}\delta_{[\sigma],y'}\sigma(e_\lambda f), \qquad w \in {}^\lambda\mathfrak{S}^\mu, f \in B_{\mathcal{R}}^{\mathfrak{S}_\nu}.$$

A lengthy computation completely analogous to the one in the proof of [MM22, Theorem 4.10] shows that in terms of this basis, $M_\lambda f H_w S_\mu$ has the highest term $\xi_{w,fe_\mu\beta_w^{-1}}$, where

$$\beta_w = \prod\nolimits_{(i,j)\in N_\lambda \cup w(N_\mu)} (x_i - x_j) \prod\nolimits_{(i,j)\in P_\lambda \cap w(P_\mu)} P_{ij}.$$

Since $e_\mu\beta_w^{-1}$ is invertible, we see that the set (4.19) is related to a subset of (3.2) by an upper-triangular (in w) change of basis. This yields that its elements are linearly independent, and so we may conclude. ☐

5. Laurel Schur algebras

5.1. **Removing invertibility assumption.** Without the invertibility of m_λ, the subalgebra $\mathbf{S}^T_{\lambda,\lambda}$ is too small to satisfy double centralizer proprety. For instance, it does not even contain the identity map $M_\lambda \mathbf{H}^T_d \to M_\lambda \mathbf{H}^T_d$. Indeed, for $\lambda = (d)$ we have ${}^\lambda\mathfrak{S}^\mu = \{1\}$, $\mathfrak{S}_\nu = \mathfrak{S}_\delta = \mathfrak{S}_d$, and so all elements of $\mathbf{S}^T_{(d),(d)}$ are of the form

$$M_{(d)} f S_{(d)} = f M_{(d)} S_{(d)} = f m_{(d)}, \qquad f \in T^{\mathfrak{S}_d}.$$

This illustrates the failure of double centralizer property as stated in Theorem 4.6. In order to remove the condition on m_λ, we will exploit the extra structure afforded by subdivision of compositions. Let us begin by proving some properties of PQWP algebra $\mathbf{H}^T_d = B \wr \mathcal{H}(d)$.

For any $\lambda \in \Lambda$ and a refinement $\nu \vDash \lambda$, write $w'_\circ := w_\circ^{({}^\nu\mathfrak{S}_\lambda)}$ and $w''_\circ := w_\circ^{(\mathfrak{S}^\nu_\lambda)}$ for short; see Section 1.2 for the notation. Define

$$K^\nu_\lambda := \sum\nolimits_{w \in {}^\nu\mathfrak{S}_\lambda} H_w \alpha_{ww'_\circ}, \quad \widetilde{K}^\nu_\lambda := \sum\nolimits_{w \in \mathfrak{S}^\nu_\lambda} \alpha^*_{w''_\circ w} H_w. \tag{5.1}$$

Example 5.1. Let $\lambda = (3)$. Then,

$$\begin{aligned} K_{(3)} &= H_1H_2H_1 + \alpha_1 H_2H_1 + \alpha_2 H_1H_2 + \alpha_1\alpha_{13}H_2 + \alpha_2\alpha_{13}H_1 + \alpha_1\alpha_{13}\alpha_2, \\ &= (H_1H_2 + \alpha^*_{s_1}H_2 + \alpha^*_{s_1s_2})(H_1 + \alpha_1) = (H_1 + \alpha_1)(H_2H_1 + H_2\alpha_{s_1} + \alpha_{s_2s_1}), \end{aligned}$$

Indeed,

$$K^{(2,1)}_{(3)} = H_2H_1 + H_2\alpha_{s_1} + \alpha_{s_2s_1}, \quad \widetilde{K}^{(2,1)}_{(3)} = H_1H_2 + \alpha^*_{s_1}H_2 + \alpha^*_{s_1s_2},$$

and hence $y_{(3)}H_1 = y_{(3)}\overline{\alpha}_1$.

Lemma 5.2. *Suppose that $A \equiv (\lambda, g, \mu)$, $\nu = \delta^r(\lambda, g, \mu)$, and $\delta = \delta^c(\lambda, g, \mu)$. Then,*

$$K_\mu = K_\delta K^\delta_\mu, \quad K_\lambda = \widetilde{K}^\nu_\lambda K_\nu.$$

In particular, $K_A := K_\lambda H_g K_\mu^\delta = \widetilde{K}_\lambda^\nu H_g K_\mu$ *is well-defined.*

Proof. It suffices to consider a transposition $s_i \in \mathfrak{S}_\lambda$. Let I be the composition such that $\mathfrak{S}_I = \langle s_i \rangle$. Denote by $\mathfrak{S}_\lambda^I$ and ${}^I\mathfrak{S}_\lambda$ the set of shortest left and right coset representatives of $\mathfrak{S}_I \subseteq \mathfrak{S}_\lambda$ with longest elements $w_\circ^l$ and $w_\circ^r$, respectively. Then,

$$K_\lambda = \sum\nolimits_{w \in \mathfrak{S}_\lambda^I} \alpha_{w_\circ^l w} H_w (H_i + \alpha_i) = \sum\nolimits_{w \in {}^I\mathfrak{S}_\lambda} (H_i + \alpha_i) H_w \alpha^*_{w w_\circ^r}, \tag{5.2}$$

and we are done. ☐

Lemma 5.3. *Let* $F \in \mathbf{H}_d^T$. *Then,* $F \in \mathbf{H}_d^T K_\lambda$ *if and only if* $FH_i = F\overline{\alpha}_i$ *for all* i *with* $s_i \in \mathfrak{S}_\lambda$.

Proof. Fix an i such that $s_i \in \mathfrak{S}_\lambda$, and let I be the composition such that $\mathfrak{S}_I = \langle s_i \rangle$. Thanks to (C1), $0 = (H_i + \alpha_i)(H_i - \overline{\alpha}_i)$, and hence

$$(H_i + \alpha_i)H_i = (H_i + \alpha_i)\overline{\alpha}_i. \tag{5.3}$$

The necessity follows from (5.2), since $K_\lambda H_i = K_\lambda \overline{\alpha}_i$. For sufficiency, use Theorem 2.12 to write $F = F_1 + F_2 H_i$, where F_1, F_2 are linear combinations of elements bH_w, $b \in B$, $w \in \mathfrak{S}^I$. Then

$$0 = (F_1 + F_2 H_i)(\overline{\alpha}_i - H_i) = F_1 \overline{\alpha}_i - F_2 R_i + (F_2 \alpha_i - F_1) H_i,$$

and so $F_1 = F_2 \alpha_i$. Thus, $F = F_2(\alpha_i + H_i)$. Doing the same computation for all i with $s_i \in \mathfrak{S}_\lambda$, we conclude that $F = F' K_\lambda$, where F' is a linear combinations of elements bH_w, $b \in B$, $w \in \mathfrak{S}^\lambda$. ☐

Proposition 5.4. *Let* $\lambda, \mu \in \Lambda$. *For each* $g \in {}^\lambda\mathfrak{S}^\mu$, *let* $\nu(g) := \delta^r(\lambda, g, \mu)$, $\delta(g) := \delta^c(\lambda, g, \mu)$, *and pick* $\Bbbk$*-bases* $\overline{\mathbb{B}}_g$ *of* $T^{\mathfrak{S}_{\nu(g)}}$ *and* $\overline{\mathbb{B}}'_g$ *of* $T^{\mathfrak{S}_{\delta(g)}}$, *respectively. Then:*

(a) The set $\{K_{A,b} := K_\lambda b H_g K_\mu^{\delta(g)} \mid g \in {}^\lambda\mathfrak{S}^\mu, b \in \overline{\mathbb{B}}_g\}$ *is a* $\Bbbk$*-basis of* $K_\lambda \mathbf{H}_d^T \cap \mathbf{H}_d^T K_\mu$, *and so is the set* $\{\widetilde{K}_\lambda^{\nu(g)} H_g b K_\mu \mid g \in {}^\lambda\mathfrak{S}^\mu, b \in \overline{\mathbb{B}}'_g\}$.

(b) Let $\alpha_A := \prod_{(i,j) \in (N_\lambda \cap g(N_\mu)) \setminus \mathrm{Inv}(g)} \alpha_{ij}$. *For any* $\lambda, \mu \in \Lambda$, *we have*

$$K_{(d)} = \sum\nolimits_{g \in {}^\lambda\mathfrak{S}^\mu} \alpha_A K_\lambda^{\nu(g)} H_g K_\delta K_\mu^{\delta(g)}. \tag{5.4}$$

Proof. For (a), it follows from Theorem 5.2 that $K_{A,b} \in K_\lambda \mathbf{H}_d^T \cap \mathbf{H}_d^T K_\mu =: \mathbb{H}$. Let $\mathbb{H}' \subseteq \mathbb{H}$ be the subspace spanned by elements of the form $K_{A,b}$. We want to show that any $h \in \mathbb{H}$ lies in $\mathbb{H}'$. Since $h \in K_\lambda \mathbf{H}_d^T$, we can write

$$h = \sum\nolimits_{w \in {}^\lambda\mathfrak{S}} K_\lambda b_w H_w \quad \text{for some} \quad b_w \in B^{\otimes d}.$$

Pick any $x \in {}^\lambda\mathfrak{S}$ with $b_x \neq 0$. Suppose that $x \notin {}^\lambda\mathfrak{S}^\mu$, then one can pick an $s = s_i \in \mathfrak{S}_\mu$ with $xs < x$ and $xs \in {}^\lambda\mathfrak{S}$. Since $h \in \mathbf{H}_d^T K_\mu$, by Theorem 5.3 we have $hH_i = h\overline{\alpha}_i$, and hence by comparing the coefficients of H_x on both sides of $\sum_w K_\lambda b_w H_w \overline{\alpha}_i = \sum_w K_\lambda b_w H_w H_i$ yields that $b_{xs} \neq 0$. Therefore, one can repeat this procedure to find a representative $g \in {}^\lambda\mathfrak{S}^\mu$ with $b_g \neq 0$.

Let $s = s_i \in \mathfrak{S}_{\delta(g)}$. On one hand we have $K_\lambda b_g H_g H_i = K_\lambda b_g H_g \overline{\alpha}_i$ by Theorem 5.3, and on the other hand,

$$\begin{aligned} K_\lambda b_g H_g H_i &= K_\lambda b_g H_j H_g = K_\lambda (H_j \sigma_j(b_g) + \rho_j(b_g)) H_g \\ &= K_\lambda \sigma_j(b_g) H_g \overline{\alpha}_i + K_\lambda \rho_j(b_g) H_g, \end{aligned}$$

where $j = g(i)$. This implies that $\overline{\alpha}_j \sigma_j(b_g) + \rho_j(b_g) = \overline{\alpha}_j b_g$, and so

$$0 = (\beta_j - \overline{\alpha}_j(x_j - x_{j+1}))(b_g - \sigma_j(b_g)) = \sigma_j(\alpha_j(x_j - x_{j+1}) + \beta_j)(b_g - \sigma_j(b_g)).$$

Since $\alpha_j(x_j - x_{j+1}) + \beta_j$ is not a zero divisor by (C3), it follows that $b_g = \sigma_j(b_g)$. We proved that $b_g \in T^{\mathfrak{S}_{\delta(g)}}$, and so $h - K_{A,b_g} \in \mathbb{H}'$. Proceeding by recurrence, we obtain that h is a sum of terms of the form K_{A,b_g}, $b_g \in T^{\mathfrak{S}_{\delta(g)}}$, and so $h \in \mathbb{H}'$.

The linear independence follows from $\Bbbk$-linear independence of the set $\{K_A \mid g \in {}^\lambda\mathfrak{S}^\mu\}$. Recall the longest elements $w_\circ^\lambda = w_\circ$, $w_\circ'' = w_\circ^{{}^\delta\mathfrak{S}_\mu}$, $w_\circ^A = w_\circ^{\mathfrak{S}_\lambda g \mathfrak{S}_\mu}$. Then the elements K_A are linearly independent, because their highest terms $H_{w_\circ^\lambda} H_g H_{w_\circ''} = H_{w_\circ^A}$ are.

Finally, the proof of (b) is a direct computation. Namely, recall the definitions (4.9,5.1) of all the terms. Since all the coefficients are products of α_{ij}'s over non-inversions, it suffices to check the equality of coefficients at each H_g, $g \in {}^\lambda\mathfrak{S}^\mu$. There, we conclude by noticing that

$$L_{(d)} \setminus \mathrm{Inv}(g) = (L_\lambda \cup g(L_\mu)) \sqcup (N_\lambda \cap g(N_\mu)) \setminus \mathrm{Inv}(g),$$

and that the coefficient of H_g on the right hand side of (5.4) is precisely $\prod_{(i,j)\in L_\lambda \cup g(L_\mu)} \alpha_{ij}$. □

5.2. **Laurel Schur algebras.**

Definition 5.5. Let $\lambda, \nu \in \Lambda$ with $\nu \vDash \lambda$. Define *partial splits and merges* by

$$S_{\nu\lambda} \in \mathcal{R}_G(Y_\nu \times Y_\lambda),\ S_\lambda(x,y) := \delta_{p_{\nu\lambda}(x),y} e(y); \quad M_{\lambda\nu} \in A_{\lambda\nu},\ M_\lambda(y,x) := \delta_{y,p_{\nu\lambda}(x)} e(y).$$

where $p_{\nu\lambda} : \mathfrak{S}_d/\mathfrak{S}_\nu \to \mathfrak{S}_d/\mathfrak{S}_\lambda$ is the natural projection. Define a subalgebra

$$\overline{\mathrm{S}}^{\mathrm{BLM}} = \overline{\mathrm{S}}_d^T := \langle S_{\nu\lambda}, M_{\lambda\nu}, t_\lambda \mid \nu \vDash \lambda \in \Lambda, t \in T^{\mathfrak{S}_\lambda} = T_{\mathfrak{S}_d}(Y_\lambda) \rangle \subseteq \mathrm{S}_d, \tag{5.5}$$

which we call the *laurel Schur algebra*.

Lemma 5.6. *Let $\nu \vDash \mu \vDash \lambda$. We have $S_{\nu\mu}S_{\mu\lambda} = S_{\nu\lambda}$, $M_{\lambda\mu}M_{\mu\nu} = M_{\lambda\nu}$. In particular, $S_{\lambda\lambda} = M_{\lambda\lambda}$ is an idempotent in $\overline{\mathrm{S}}_d^T$.*

Proof. The computation is identical for splits and merges, so we only write it for the former:

$$S_{\nu\mu}S_{\mu\lambda}(x,y) = \sum\nolimits_z \delta_{p_{\nu\mu}(x),z}\delta_{p_{\mu\lambda}(z),y} e(z)e(z)^{-1}e(y) = \delta_{p_{\nu\lambda}(x),y}e(y) = S_{\nu\lambda}. \qquad □$$

The following result is proved completely analogously to Theorem 4.5; we will not use it.

Lemma 5.7. *Let $\lambda \vDash d$. Then, $S_\lambda^{(d)} M_{(d)}^\lambda = \begin{bmatrix} d \\ \lambda \end{bmatrix}_{(\alpha,S)}$.* □

It is clear from the definition that $\mathrm{S}_d^T \subseteq \overline{\mathrm{S}}_d^T$.

Proposition 5.8. *We have $\mathrm{S}_d^T = \overline{\mathrm{S}}_d^T$ if m_λ is invertible for all λ.*

Proof. We have

$$M_{\lambda\nu} = M_{\lambda\nu}M_\nu S_\nu m_\nu^{-1} = M_\lambda S_\nu m_\nu^{-1}, \qquad S_{\nu\lambda} = m_\nu^{-1} M_\nu S_\lambda,$$

by Theorem 5.6, so that it only remains to prove that any $t \in T_{\mathfrak{S}_d}(Y_\lambda)$ belongs to S_d^T. However, $t = M_\lambda S_\lambda m_\lambda^{-1} t = M_\lambda m_\lambda^{-1} t' S_\lambda$, where t' is the image of t in $T = T_{\mathfrak{S}_d}(\mathfrak{S}_d)$. We are done. □

The goal of this section is to prove the following double centralizer property without an invertibility assumption.

Theorem 5.9. *We have $\mathrm{End}_{\overline{\mathrm{S}}_d^T}(\mathbf{C}^T) = B \wr \mathcal{H}(d)$, $\mathrm{End}_{B\wr\mathcal{H}(d)}(\mathbf{C}^T) = \overline{\mathrm{S}}_d^T$. In particular, $B \wr \mathcal{H}(d) = S_{\omega\omega} * \overline{\mathrm{S}}_d^T * S_{\omega\omega}$, and the Schur functor is given by $\overline{\mathrm{S}}_d^T$–mod $\to B \wr \mathcal{H}(d)$–mod, $M \mapsto S_{\omega\omega} * M$.*

By Theorem 3.8, we have a natural inclusion

$$\psi = \psi_\lambda^L \circ \psi_\mu^R : \mathbf{S}_{\lambda\mu} \hookrightarrow \mathbf{H}_d, \qquad x \mapsto S_\lambda x M_\mu.$$

Let us denote by $\overline{\mathbf{A}}_{\lambda\mu}$ the intersection $K_\lambda \mathbf{H}_d^T \cap \mathbf{H}_d^T K_\mu$. Since $K_\lambda = S_\lambda M_\lambda$, all such elements belong to the image of ψ; we will therefore implicitly identify $\overline{\mathbf{A}}_{\lambda\mu}$ with its preimage in $\mathbf{S}_{\lambda\mu}$ under ψ. Note that we can alternatively write

$$\overline{\mathbf{A}}_{\lambda\mu} = \psi_\lambda^L(M_\lambda \mathbf{H}_d^T) \cap \psi_\mu^R(\mathbf{H}_d^T S_\mu).$$

Let $\lambda, \mu \in \Lambda$, and w, ν, δ as in Theorem 5.4. Consider the element

$$\widetilde{H}_w := H_w K_\nu. \tag{5.6}$$

We have $\widetilde{H}_w = (H_w S_\nu) M_\nu$, but also by braid relations $\widetilde{H}_w = K_\delta H_w = S_\delta(M_\delta H_w)$; therefore $\widetilde{H}_w \in \overline{\mathbf{A}}_{\delta\nu}$.

Lemma 5.10. *For any $\lambda, \mu \in \Lambda$ the vector space $\overline{\mathbf{A}}_{\lambda\mu} \subseteq \mathrm{S}_d$ is spanned by elements of the form $M_{\lambda\nu} b \widetilde{H}_w S_{\delta\mu}$, where $w \in {}^\lambda\mathfrak{S}^\mu$, $b \in T^{\mathfrak{S}_\nu}$, and ν, δ are as in Section 4.4.*

Proof. Applying ψ, this follows from Theorem 5.4(a). □

Lemma 5.11. *Let $\lambda = (d_1, d_2)$, $\mu = (d_2, d_1)$. For any $0 \le i \le \min(d_1, d_2)$ denote $\nu_i = (i, d_1 - i, d_2 - i, i)$, $\delta_i = (i, d_2 - i, d_1 - i, i)$, and $w_i \in \mathfrak{S}_d$ the shuffle sending ν_i' to ν_i. Denote $c_i = \prod_{1 \le i', j' \le i} \alpha_{i', d-i+j'}$. Then we have $S_{\lambda,(d)} M_{(d),\mu} = \sum_{i=0}^{\min(d_1,d_2)} c_i M_{\lambda,\nu_i} \widetilde{H}_{w_i} S_{\delta_i,\mu}$.*

Proof. It suffices to check this equiality after applying ψ. Note the following simple equilities:

$$S_\nu S_{\nu\lambda} M_\lambda = K_\lambda = S_\nu M_\nu K_\lambda^{(\nu)} \quad \Rightarrow \quad S_{\nu\lambda} M_\lambda = M_\nu K_\lambda^{(\nu)},$$
$$S_\lambda M_{\lambda\nu} M_\nu = K_\lambda = K_\lambda^{(\nu)} S_\nu M_\nu \quad \Rightarrow \quad S_\lambda M_{\lambda\nu} = K_\lambda^{(\nu)} S_\nu.$$

Using these and the associativity equations of Theorem 5.6, we get in the image of ψ

$$\begin{aligned} K_{(d)} &= \sum_{i=0}^{\min(d_1,d_2)} c_i S_\lambda M_{\lambda,\nu_i} \widetilde{H}_{w_i} S_{\delta_i,\mu} M_\mu \\ &= \sum_{i=0}^{\min(d_1,d_2)} c_i K_\lambda^{(\nu_i)} S_{\nu_i} \widetilde{H}_{w_i} M_{\delta_i} K_\mu^{(\delta_i)} = \sum_{i=0}^{\min(d_1,d_2)} c_i K_\lambda^{(\nu_i)} H_{w_i} K_\delta K_\mu^{(\delta_i)}. \end{aligned}$$

One easily checks that $c_i = \alpha_A$ for $A = (\lambda, w_i, \mu)$, so we can conclude by Theorem 5.4(b). □

5.3. Proof of double centralizer property.

Proof of Theorem 5.9. The actions of $B \wr \mathcal{H}(d) = \mathbf{H}_d^T$ and $\overline{\mathbf{S}}_d^T$ on $\mathbf{C}$ manifestly commute. Moreover, the action of $\overline{\mathbf{S}}_d^T$ descends to $\mathbf{C}^T$. Indeed,

$$t_\lambda(M_\lambda F) = M_\lambda(\widetilde{t}_\lambda F); \qquad M_{\lambda\nu}(M_\nu F) = M_\lambda F, \qquad S_{\nu\lambda}(M_\lambda F) = M_\nu(K_\lambda^{(\nu)} F).$$

It follows that $\overline{\mathbf{S}}_d^T \subset \mathrm{End}_{\mathbf{H}_d^T}(\mathbf{C}^T)$. The first equality also immediately follows:

$$\mathbf{H}_d^T = \mathrm{End}_{\mathbf{S}_d^T}(\mathbf{C}^T) \supseteq \mathrm{End}_{\overline{\mathbf{S}}_d^T}(\mathbf{C}^T) \supseteq \mathbf{H}_d^T \quad \Rightarrow \quad \mathrm{End}_{\overline{\mathbf{S}}_d^T}(\mathbf{C}^T) = \mathbf{H}_d^T.$$

It remains to show the inclusion $\mathrm{End}_{\mathbf{H}_d^T}(\mathbf{C}^T) \subseteq \overline{\mathbf{S}}_d^T$. As in the proof of Theorem 3.11, a map P of $\mathbf{H}_d^T$-modules $\mathbf{C}_\lambda^T \to \mathbf{C}_\mu^T$ is completely determined by the element $P(M_\lambda)$. Moreover, it has to satisfy the conditions of Theorem 5.3 by Theorem 4.9:

$$P(M_\lambda)H_i = P(M_\lambda H_i) = P(M_\lambda(\sigma_i(\alpha_i) + S_i)) = P(M_\lambda)(\sigma_i(\alpha_i) + S_i).$$

Therefore $\mathrm{End}_{\mathbf{H}_d^T}(\mathbf{C}^T) \subseteq \overline{\mathbf{A}}_{\lambda\mu}$. By Theorem 5.10, every element of $\overline{\mathbf{A}}_{\lambda\mu}$ is written as a product of partial splits, partial merges, elements of $T^{\mathfrak{S}_\nu}$ and $\widetilde{H}_w$. Note that $\widetilde{H}_w$ belongs to $\overline{\mathbf{S}}_d^T$. Indeed, let us write w as a reduced expression $s_{i_1} \dots s_{i_l}$, where s_{i_j} are elementary transpositions in $\mathfrak{S}_{|\nu|}$. By definition, $\widetilde{H}_w = S_\nu(M_\nu H_w) = (H_w S_\delta) M_\delta$ as an element of $\mathbf{A}_{\nu\delta}$. Given another $\widetilde{H}_{w'} = S_\delta(M_\delta H_{w'}) = (H_{w'} S_{\delta'}) M_{\delta'}$, we can compute the product $\widetilde{H}_w \widetilde{H}_{w'}$ inside $\mathbf{S}_d$, but as an element of $\overline{\mathbf{A}}_{\nu\delta}$, as follows:

$$\widetilde{H}_w \widetilde{H}_{w'} = (H_w S_\delta)(M_\delta H_{w'}) = H_w K_\delta H_{w'} = H_w H_{w'} K_{\delta'} = H_{ww'} K_{\delta'} = \widetilde{H}_{ww'}.$$

Reasoning by induction, we obtain $\widetilde{H}_w = \widetilde{H}_{s_{i_1}} \dots \widetilde{H}_{s_{i_l}}$. Theorem 5.11 implies that each $\widetilde{H}_{s_{i_j}}$ is expressed inductively in terms of partial splits and merges:

$$\widetilde{H}_{s_{i_j}} = \widetilde{H}_{w_0} = S_{\lambda,(d)} M_{(d),\mu} - \sum_{i=1}^{\min(d_1,d_2)} c_i M_{\lambda,\nu_i} \widetilde{H}_{w_i} S_{\delta_i,\mu}.$$

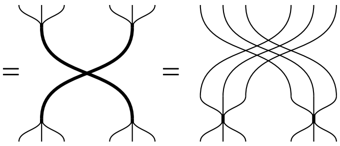
Therefore $\overline{\mathbf{A}}_{\lambda\mu} \subseteq \overline{\mathbf{S}}_d^T$, and we may conclude. □

Corollary 5.12. *Let $\lambda, \mu \in \Lambda$. For each $g \in {}^\lambda\mathfrak{S}^\mu$, let $\nu(g) := \delta^r(\lambda, g, \mu)$, and pick a $\Bbbk$-basis $\overline{\mathbb{B}}_g$ of $T^{\mathfrak{S}_{\nu(g)}}$. Then, the following set forms a $\Bbbk$-basis of $\overline{\mathbf{S}}_{\lambda,\mu}^T$:*

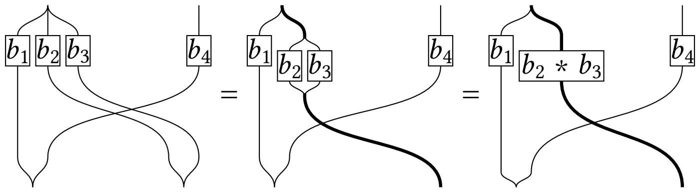
$$\{M_{\lambda\nu} b \widetilde{H}_g S_{\delta\mu} \mid g \in {}^\lambda\mathfrak{S}^\mu, b \in \overline{\mathbb{B}}_g\}. \tag{5.7}$$

Proof. Thanks to Theorem 5.10, it suffices to check linear independence. This is done in the same way as in Theorem 4.10. □

In order to better explain the difference between coil Schur $\mathbf{S}_d^T$ and laurel Schur $\overline{\mathbf{S}}_d^T$, let us represent their bases diagrammatically. We read algebra elements from right to left, and diagrams from bottom to top. We represent the idempotents 1_λ, see (3.5), by drawing strands of thicknesses $\lambda_1, \ldots, \lambda_k$, elements $t \in T$ by coupons on strands, splits and merges by splits and merges, and the elements $\widetilde{H}_w$ by crossings of thick strands. First of all, note that our definition (5.6) translates to "splits/merges go past crossings". For example:

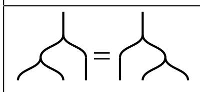
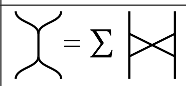
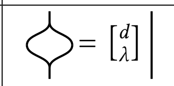
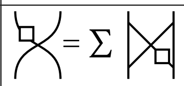
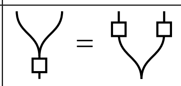
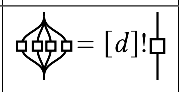
$S_{(3,3)}\widetilde{H}_w M_{(3,3)} =$  $= H_w K_{(3,3)}.$

Below is a typical presentation of a basis element (4.19) in coil Schur, and its translation into a basis element (5.7) in laurel Schur, using Theorem 5.6 and the equation above. In this example $A := \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ with $\lambda = (3,1)$, $\mu = (2,2)$, $\delta = (1,1,2)$, $\nu = (1,2,1)$, $g = s_3 s_2$, $b = b_1 \otimes b_2 \otimes b_3 \otimes b_4 \in B^{\otimes 4}$, and we denote $b_2 * b_3 := M_{\lambda\nu}(b_2 \otimes b_3) S_{\nu\lambda}$ for simplicity:



In particular, only coupons valued in $M_\nu T S_\nu$ can appear on thick strands for the elements in coil Schur, while in laurel Schur coupons belong to a slightly larger space $T^{\mathfrak{S}_\nu}$.

Remark 5.13. We do not pursue a description of $\overline{\mathbf{S}}_d^T$ in generators and relations here, but we expect a result similar to [SW24a, SW24b]. However, we end up proving analogues of most of the defining relations in *loc. cit.*; here is a schematic comparison (omitting indices and coefficients):

(2.3)	(2.4)	(2.5)	(2.6)	(2.7)	(2.8)
					
Theorem 5.6	Theorem 5.11	Theorem 5.7	???	Theorem 3.9	Theorem 4.5

The relation (2.6) is significantly more complicated in our context, because of the presence of algebra F in the coefficients and great freedom of choice of β. However, note that computation analogous to (4.7) gives a formula for the action of partial merges on the polynomial representation:

$$M_{\lambda\nu}(f) = \partial_{\lambda\nu}\left(f \prod\nolimits_{(i,j) \in L_\lambda \setminus L_\nu} P_{ij}\right),$$

where $\partial_{\lambda\nu}$ is the Demazure operator corresponding to the longest element $w_\circ \in \mathfrak{S}_\lambda^\nu$. We expect to be able to compute an analogue of (2.6) using these formulas, and thus to obtain a presentation of $\mathbf{S}_d^T$ and $\overline{\mathbf{S}}_d^T$ by generators and relations.

5.4. **Relaxing conditions** (C1)–(C3)**.** When the elements α, β are not central, Theorem 4.2 immediately fails:

$$Hb - \sigma(b)H = \xi_{1, \frac{\beta b - \sigma(b)\beta}{x_1 - x_2}} + \xi_{\sigma, \frac{P\sigma(b) - \sigma(b)P}{x_1 - x_2}} = \rho(b) + \xi_{\sigma, \frac{P\sigma(b) - \sigma(b)P}{x_1 - x_2}} \neq \rho(b).$$

This suggests that our approach via convolution algebras is not viable in general. Instead, one should take a version of Theorem 4.4, with all products *taken in correct order*, as the definition of quasi-idempotents K_λ. We expect that with enough bookkeeping of product orderings one can show that $K_\lambda^2 = m_\lambda K_\lambda$, which would imply the analogue of Theorem 4.6. However, we wanted to highlight the very general Theorem 3.11 as a result of independent interest.

Conjecture 5.14. *Assume that $B \wr \mathcal{H}_d$ is a PQWP satisfying* (C1) *and* (C3)*. Let us write $\mathbf{H}_d = B \wr \mathcal{H}_d$. For any composition $\lambda \vDash d$, define K_λ by the formula* (4.9)*, where the product is taken in lexicographic order. Consider the right $\mathbf{H}_d$-module $\mathbf{C}^T := \bigoplus_\lambda K_\lambda \mathbf{H}_d$. Furthermore, let $\overline{\mathbf{S}}_d^T := \bigoplus_{\lambda,\mu} \overline{\mathbf{S}}_{\lambda,\mu}^T$, $\overline{\mathbf{S}}^T := K_\lambda \mathbf{H}_d \cap \mathbf{H}_d K_\mu$,* *equipped with the product*

$$xK_\mu * K_\mu y := xK_\mu y, \qquad xK_\mu \in \overline{\mathbf{S}}^T_{\lambda,\mu}, \quad K_\mu y \in \overline{\mathbf{S}}^T_{\mu,\nu}.$$

Then the statement of Theorem 5.9 holds.

On the other hand, the existence of solutions of (C1) seems to be crucial to get the theory going. Indeed, suppose we want to extend the left action of $B^{\otimes d}$ on itself to the whole $B \wr \mathcal{H}(d)$, that is to construct a polynomial representation. By wreath relation, this action is completely determined by $\gamma_i := H_i(1)$. However, using the quadratic relation

$$0 = (H_i^2 - SH_i - R)(1) = H_i(\gamma_i) - S\gamma_i - R = (\sigma_i(\gamma_i) - S_i)\gamma_i - R_i,$$

and so $(-\gamma_i)$ satisfies the condition (C1).

Similarly, when the condition (C3) fails the polynomial representation ceases to be faithful. Indeed, let us fix $\gamma = -\alpha$, and assume that $\zeta P = 0$ for some $\zeta \in B \otimes B$. Then

$$(\zeta(x_2 - x_1)H + \zeta\beta)(f) = \zeta(x_1 - x_2)\sigma(f)\alpha - \zeta\beta(f - \sigma(f)) + \zeta\beta f = \zeta P\sigma(f) = 0$$

for all $f \in B \otimes B$.

6. Schurification of quantum wreath products à la Dipper–James

In this section, we translate our Schurifications into a different flavor, which is closer to [DJ89]. We will define the wreath Schur algebra $\mathbf{S}^{\mathrm{DJ}} = \mathbf{S}_{n,d} := \mathrm{End}_{\mathbf{H}_d}(V_n^{\otimes d})$ algebraically, and use the double centralizer property on the convolution side to prove the double centralizer property on the algebraic side when $n = d$. Finally, we prove the case $n > d$ by explicitly constructing idempotents. The aforementioned algebras are related via the following diagram:

$$\begin{array}{ccccc}
\overline{\mathbf{S}}^T_d & \curvearrowright & \bigoplus_{\lambda\in\Lambda} M_\lambda \mathbf{H}^T_d & \curvearrowleft & \mathbf{H}^T_d \\
\downarrow & & \downarrow & & \| \\
\mathbf{S}_{d,d} & \curvearrowright & V_d^{\otimes d} \simeq \bigoplus_{\lambda\in\Lambda_{d,d}} M^\lambda & \curvearrowleft & B \wr \mathcal{H}(d) \\
\downarrow & & \downarrow & & \| \\
\mathbf{S}_{n,d} & \curvearrowright & V_n^{\otimes d} & \curvearrowleft & B \wr \mathcal{H}(d)
\end{array}$$

6.1. A tensor module over affine Hecke algebras.

Let us recall the action of the affine Hecke algebra of type A on a tensor space appearing in [KMS95]. Renormalizing Theorem 2.10 by $v = q^{1/2}$, the affine Hecke algebra is a PQWP with $B = \Bbbk[x^{\pm 1}]$, $S = (v - v^{-1})(1 \otimes 1)$, and $R = 1 \otimes 1$. Consider the set $I_n = \{1, \dots, n\}$ together with its natural total order, and let $V_n = \bigoplus_{i \in I_n} v_i B$ be a free right B-module with basis $\{v_i\}_{i \in I_n}$. We further consider the right $B^{\otimes d}$-module $V_n^{\otimes d}$. It has an obvious $\Bbbk$-basis, given by elements

$$v_{\underline{i}} x_{\underline{k}}, \qquad \underline{i} = (i_1, \dots, i_d) \in I_n^d, \quad \underline{j} = (j_1, \dots, j_d) \in \mathbb{Z}^d,$$

where $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_d}$, $x_{\underline{j}} = x^{j_1} \otimes \cdots \otimes x^{j_d}$. We have a natural right $\mathfrak{S}_d$-action on both I_n^d and $B^{\otimes d}$ and $V_n^{\otimes d}$ by permuting factors; we will denote it by $- \cdot \sigma$. The action of each $H_k \in B \wr \mathcal{H}(d)$ on $V_n^{\otimes d}$ in [KMS95, (32)] can be rephrased as follows:

$$(v_{\underline{i}} x_{\underline{j}})H_k := \begin{cases} v_{\underline{i}\cdot s_k}(x_{\underline{j}} \cdot s_k) - (v - v^{-1})v_{\underline{i}}\partial_k(x_{\underline{j}})x_{k+1} & \text{if } i_k < i_{k+1}; \\ v v_{\underline{i}}(x_{\underline{j}} \cdot s_k) - (v - v^{-1})v_{\underline{i}}\partial_k(x_{\underline{j}})x_{k+1} & \text{if } i_k = i_{k+1}; \\ v_{\underline{i}\cdot s_k}(x_{\underline{j}} \cdot s_k) - (v - v^{-1})v_{\underline{i}}\partial_k(x_{\underline{j}}x_{k+1}) & \text{if } i_k > i_{k+1}. \end{cases}$$

This suggests a uniform construction for other PQWPs.

6.2. A tensor module over polynomial quantum wreath products.

Let $B \wr \mathcal{H}(d)$ be a PQWP satisfying (C1)–(C2), where B is the ring of (Laurent) polynomials over a unitary algebra F. As before, consider the free right B-module $V_n = \bigoplus_{i \in I_n} v_i B$, and the right $B^{\otimes d}$-module $V_n^{\otimes d}$. Given a $\Bbbk$-basis $\mathbb{B}$ of

$B^{\otimes d}$, we have an obvious $\Bbbk$-basis of $V_n^{\otimes d}$:

$$\{v_{\underline{i}} b \ : \ \underline{i} \in I_n^d, b \in \mathbb{B}\}.$$

Let $\overline{\alpha} := \sigma(\alpha) + S$ as in Section 4.3; note that $-\overline{\alpha}$ satisfies the equation (C1). We define the right action of each Hecke-like generator $H_k \in B \wr \mathcal{H}(d)$ on $V_n^{\otimes d}$ by

$$(6.1) \qquad (v_{\underline{i}} b)H_k = \begin{cases} v_{\underline{i}\cdot s_k}(b \cdot s_k) + v_{\underline{i}}\partial_k^\beta(b) & \text{if } i_k < i_{k+1}; \\ v_{\underline{i}}\overline{\alpha}_k(b \cdot s_k) + v_{\underline{i}}\partial_k^\beta(b) & \text{if } i_k = i_{k+1}; \\ v_{\underline{i}\cdot s_k} R_k(b \cdot s_k) + v_{\underline{i}}(\partial_k^\beta(b) + S_k(b \cdot s_k)) & \text{if } i_k > i_{k+1}. \end{cases}$$

Note that $\partial(b)\beta = \partial^\beta(b)$, $\partial(b\beta) = \partial^\beta(b) + S\sigma(b)$ by (C2) and Leibniz rule; therefore for affine Hecke algebras, (6.1) specializes to the formula from [KMS95].

Proposition 6.1. *Let $B\wr\mathcal{H}(d)$ be a PQWP satisfying (C1)–(C2). The formulas (6.1) define a $B\wr\mathcal{H}(d)$-action on $V_n^{\otimes d}$.*

Proof. For quadratic and wreath relations, the verification reduces to the case $d = n = 2$. If $i_1 = i_2$, the action of H preserves $v_{\underline{i}}$. Dropping it from the notation, we have

$$\begin{aligned} P(Hb) &= \overline{\alpha}\sigma(P)b + \rho(P)b \overset{\text{Leibniz}}{=} \overline{\alpha}\sigma(P)b + \rho(\sigma(b)P) + P\rho(b) \\ &= P(\sigma(b)H + \rho(b)); \\ PH^2 &= (\overline{\alpha}\sigma(P) + \rho(P))H = \overline{\alpha}\sigma(\overline{\alpha}) + \overline{\alpha}(\sigma\rho + \rho\sigma)(P) + \rho^2(P) \\ &\overset{(P4)}{=} (\overline{\alpha}\sigma(\overline{\alpha}) - \overline{\alpha}S)P + \overline{\alpha}S\sigma(P) + S\rho(P) = PR + P(SH). \end{aligned}$$

If $i_1 < i_2$, then $v_{\underline{i}} = v_{12}$, and so we have

$$\begin{aligned} v_{12}P(Hb) &= v_{21}\sigma(P)b + v_{12}\rho(P)(b) = v_{21}\sigma(P)b + v_{12}\rho(\sigma(b)P) + v_{12}P\rho(b) \\ &= v_{12}(\sigma(b)H + \rho(b)); \\ v_{12}PH^2 &= (v_{21}\sigma(P) + v_{12}\rho(P))H = v_{12}RP + v_{21}\rho\sigma(P) + v_{21}SP + v_{21}\sigma\rho(P) + v_{12}\rho^2(b) \\ &= v_{12}RP + v_{12}S\rho(P) + v_{21}S\sigma(b) = v_{12}PR + v_{12}P(SH). \end{aligned}$$

The case $i_2 > i_1$ is checked in an analogous fashion.

It remains to check the braid relations. It is clear from definition that $v_{\underline{i}}PH_iH_j = v_{\underline{i}}PH_jH_i$ when $|i - j| > 1$. Thus it remains to check the cubic braid relation, for which we can assume that $d = n = 3$. Using wreath relations, we can write

$$(v_{\underline{i}}P)H_1H_2H_1 = v_{\underline{i}}H_1H_2H_1(s_1s_2s_1)(P) + \sum\nolimits_{w < s_1s_2s_1} v_{\underline{i}}H_wP_w$$

for some $P_w \in B^{\otimes 3}$ expressed in terms of P, σ_i and ρ_i. One can similarly rewrite $(v_{\underline{i}}P)H_2H_1H_2$ as a sum of $v_{\underline{i}}H_2H_1H_2(s_2s_1s_2)(P)$ and lower terms. One checks directly that, thanks to relations (P5)–(P7), the lower terms coincide on the nose. Thus it suffices to check the braid relation on vectors $v_{\underline{i}}$. In this case, (6.1) simplifies to

$$(6.2) \qquad v_{\underline{i}}H_k = \begin{cases} v_{\underline{i}\cdot s_k} & \text{if } i_k < i_{k+1}; \\ v_{\underline{i}\cdot s_k}\overline{\alpha}_k & \text{if } i_k = i_{k+1}; \\ v_{\underline{i}\cdot s_k}R_k + v_{\underline{i}}S_k & \text{if } i_k > i_{k+1}. \end{cases}$$

By [Eli22, Lemma 5.1], a minimal set of the rank three ambiguities corresponds to the following:

$$\begin{aligned} v_{(1,2,3)}(H_1H_1)H_2H_1 &= v_{(1,2,3)}H_1(H_2H_1H_2), \\ v_{(1,2,3)}H_1H_2(H_1H_2H_1) &= v_{(1,2,3)}H_1(H_2^2)H_1H_2. \end{aligned}$$

That is, checking $v_{\underline{i}}H_1H_2H_1 = v_{\underline{i}}H_2H_1H_2$ can be reduced to checking it for $\underline{i} \in \{(1,3,2),(2,3,1)\}$ and for the case $1 \le i_k \le 2$.

By a direct computation, the equality $v_{(1,3,2)}H_1H_2H_1 = v_{(1,3,2)}H_2H_1H_2$ holds if and only if the coefficients of $v_{\underline{i}}$, $\underline{i} \in (1,2,3)\cdot\mathfrak{S}_3$ on both sides coincide. This is equivalent to conditions (P8)–(P9). By a similar computation, the equality $v_{(2,3,1)}H_1H_2H_1 = v_{(2,3,1)}H_2H_1H_2$ holds if and only the following hold, if $Y = S$ or R:

$$S_1\sigma_1\sigma_2(Y_1) + \rho_1\sigma_2(Y_1) = \sigma_1(Y_2)S_1 + \rho_1(Y_2), \quad R_1\sigma_2(Y_1) = \sigma_1(Y_2)R_1. \tag{6.3}$$

The first equality of (6.3) follows from (P4) and (P8). Namely, we have $\rho_1\sigma_2(Y_1) = 0$ and

$$\begin{aligned} S_1\sigma_1\sigma_2(Y_1) &= l_{S_1}\sigma_1(\sigma_2(Y_1)) = (r_{S_1}\sigma_1^2 + \rho_1\sigma_1 + \sigma_1\rho_1)(\sigma_2(Y_1)) \\ &= \sigma_2(Y_1)S_1 + \rho_1(\sigma_1\sigma_2(Y_1)) + \sigma_1(\rho_1\sigma_2(Y_1)) = \sigma_2(Y_1)S_1 + \rho_1(Y_2). \end{aligned}$$

The second equality in (6.3) follows from centrality of R. Finally, if $1 \le i_k \le 2$, then any such case is a degenerate case of the corresponding rank three calculation since at least two of the tensor factors in $v_{\underline{i}}$ agree. If all i_k's are the same, then the braid relation holds if and only if $\overline{\alpha}_1\overline{\alpha}_{13}\overline{\alpha}_2 = \overline{\alpha}_2\overline{\alpha}_{13}\overline{\alpha}_1$, which holds thanks to (C2). For the most complicated degenerate case, it requires to verify that $v_{(2,2,1)}H_1H_2H_1 = v_{(2,2,1)}H_2H_1H_2$, Equivalently, it suffices to check that

$$R_1R_{13}\overline{\alpha}_2 = \overline{\alpha}_2R_{13}R_1, \quad S_1R_{13}\overline{\alpha}_2 = R_2\overline{\alpha}_{13}S_1, \quad \overline{\alpha}_1S_{13}\overline{\alpha}_2 = S_{13}R_1 + S_2\overline{\alpha}_{13}S_1, \tag{6.4}$$

which follows from combining the fact that S is weak Frobenius and (P2). This concludes the verification of braid relations. ☐

Definition 6.2. We call the centralizing algebra $\mathsf{S}^{\mathrm{DJ}} = \mathsf{S}_{n,d} := \mathsf{End}_{B\wr\mathcal{H}(d)}(V_n^{\otimes d})$ the *wreath Schur* algebra.

6.3. **An analog of permutation modules.** Suppose that M is a $B^{\otimes d}$-module, N is a $\Bbbk$-vector space. The conditions on Q such that $N\otimes M$ has a module structure over an arbitrary quantum wreath product $B\wr\mathcal{H}(d)$ have been developed [LNX]. Here, we provide a special case of the theory therein.

Let $B\wr\mathcal{H}(d)$ be a PQWP such that (C1)–(C2) holds. Recall from (5.3) that $(H_i + \alpha_i)$ is an eigenvector with respect to the right multiplication by H_i, and the corresponding eigenvalue is $\overline{\alpha}_i = S_i+\sigma_i(\alpha_i) \in B^{\otimes d}$. We want to construct an analog N^λ of the permutation module $x_\lambda\mathcal{H}_q(\mathfrak{S}_d)$ using eigenvectors of this form.

For $A \equiv (\lambda, g, \mu)$, we further write $\delta := \delta^c(A)$, $G(A) := {}^\delta\mathfrak{S}_\mu$, and recall the elements

$$K_\lambda = \sum\nolimits_{w\in\mathfrak{S}_\lambda} \alpha_w H_{ww_\circ^\lambda}, \quad K_\mu^\delta = \sum\nolimits_{w\in G(A)} H_w\alpha_{ww_\circ^\delta w_\circ^\mu}, \quad K_A = K_\lambda H_g K_\mu^\delta$$

defined by (4.16), (5.1) and Theorem 5.2 respectively. In order to make the notation closer to Dipper-James contruction, we will write

$$y_\lambda := K_\lambda, \quad y_\mu^\delta := K_\mu^\delta, \quad y_A := K_A$$

until the end of this section. For any $\lambda \in \Lambda_{n,d}$, we define a subspace

$$N^\lambda := \mathrm{Span}_\Bbbk\{y_\lambda H_g \in B\wr\mathcal{H}(d) \mid g \in {}^\lambda\mathfrak{S}\},$$

on which the right multiplication of $B\wr\mathcal{H}(d)$ induces a structure map

$$\tau^\lambda : N^\lambda \otimes \Bbbk\mathfrak{S}_d \to N^\lambda \otimes F^{\otimes d}, \quad y_\lambda H_\eta \otimes w \mapsto \sum\nolimits_{g\in{}^\lambda\mathfrak{S}} (y_\lambda H_g) \otimes b_{\eta,w}^g, \tag{6.5}$$

where $b_{\eta,w}^g$ are the coefficients appearing in $y_\lambda H_\eta H_w = \sum_{g\in{}^\lambda\mathfrak{S}} y_\lambda H_g b_{\eta,w}^g$.

Example 6.3. Let $d = 2$. Then, $\mathfrak{S}_{(1,1)} = 1 = {}^{(2)}\mathfrak{S}$ and ${}^{(1,1)}\mathfrak{S} = \mathfrak{S}_2 = \mathfrak{S}_{(2)}$. Hence, $y_{(1,1)} = 1$, $y_{(2)} = H_1 - \overline{\alpha}$. Note that right multiplication by H_1 does not preserve $N^{(1,1)} = \mathrm{Span}_\Bbbk\{y_{(1,1)}, y_{(1,1)}H_1\}$. More precisely, the structure map (6.5) for $\lambda = (1,1)$ is given explicitly by

$$y_{(1,1)} \otimes s_1 \mapsto y_{(1,1)}H_1 \otimes 1, \quad y_{(1,1)}H_1 \otimes s_1 \mapsto y_{(1,1)}H_1 \otimes S_1 + y_{(1,1)} \otimes R_1.$$

Similarly, the structure map for $N^{(2)} = \Bbbk y_{(2)}$ is given by $v \otimes s_1 \mapsto v \otimes \overline{\alpha}$.

For any $\underline{i} \in I_n^d$, denote by $\underline{i}^+ \in I_n^d$ the non-decreasing rearrangement of $\underline{i}$, and let $w(\underline{i}) \in {}^\lambda\mathfrak{S}$ be such that $\underline{i}^+ \cdot w(\underline{i}) = \underline{i}$. Then, $\underline{i}^+$ is of the form $(1^{\lambda_1}, \dots, n^{\lambda_n})$ for some $\lambda = \lambda(\underline{i}) \in \Lambda_{n,d}$. Write $v_\lambda^+ := v_{\underline{i}^+}$.

Note that V_n decomposes into a direct sum $U_1 \oplus \cdots \oplus U_n$, where $U_i := v_iB$. For $\lambda \in \Lambda_{n,d}$, let $U^\lambda := v_\lambda^+ B^{\otimes d} = U_1^{\otimes\lambda_1} \otimes\cdots\otimes U_n^{\otimes\lambda_n}$ be a free right $B^{\otimes d}$-module by factorwise multiplication. Define a vector space $M^\lambda := N^\lambda \otimes U^\lambda$. It inherits a right $B^{\otimes d}$-action from U^λ. Furthermore, using the structure map (6.5), define a right action of H_k, $1 \le k \le d-1$ on M^λ by

$$(y_\lambda H_\eta \otimes P)H_k = y_\lambda H_\eta \otimes \partial_k^\beta(P) + \sum\nolimits_{g \in {}^\lambda\mathfrak{S}} (y_\lambda H_g) \otimes b^g_{\eta, s_k}(P \cdot s_k).$$

We have a vector space isomorphism

$$\bigoplus\nolimits_{\lambda \in \Lambda_{n,d}} M^\lambda \simeq V_n^{\otimes d}; \qquad y_\lambda H_g \otimes v_\lambda^+ P \mapsto v_{\underline{i}^+ \cdot g} P, \quad y_\lambda H_{g(\underline{i})} \otimes v_{\underline{i}^+} P \mapsfrom v_{\underline{i}} P. \tag{6.6}$$

Proposition 6.4. *Let $B \wr \mathcal{H}(d)$ be a PQWP satisfying (C1)–(C2).*

(a) The map (6.6) is compatible with the right action of H_k, $1 \le k \le d-1$. In particular, M^λ is an $B \wr \mathcal{H}(d)$-submodule of $V_n^{\otimes d}$;

(b) Let λ' be the strict composition obtained from $\lambda \in \Lambda_{n,d}$ by removing zeroes. Then $M^\lambda \equiv \mathbf{C}^T_{\lambda'}$, where $\mathbf{C}^T_{\lambda'} = M_{\lambda'} \mathbf{H}^T_d$ is the direct factor of the bimodule in Theorem 4.6, via

$$y_\lambda H_g \otimes v_\lambda^+ P \mapsto M_{\lambda'} H_g P, \qquad P \in B^{\otimes d}, g \in {}^\lambda\mathfrak{S}.$$

Proof. It is easy to see that the action (6.6) is obtained by rewriting (6.1)–(6.2) under the isomorphism (6.6), hence the first claim. The second claim follows by direct comparison, recalling that the map ψ^L_λ from Theorem 3.8 induces an isomorphism of right modules $M_{\lambda'} \mathbf{H}^T_d \to K_{\lambda'} \mathbf{H}^T_d$. □

6.4. **Schur duality and a basis of wreath Schur.** Let us describe a basis of wreath Schur algebra $\mathsf{S}_{n,d}$ in terms of homomorphisms between permutation modules M^λ. For $A \equiv (\lambda, g, \mu)$ and $P \in B^{\otimes d}$, let $\theta_{A,P} \in \mathrm{Hom}_{\Bbbk}(M^\mu, M^\lambda)$ be the following map:

$$\theta_{A,P} : M^\mu \to M^\lambda, \quad (y_\mu \otimes v_\mu^+) \cdot h \mapsto \Big(\sum\nolimits_{w \in \mathfrak{S}_\mu, w' \in {}^\lambda\mathfrak{S}} y_\lambda H_{w'} \otimes v_\lambda^+ b^{w'}_{g,w} P_w \Big) \cdot h, \tag{6.7}$$

where P_w are defined by $P y^\delta_\mu = \sum_{w \in \mathfrak{S}_\mu} H_w P_w$. The statement below follows immediately from Theorem 5.4 in view of identifications in Theorem 6.4.

Proposition 6.5. *For each $\nu \vDash d$, fix a $\Bbbk$-basis $\overline{\mathbb{B}}_\nu$ of $(B^{\otimes d})^{\mathfrak{S}_\nu}$. Then, $\mathsf{S}_{n,d}$ has the following basis:*

$$\{\theta_{A,P} \mid A \in \Theta_{n,d}, P \in \overline{\mathbb{B}}_{\delta^c(A)}\}.$$ □

Note that $P = 1$, we recover the Dipper–James elements $\theta_{A,P} = y_A$. In general, the situation is more complicated.

Example 6.6. Suppose that $A := \left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)$ with $\lambda = (2,2)$, $\mu = (3,1)$, $\delta := \delta^c(A) = (1,2,1)$, and $g = s_2 s_3$, and $G(A) = {}^{\delta^c}\mathfrak{S}_\mu$. Then,

$$\mathfrak{S}_\lambda = \langle s_1, s_3 \rangle, \quad G(A) = \{g \in \langle s_1, s_2 \rangle \mid s_2 g > g\} = \{e, s_1, s_1 s_2\},$$

where the longest element in $G(A)$ is $s_1 s_2 = w^\delta_\circ w^\mu_\circ$. Pick $w := \kappa^{-1}(s_1, s_1) = s_1 g s_1$. Then, $w^A_\circ w = s_1 s_2$, $H_g = H_2 H_3$, and $y^\delta_\mu = (H_1 H_2 + H_1 \alpha_2 + \alpha_{s_1 s_2})$. Let $f \in F$, $P := f_1 x_2 x_3 = f \otimes (x_1 x_2) \otimes 1$; then

$$\begin{aligned} \theta_{A,P}(y_\mu \otimes v_\mu^+) &= y_\lambda H_2 H_3 H_1 H_2 \otimes v_\lambda^+ x_1 x_2 f_3 + y_\lambda H_2 H_3 H_1 \otimes v_\lambda^+ x_1 f_2 (x_3 \alpha_2 - \beta_{23}) \\ &\quad + y_\lambda H_2 H_3 \otimes v_\lambda^+ f_1 (\beta_{13}\beta_2 + x_2 x_3 \alpha_{s_1 s_2} - \beta_1 \alpha_2 x_3 - x_2 \overline{\alpha}_2 \beta_{13} - \rho_2(\beta_1) x_3). \end{aligned}$$

Theorem 6.7. *Assume that $n \ge d$. Then $B \wr \mathcal{H}(d) \cong \mathrm{End}_{\mathsf{S}_{n,d}}(V_n^{\otimes d})$, and so Schur duality holds for the pair $(\mathsf{S}_{n,d}, B \wr \mathcal{H}(d))$.*

Proof. We have an obvious inclusion $B \wr \mathcal{H}(d) \subseteq \mathrm{End}_{\mathsf{S}_{n,d}}(V_n^{\otimes d})$. For the opposite inclusion, let $\Lambda'_{n,d} \subset \Lambda_{n,d}$ be the subset of non-decreasing compositions, and consider the following $B \wr \mathcal{H}(d)$-submodule $V' \subset V_n^{\otimes d}$:

$$V' = \bigoplus\nolimits_{\lambda \in \Lambda'_{n,d}} M^\lambda.$$

Note that $\Lambda'_{n,d}$ is in bijection with the set of strong compositions of d. Applying Theorem 6.4(b), we get an isomorphism of $B \wr \mathcal{H}(d)$-modules $V' \simeq \mathbf{C}^T$. In particular, by Theorem 5.9 we have $\mathrm{End}_{\mathsf{S}_{n,d}}(V_n^{\otimes d}) \subseteq \mathrm{End}_{\overline{\mathsf{S}}^T_d}(V') = B \wr \mathcal{H}(d)$, and so we may conclude. □

Corollary 6.8. *Suppose that $B \wr \mathcal{H}(d)$ is a PQWP satisfying (C1)–(C3). The wreath and laurel Schur algebras are Morita-equivalent, provided that $n \ge d$.*

Proof. By Theorem 6.4(b), both the tensor space $V_n^{\otimes d}$ and the polynomial representation $\mathbf{C}^T$ have the same direct summands as right $B \wr \mathcal{H}(d)$-modules. Let $\epsilon \in \mathrm{S}_{n,d}$ be the idempotent corresponding to the split inclusion $V' \simeq \mathbf{C}^T \subset V_n^{\otimes d}$. Then $\overline{\mathrm{S}}_d^T \simeq \epsilon \mathrm{S}_{n,d}\epsilon$, and Morita theory implies that $\mathrm{S}_{n,d}$–mod $\to \overline{\mathrm{S}}_d^T$–mod, $M \mapsto M\epsilon$ is an equivalence of categories. □

7. Examples and applications

7.1. **Affine Hecke algebras.** The affine Hecke algebra of type A is a PQWP with the following parameters:

$$B = \Bbbk[x^{\pm 1}], \quad S = (q-1)(1 \otimes 1), \quad R = q(1 \otimes 1), \quad \alpha = 1 \otimes 1, \quad \beta = (1-q)x_2.$$

It is well-known [GRV94] that the corresponding Schur algebra $\widehat{S}_q(n,d)$ can be realized as a convolution algebra for affine partial flags (such a realization corresponds to the Coxeter presentation). $\widehat{S}_q(n,d)$ has a Morita equivalent version given by the K-theoretic convolution algebra

$$\widehat{S}_q^K(d) := \sum\nolimits_{\lambda,\mu \vDash d} K^{\mathrm{GL}_d \times \mathbb{C}^*}(T^*(\mathrm{GL}_d/P_\lambda) \times_{\mathcal{N}_d} T^*(\mathrm{GL}_d/P_\mu)),$$

defined as in [CG97, Ch. 5], where $P_\lambda \subseteq \mathrm{GL}_d$ are standard parabolic subgroups, and $\mathcal{N}_d \subseteq \mathfrak{gl}_d$ the nilcone. The Schur duality is known for both $\widehat{S}_q(n,d) \equiv \mathrm{S}_{n,d}$ and $\widehat{S}_q^K(d) \equiv \overline{\mathrm{S}}_d^T$. Our work only produces a new twisted convolution algebra construction for $\widehat{S}_q(n,d)$, which can be related to the K-theoretic convolution via equivariant localization, as in [MM22]. The basis $\theta_{A,P}$ is different from Dipper–James basis, and rather recovers the basis in [MS19, Prop. 4.17].

7.2. **Degenerate affine Hecke algebras.** The degenerate affine Hecke algebra $\mathcal{H}_d^{\deg}$ of type A is a PQWP with the following parameters:

$$B = \Bbbk[x], \quad S = 0, \quad R = 1 \otimes 1 = \alpha = \beta.$$

A notion of degenerate affine Schur algebras has not been explicitly studied, to our knowledge (however, see [BK08, SW11] for cyclotomic versions in characteristic 0).

The closest but different notion appeared in the (higher) Schur duality of Arakawa–Suzuki [AS98], which states that for any $\mathfrak{gl}_n$-module M, there is a $\mathcal{H}_d^{\deg}$-action on $M \otimes (\Bbbk^n)^{\otimes d}$ such that the x_i-actions involve permuting tensor factors in $(\Bbbk^n)^{\otimes d}$. A functor that connects the representation theory of Yangians $\mathrm{Y}(\mathfrak{gl}_n)$ and of $\mathcal{H}_d^{\deg}$ is given. However, it was not about double centralizer property, nor the Schur algebras were studied. In contrast, our $\mathcal{H}_d^{\deg}$-action is defined on the tensor space $(\Bbbk^{\mathbb{Z}})^{\otimes d} \simeq (\Bbbk^n[x])^{\otimes d}$, such that the tensor factors do not permute under the actions of x_i's.

Corollary 7.1. *There is a double centralizer property between $\mathcal{H}_d^{\deg}$ and the corresponding wreath Schur algebra $\mathrm{S}_{n,d}$ on the tensor space $V_n^{\otimes d}$ for $n \geq d$ over a field $\Bbbk$ of any characteristic.*

We expect that there exists an explicit quotient map $\mathrm{Y}(\mathfrak{gl}_n) \twoheadrightarrow \mathrm{S}_{n,d}$ for any d. In contrast, while the laurel Schur algebra $\overline{\mathrm{S}}_d^T$ also enjoys a double centralizer property (on a submodule of $V_n^{\otimes d}$), we suspect that its connection with the Yangians is less transparent.

7.3. **Pro-p Iwahori Hecke algebras.** Denote by $\mathcal{H}(q_s, c_s)$ the generic pro-p Iwahori Hecke algebras with respect a p-adic group G and choice of parameters q_s and c_s. Consider the case $G = \mathrm{GL}_d(\mathbb{Q}_p)$, $q_s = 1$, and $c_{s_i} = (q - q^{-1})e_i$ for some idempotent $e \in (\frac{\Bbbk[t]}{(t^{p-1}-1)})^{\otimes 2}$ for all i. The quadratic relation does split since

$$H^2 - (q-q^{-1})eH - 1 = (H + (q^{-1}+1)e - 1)(H - (q+1)e + 1). \tag{7.1}$$

Assume that e is weak Frobenius. Then, $\mathcal{H}(q_s, c_s)$ is a PQWP with the following parameters:

$$B = \frac{\Bbbk[t]}{(t^{p-1}-1)}[x^{\pm 1}], \quad S = (q-q^{-1})e, \quad R = (1 \otimes 1), \quad \alpha = (1+q^{-1})e - 1 \otimes 1, \quad \beta = (q^{-1}-q)ex_2.$$

When $e = \frac{1}{p-1}\sum_{j=1}^{p-1} t^j \otimes t^{-j}$, such an algebra is isomorphic to the affine Yokonuma algebra [CS16], a quantization of the group algebra of $(C_m \times \mathbb{Z}) \wr \mathfrak{S}_d$.

Corollary 7.2. *Consider the pro-p Iwahori Hecke algebras $\mathcal{H} = \mathcal{H}(e)$ for $\mathrm{GL}_d(\mathbb{Q}_p)$ at the specialization $q_s = 1$, and $c_{s_i} = (q - q^{-1})e_i$ for some weak Frobenius idempotent e. Then, there is a double centralizer property between $\mathcal{H}(e)$ and the corresponding wreath Schur algebra $\mathbf{S}_{n,d}$ on the tensor space $V_n^{\otimes d}$, if $n \geq d$.*

We expect our results to be useful to understand a Schur duality involving the pro-p Iwahori Hecke algebra and its Gelfand–Graev representation from [GGK24], essentially computing the endomorphism ring of the Gelfand–Graev representation over $\mathcal{H}$ (working at the Iwahori level instead of the pro-p level, one obtains the Schur duality present in Section 7.1).

In particular, Theorem 7.2 holds for the affine Yokonuma algebras in which $e = \frac{1}{p-1}\sum_j t^j \otimes t^{-j}$. Since these algebras arise from the constructing knot invariants in the solid torus, and hence we are curious whether there can be any applications to knot theory.

We also remark that the Schur algebras for finite Yokonuma algebras with respect to a different action has been studied in an unpublished manuscript [Cui14] due to Cui.

7.4. **Affine zigzag algebras and curve Schur algebras.** Let Q be a Dynkin quiver. The affine zigzag algebras $\mathcal{Z}_d(Q)$ were studied in [KM19] in relation to (imaginary) semicuspidal categories for quiver Hecke algebras for the associated affine quiver $Q^{(1)}$. These algebras are particular cases of Savage algebras, and as such satisfy conditions (C1) and (C3), as explained in Theorem 4.1. Since the zigzag algebra of Q is commutative only in type A_1 (namely, $Z_{A_1} \simeq H^*(\mathbb{P}^1) = \Bbbk[c]/c^2$), the condition (C2) only holds in this case.

For the affine zigzag algebra $\mathcal{Z}_d(A_1)$, the coil and laurel Schur algebras appeared in [MM22]. There, the authors defined a notion of curve Hecke algebra $\mathcal{H}_d^C$ and curve Schur algebra $\mathcal{S}_d^C$ for any smooth projective curve C. The latter admitted two distinct $\mathbb{Z}$-forms, in the notations of *loc. cit.*, $\mathbf{S}_d^T \equiv \widetilde{\mathcal{S}}_d^{\mathbb{P}^1} \subseteq \overline{\mathbf{S}}_d^T \equiv \mathcal{S}_d^{\mathbb{P}^1}$, which gave rise to very different reductions modulo p. For instance, the reduction of coil Schur algebra $\widetilde{\mathcal{S}}_d^{\mathbb{P}^1}$ controls the semicuspidal category of type $A_1^{(1)}$ in small characteristic. Our Theorem 5.9 establishes Schur duality between $\mathcal{S}_d^C$ and $\mathcal{H}_d^C$, but suggests that one should not expect a double centralizer description of $\widetilde{\mathcal{S}}_d^{\mathbb{P}^1}$ in small characteristic.

Admitting Theorem 5.14, one could expect that coil Schur algebra $\mathbf{S}_d^T$ describes the semicuspidal category for other (simply-laced) affine types. It turns out that this is not quite the right answer. One way to see this is by looking at the idempotents. Suppose that the algebra F has a complete set of idempotents parameterized by I. By construction both $\mathbf{S}_d^T$ and $\overline{\mathbf{S}}_d^T$ have at least as many idempotents as the size of the set

$$\{(\lambda, \mathbf{i}) : \lambda \vDash d, \mathbf{i} \in I^d/\mathfrak{S}_\lambda\}.$$

On the other hand, Gelfand-Graev idempotents in the semicuspidal algebra are parameterized by a smaller set

$$\{(\lambda, \varphi) : \lambda \vDash d, \varphi \in I^{|\lambda|}\}.$$

Diagrammatically, this means only allowing thick strands of "pure color". So, while the answer should still be contained in $\overline{\mathbf{S}}_d^T$, it is smaller than the coil algebra $\mathbf{S}_d^T$.

Conjecture 7.3. *Assume that Theorem 5.14 holds. Let Q be a Dynkin quiver, $Q^{(1)}$ the corresponding affine quiver, and $R(d\delta)$ the quiver Hecke algebra, where $n\delta$ is an imaginary root of $\widehat{\mathfrak{g}}_Q$. Let F be the zigzag algebra of Q over $\mathbb{Z}$, and $\Delta \in F \otimes F$ the Frobenius element. Consider the laurel Schur algebra $\overline{\mathbf{S}}_d^T$ with parameters $\alpha = 1$, $\beta = \Delta$. For any $\lambda \vDash d$, denote*

$$T_\lambda := \bigotimes_k \bigoplus_{i \in I} (e_i F[x] e_i)^{\otimes \lambda_k},$$

where $\{e_i\}_{i\in I}$ is the set of idempotents in F. Consider the subalgebra $\mathbf{S}'_d \subset \overline{\mathbf{S}}_d^T$ with basis of the form (5.7), where the set $\overline{\mathbb{B}}_g$ runs over a $\mathbb{Z}$-basis of $\mathbb{Z}[x_1, \dots, x_d]^{\mathfrak{S}_\nu}(M_\nu T_\nu S_\nu)$. Then in any characteristic p, the semicuspidal algebra $C(d\delta)$ is Morita equivalent to the reduction of $\mathbf{S}'_d$ modulo p.

While one of our reasons to leave Theorem 5.14 unproven is an ongoing infestation with seventh deadly sin, a more pragmatic reason is the hope that we can still realize these smaller subalgebras as twisted convolution algebras, exploiting the fact that idempotent truncations of zigzag algebras are commutative. This was recently achieved in [MM] in the smallest non-trivial case, where the authors relate the semicuspidal category for quiver *Schur* algebra of type $A_1^{(1)}$ to "pure color" idempotent truncation of the coil Schur algebra associated to the *extended* zigzag algebra of type A_1.

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