

$P = W$ VIA \mathcal{H}_2

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ABSTRACT. Let \mathcal{H}_2 be the Lie algebra of polynomial Hamiltonian vector fields on the symplectic plane. Let X be the moduli space of stable Higgs bundles of fixed relatively prime rank and degree, or more generally the moduli space of stable parabolic Higgs bundles of arbitrary rank and degree for a generic stability condition. Let $H^*(X)$ be the cohomology with rational coefficients. Using the operations of cup-product by tautological classes and Hecke correspondences we construct an action of \mathcal{H}_2 on $H^*(X)[x, y]$, where x and y are formal variables. We show that the perverse filtration on $H^*(X)$ coincides with the filtration canonically associated to $\mathfrak{sl}_2 \subset \mathcal{H}_2$ and deduce the $P = W$ conjecture of de Cataldo-Hausel-Migliorini.

1. INTRODUCTION

1.1. **The $P = W$ conjecture.** Let $r > 0$ and d be relatively prime integers and let C be a smooth projective algebraic curve over \mathbb{C} . Following Hitchin [Hit87] and Simpson [Sim90, Sim92], *Higgs bundles* are pairs (\mathcal{E}, θ) such that \mathcal{E} is an algebraic vector bundle on C and $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega$, where Ω denotes the line bundle of differential forms on C . A Higgs bundle is *stable* if for all proper subbundles $\mathcal{E}' \subset \mathcal{E}$ of rank r' and degree d' satisfying $\theta(\mathcal{E}') \subset \mathcal{E}' \otimes \Omega$ we have the inequality

$$\frac{d'}{r'} < \frac{d}{r}.$$

Let $M_{r,d}$ be the moduli space of stable Higgs bundles on C of rank r and degree d , which is a (smooth) quasi-projective symplectic variety of dimension $2(g-1)r^2 + 2$. By the non-abelian Hodge correspondence ([Hit87, Sim91, Sim92]), the space $M_{r,d}$ is diffeomorphic to the moduli space $M_{r,d}^{\text{Betti}}$ parameterizing certain twisted local systems on C . That latter space is an affine algebraic variety, but the complex structure is different from the one of $M_{r,d}$. By [Del71], the cohomology with complex coefficients $H^*(M_{r,d}^{\text{Betti}})$ is equipped with a natural *weight filtration*. It is natural to ask for a description of the weight filtration on $H^*(M_{r,d}) = H^*(M_{r,d}^{\text{Betti}})$ in terms of the algebraic geometry of the space $M_{r,d}$.

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A conjectural answer to this question was suggested in [dCHM12]. The moduli space $M_{r,d}$ is endowed with the *Hitchin map* $\chi : M_{r,d} \rightarrow \mathbb{C}^{(g-1)r^2+1}$ which reads off the coefficients of the characteristic polynomial of θ . It turns out that χ is proper and applying the decomposition theorem [BBD82] one constructs the perverse filtration, starting in degree $N = (g-1)r^2 + 1$. The $P = W$ conjecture then claims that the weight filtration and the perverse filtration are essentially equal:

Conjecture ($P = W$, [dCHM12]). For $i = 0, 1, \dots, (g-1)r^2 + 2$, we have

$$P_{i-N}H^*(M_{r,d}) = W_{2i}H^*(M_{r,d}^{\text{Betti}}).$$

In this paper, we prove this conjecture as well as its analog in which one replaces stable Higgs bundles by stable parabolic Higgs bundles. Our approach goes through a variant of the $P = W$ conjecture known as the $P = C$ conjecture. As proven by [Mar02], the cohomology ring $H^*(M_{r,d})$ is generated by the tautological classes, which are the components of the Künneth decomposition of the Chern classes of the tautological sheaf on $M_{r,d} \times C$. By [She16] the weight filtration may be entirely described in terms of tautological classes. Namely, assign a weight to each tautological class so that the classes coming from the i -th Chern class have weight $2i$. Then W_m is spanned by products of tautological classes of total weight $\leq m$.

Our main result shows that the same kind of description holds for P :

Theorem ($P = C$ conjecture). *The subspace $P_m H^*(M_{r,d})$ is the span of products of tautological classes of total weight $\leq 2(m + N)$, where $N = (g-1)r^2 + 1$.*

This immediately implies

Corollary. *The $P = W$ conjecture holds for the spaces $M_{r,d}$.*

As mentioned above, we work in a more general context of moduli spaces of stable parabolic Higgs bundles of arbitrary rank and degree for a generic stability condition, and prove the corresponding result in that context. In fact, it is essential for our proof to consider both parabolic and non-parabolic setups at once.

The $P = W$ conjecture in the case of rank 2 was proved already in [dCHM12]. The case when C has genus 2 was established in [dCMS22]. While the present paper was in the final stages of preparation, we learned of a proof of $P = W$ by Maulik and Shen using a different approach, see [MS22]. Since then, a third proof has appeared, see [MSY]. We refer the interested reader to [Hos24] for a survey and comparison of these proofs.

1.2. The algebra \mathcal{H}_2 . Let M be the moduli space $M_{r,d}$, or more generally the moduli space of parabolic Higgs bundles. Our proof of $P = W$ proceeds by constructing an action of an interesting algebra on $H^*(M)$.

The Lie algebra \mathcal{H}_2 of polynomial Hamiltonian vector fields on the plane with respect to the standard symplectic form has the following description. A basis is given by the fields $V_{m,n}$ with Hamiltonian $x^m y^n$; explicitly, we have

$$V_{m,n} = ny^{n-1}x^m \frac{\partial}{\partial x} - mx^{m-1}y^n \frac{\partial}{\partial y}.$$

The Lie bracket is given as follows:

$$[V_{m,n}, V_{m',n'}] = (m'n - mn')V_{m+m'-1, n+n'-1}.$$

It is expected by physicists (Lev Rozansky, private communication) that \mathcal{H}_2 acts on many geometric invariants. For instance, in [GHM21], in order to prove a Lefschetz property, certain operators acting on the so-called y -ified Khovanov-Rozansky homologies of links were constructed and it was speculated that these operations satisfy the relations of \mathcal{H}_2 . This and some other similarities with the situation of homologies of links gave us a hint that \mathcal{H}_2 may act on $H^*(M)$. It is easy to check however, that \mathcal{H}_2 does not have any non-zero finite-dimensional representation, so there is no hope that it acts directly on $H^*(M)$. Nevertheless, \mathcal{H}_2 *does* naturally act on the infinite-dimensional space $H^*(M)[x, y]$, where x and y are formal variables satisfying some natural commutation relations with $V_{m,n}$.

1.3. The main construction. The construction of the action of \mathcal{H}_2 begins with an explicit presentation of the cohomological Hall algebra (CoHA) of zero-dimensional sheaves on a smooth surface. This algebra describes the relations satisfied by all (punctual) Hecke correspondences on a given surface S . As shown in [MMSV23], it is isomorphic to a deformed $W_{1+\infty}$ -algebra, modeled on the cohomology ring $H^*(S)$. The presentation drastically simplifies when the surface satisfies some cohomological vanishing properties, such as for open symplectic surfaces (which is the case of interest for Higgs bundles).

Now let $S = T^*C$ for C a smooth complex projective curve. The CoHA of zero-dimensional sheaves on S naturally acts on the (co)homology of the stack $\mathcal{Coh}_r = \bigsqcup_d \mathcal{Coh}_{r,d}$ of all Higgs bundles on C (of fixed rank and varying degree), but as Hecke correspondences do not preserve (semi)stability, there is no natural action on the cohomology of $M_r = \bigsqcup_d M_{r,d}$ or its parabolic version, much less so on each individual $H^*(M_{r,d})$. To circumvent this problem, we consider the *elliptic locus* $M_r^{\text{ell}} \subset M_r$, where the spectral curve is reduced and irreducible (which automatically implies stability). This property being

preserved under Hecke correspondences, the *Hecke operators* are well defined on $H^*(M_r^{\text{ell}})$, see Section 4. Using [MMSV23], the Hecke operators and the commutation relations between them and the tautological classes are computed explicitly. In particular, the action on the subring $H_{\text{taut}}^*(M_r^{\text{ell}}) \subset H^*(M_r^{\text{ell}})$ generated by the tautological classes is described in terms of the so-called Fock space representation (see Theorems 5.2, 5.6 formulated in a more general case of sheaves on a surface, and Corollary 7.2 and Theorem 7.8 for Higgs bundles).

Once we know the commutation relations between the Hecke operators, we use a specific degree 1 Hecke operator to canonically identify the cohomology groups of $M_{r,d}^{\text{ell}}$ for all values of the degree d , and then follow a certain degeneration procedure, analogous to the passage from the trigonometric Cherednik algebra to its rational version, see Section 6. We define in this way a family of operators on $H^*(X_{r,d}^{\text{ell}})$ for any *fixed* d , satisfying relations which look almost like those of \mathcal{H}_2 , see Corollary 7.3 and Proposition 7.4.

This leads us to the construction of an \mathfrak{sl}_2 -triple $(\mathfrak{e}, \mathfrak{h}, \mathfrak{f})$ acting on $H^*(M_{r,d}^{\text{ell}})$, see Proposition 7.9. The operator \mathfrak{e} is the multiplication by a tautological class, while \mathfrak{f} is a complicated Hecke operator. After applying a suitable gauge transformation (Proposition 7.10) we may even assume that \mathfrak{e} is the multiplication by the class of a divisor ample relatively to the Hitchin morphism. It easily follows that the perverse filtration matches the natural filtration coming from the action of this \mathfrak{sl}_2 (Proposition 8.3). We deduce that the $P = C$ conjecture holds for the pure part of the cohomology of the elliptic locus (Theorem 8.5).

In the rest of Section 8 we show how the results for the elliptic locus of the parabolic moduli space imply the desired $P = C$ statement for the entire space of stable Higgs bundles. This goes through several reduction steps: from elliptic parabolic Higgs bundles to stable parabolic Higgs bundles, to nilpotent stable parabolic Higgs bundles, and finally to stable Higgs bundles. We refer to Section 8.1 for a summary.

1.4. Some remarks. (1) The fact that the perverse filtration is controlled by Hecke operators is reminiscent of the construction in [OY16]. There, the authors consider an action of trigonometric Cherednik algebra on the cohomology of affine Springer fibers by Hecke correspondences, and degenerate it to an action of rational Cherednik algebra by taking the associated graded with respect to the perverse filtration; the latter is inherited from a realization of certain affine Springer fibers as Hitchin fibers. In our situation, the algebra generated by the operators $D_{m,n}$ can be viewed as a global version of the (spherical) trigonometric Cherednik algebra, and the passage to the operators $\tilde{D}_{m,n}$ in Section 6 can be understood as the rational degeneration.

(2) The most conceptual framework for the study of the algebras occurring in this paper is that of CoHAs of curves and surfaces, as considered in e.g. [Min20, SS20, KV19, DHS]. We expect that using the powerful structural results of Davison and Kinjo on the Lie algebra version of CoHAs (work in progress) together with the computation of the zero-dimensional CoHA in [MMSV23] one should be able to streamline our proof of $P = W$ and, for instance, avoid the use of the elliptic locus or parabolic Higgs bundles.

(3) Our action of \mathfrak{sl}_2 on $H^*(M_{r,d})$ gives rise to an action of the Weyl group S_2 , which exchanges (by conjugation) the operators of multiplication by tautological classes and the Hecke operators, as expected by mirror symmetry.

(4) The Higgs bundles appearing here are sometimes called GL_r -Higgs bundles. We expect that our results, including the algebra action and $P = W$ translate to the PGL_r case with only slight modifications.

(5) The original motivation for the $P = W$ conjecture came from the observation that the E-polynomials of character varieties and the conjectural mixed Hodge polynomials have an interesting symmetry, which would be geometrically explained by the so-called *curious Poincaré duality* or *curious hard Lefschetz theorem*, see [HRV08], [HLRV11]. For the perverse filtration, the corresponding symmetry is explained by the relative hard Lefschetz theorem, see [dCM05], [dCHM12], and Theorem 3.6 here. The curious hard Lefschetz theorem was established in [Mel19] by working on the Betti side. The present proof of $P = W$ now implies another, independent proof of the curious hard Lefschetz theorem (in the usual and the parabolic setups).

2. TOPOLOGICAL PREREQUISITES

All the varieties we consider in this paper are defined over \mathbb{C} . By a *space* we mean an algebraic variety or a (classical) Artin stack X of finite type. All our stacks are in fact of the form $\mathcal{X} = X \times B\mathbb{G}_m$ where X is an algebraic variety. We work with the cohomology $H^*(X) = H^*(X, \mathbb{Q})$, which is a ring, and the Borel-Moore homology $H_*(X) = H_*(X, \mathbb{Q})$, which is a module over $H^*(X)$. We refer to [CG97, Ch. 2] for standard properties of H^* and H_* . If X is of pure dimension n , we denote by $[X] \in H_{2n}(X)$ the fundamental class; it gives rise to a map

$$H^i(X) \rightarrow H_{2n-i}(X), \quad \alpha \rightarrow [X] \cap \alpha.$$

When X is smooth, this is an isomorphism, and we will sometimes use it implicitly to identify $H^*(X)$ and $H_{2n-*}(X)$. If X is not smooth, consider a resolution of singularities

$\pi : \tilde{X} \rightarrow X$. By the projection formula, the above map factors as

$$(2.1) \quad H^i(X) \xrightarrow{\pi^*} H^i(\tilde{X}) \cong H_{2n-i}(\tilde{X}) \xrightarrow{\pi_*} H_{2n-i}(X).$$

If X, Y are smooth spaces of dimensions n, m respectively, and $f : X \rightarrow Y$ is proper, we have the Gysin map in cohomology

$$H^i(X) \cong H_{2n-i}(X) \xrightarrow{f_*} H_{2n-i}(Y) \cong H^{2m-2n+i}(Y).$$

We denote the Gysin map by f_* as well.

2.1. Correspondences. Let Z be a space of pure dimension n , together with maps $\pi_1 : Z \rightarrow X$ and $\pi_2 : Z \rightarrow Y$ such that π_2 is proper. We call this collection of data a *correspondence*, and consider the *induced map*

$$H^i(X) \rightarrow H_{2n-i}(Y), \quad \alpha \rightarrow \pi_{2*}([Z] \cap \pi_1^* \alpha).$$

If Y is smooth of dimension m this gives a map $H^i(X) \rightarrow H^{2m-2n+i}(Y)$. The factorization (2.1) implies that Z can be replaced by a resolution of singularities without changing the induced map. If X is proper, we can decompose the induced map as follows:

$$H^i(X) \xrightarrow{\pi_X^*} H^i(X \times Y) \xrightarrow{(\pi_1 \times \pi_2)_*[Z] \cap} H_{2n-i}(X \times Y) \xrightarrow{\pi_{Y*}} H_{2n-i}(Y).$$

Notice that since π_2 is proper, $\pi_1 \times \pi_2$ is also proper. If moreover X and Y are smooth, the class $(\pi_1 \times \pi_2)_*[Z]$ corresponds to a class in $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$, which using the Poincaré duality on X and a choice of basis in $H^*(X), H^*(Y)$ can be identified with a matrix so that the induced map of the correspondence is the action of this matrix.

2.2. Purity. Let X be a smooth space. Consider any smooth compactification $\bar{X} \supset X$. The *pure part* $H_{\text{pure}}^*(X) \subset H^*(X)$ is the image of the restriction map $H^*(\bar{X}) \rightarrow H^*(X)$. This definition is independent of the choice of a smooth compactification. Indeed, for any two smooth compactifications \bar{X}, \bar{X}' , let \bar{X}'' be a resolution of singularities of the closure of X in $\bar{X} \times \bar{X}'$. Then we have a diagram

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \downarrow \iota & & \downarrow \pi_{2\iota} \\ \bar{X} & \xleftarrow{\pi_1} \bar{X}'' \xrightarrow{\pi_2} & \bar{X}' \end{array}$$

with cartesian square, and therefore by base change we have $\iota^* = (\pi_{2\iota})^* \pi_{2*}$. Thus

$$\text{Im}(\pi_1 \iota)^* \subset \text{Im} \iota^* = \text{Im}(\pi_{2\iota})^* \pi_{2*} \subset \text{Im}(\pi_{2\iota})^*,$$

from which the result follows. The following claim is standard; we include the proof for completeness.

Proposition 2.1. *Let $Z \rightarrow X \times Y$ be a correspondence between smooth spaces. The induced map in cohomology preserves the pure part. In particular, if X is proper then the image of the induced map is contained in $H_{\text{pure}}^*(Y)$.*

Proof. We may assume that Z is smooth. Hence it is enough to prove that the pure part is preserved under pullbacks and proper pushforwards by maps between smooth spaces. Let $f : Z \rightarrow X$ be such a map. Pick a smooth compactification of X , a smooth compactification of Z , and resolve singularities of the closure of Z in the corresponding product of compactifications to obtain a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow \iota_Z & & \downarrow \iota_X \\ \overline{Z} & \xrightarrow{\overline{f}} & \overline{X} \end{array}$$

where ι_Z, ι_X are smooth compactifications. Taking pullbacks implies that $f^*H_{\text{pure}}^*(X) \subset H_{\text{pure}}^*(Z)$. Next, assume $f : Z \rightarrow Y$ is proper, and consider a commutative square as above. The canonical map $j : Z \rightarrow \overline{Z} \times_{\overline{Y}} Y$ is both proper and an open embedding, hence it is an isomorphism, and the base change implies $f_*H_{\text{pure}}^*(Z) \subset H_{\text{pure}}^*(Y)$. \square

Note in particular that the Chern classes of any vector bundle (or a bounded complex of vector bundles) on a smooth space X are pure.

2.3. Diagonal class. Let X be smooth of pure dimension n and let $\overline{X} \supset X$ be a smooth compactification. The identity map $X \rightarrow X$ factors as a composition of proper maps $X \rightarrow \overline{X} \times X \rightarrow X$. By (2.1), we can rewrite the restriction map $H^*(\overline{X}) \rightarrow H_{\text{pure}}^*(X)$ as the composition

$$H^*(\overline{X}) \xrightarrow{\pi_1^*} H^*(\overline{X} \times X) \xrightarrow{[\Delta] \cap} H^*(\overline{X} \times X) \xrightarrow{\pi_2^*} H_{\text{pure}}^*(X),$$

where $\Delta \subset \overline{X} \times X$ is the diagonal. Thus the classes in $H_{\text{pure}}^*(X)$ of any Künneth decomposition of $[\Delta]$ span $H_{\text{pure}}^*(X)$.

2.4. Virtual fundamental class. Suppose X is smooth of dimension n and let \mathcal{E} be a complex vector bundle on X of rank r . We have the Thom class

$$\tau_{\mathcal{E}} \in H^{2r}(\text{Tot}_{\mathcal{E}}, \text{Tot}_{\mathcal{E}} \setminus X),$$

where $\text{Tot}_{\mathcal{E}}$ is the total space of \mathcal{E} . If $s : X \rightarrow \text{Tot}_{\mathcal{E}}$ is a section, by pullback we obtain a class in $H^{2r}(X, X \setminus Z) \cong H_{2n-2r}(Z)$, where $Z = X \cap s(X)$ is the zero set of s , which we call the *virtual fundamental class* and denote $[Z]^{\text{vir}}$. The image of $[Z]^{\text{vir}}$ under the pushforward $H_*(Z) \rightarrow H_*(X)$ is $[X] \cap c_r(\mathcal{E})$. Note that the class $[Z]^{\text{vir}}$ does depend on the realization of Z as the zero set of a vector bundle. However, if Z has pure dimension $n - r$ and the intersection $X \cap s(X)$ is generically transversal over Z , we have $[Z]^{\text{vir}} = [Z]$.

3. RELATIVE LEFSCHETZ THEORY, PERVERSE FILTRATION AND HECKE OPERATORS

In this section we review some notions of the relative Hard Lefschetz theorem of de Cataldo and Migliorini, and describe its behaviour with respect to correspondences.

3.1. Relative Lefschetz theory. Before summarizing the relative Hard Lefschetz theorem from [dCM05] we need to develop some algebraic framework.

Definition 3.1. A *Lefschetz structure* is a finite dimensional vector space V endowed with a linear endomorphism ω and a finite increasing filtration $P_{\bullet}V$ such that we have $\omega P_i V \subset P_{i+2} V$ for all i , and for all $k \geq 0$ the map

$$\omega^k : P_{-k} V / P_{-k-1} V \rightarrow P_k V / P_{k-1} V$$

is an isomorphism. A map of Lefschetz structures is a map of vector spaces compatible with the filtration and commuting with ω .

Denote by $\text{Gr } V$ the associated graded

$$\text{Gr } V = \bigoplus_i \text{Gr}_i V, \quad \text{Gr}_i V = P_i V / P_{i-1} V.$$

This is a graded vector space on which ω induces an operator of degree 2. A map of Lefschetz structures $\varphi : U \rightarrow V$ induces a map $\text{Gr } \varphi : \text{Gr } U \rightarrow \text{Gr } V$. The following is well-known.

Proposition 3.2. *The triple of a vector space V , an increasing filtration $P_{\bullet}V$ and $\omega : V \rightarrow V$ satisfying $\omega P_i V \subset P_{i+2} V$ is a Lefschetz structure if and only if there exists an action of \mathfrak{sl}_2 on $\text{Gr } V$ for which $\mathfrak{e} = \omega$ and \mathfrak{h} acts on $\text{Gr}_i V$ as the multiplication by i . For any map of Lefschetz structures $\varphi : U \rightarrow V$ the induced map $\text{Gr } \varphi$ commutes with \mathfrak{f} . In particular, the \mathfrak{sl}_2 -action above is unique. \square*

Lefschetz structures are similar to mixed Hodge structures [Del71] in the following sense:

Proposition 3.3. *The category of Lefschetz structures is abelian and the functor $V \rightarrow \text{Gr } V$ is an exact faithful functor from the category of Lefschetz structures to the category of finite dimensional representations of \mathfrak{sl}_2 .*

Proof. Let $\varphi : U \rightarrow V$ be a map of Lefschetz structures. By the snake lemma applied to the short exact sequence

$$0 \rightarrow P_i U / P_{i-k} U \rightarrow P_{i+1} U / P_{i-k} U \rightarrow P_{i+1} U / P_i U \rightarrow 0,$$

the corresponding sequence for V and the map between them induced by φ we obtain a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ker } \varphi_{P_i/P_{i-k}} \rightarrow \text{Ker } \varphi_{P_{i+1}/P_{i-k}} \rightarrow \text{Ker } \varphi_{P_{i+1}/P_i} \rightarrow \text{Coker } \varphi_{P_i/P_{i-k}} \\ \rightarrow \text{Coker } \varphi_{P_{i+1}/P_{i-k}} \rightarrow \text{Coker } \varphi_{P_{i+1}/P_i} \rightarrow 0. \end{aligned}$$

Here we denote by φ_{P_i/P_j} the induced map $P_i U / P_j U \rightarrow P_i V / P_j V$.

Lemma 3.4. *The connecting map $\text{Ker } \varphi_{P_{i+1}/P_i} \rightarrow \text{Coker } \varphi_{P_i/P_{i-k}}$ is zero.*

Proof. We proceed by induction. Notice that $\bigoplus_i \text{Ker } \varphi_{P_{i+1}/P_i} = \text{Ker } \text{Gr } \varphi$ is a representation of \mathfrak{sl}_2 , and therefore is generated over ω by *primitive* elements, i.e. elements $x \in \text{Ker } \varphi_{P_{-i}/P_{-i-1}}$ for $i \geq 0$ satisfying $\omega^{i+1} x = 0$. The image of such x under the connecting map is some $y \in \text{Coker } \varphi_{P_{-i-1}/P_{-i-1-k}}$ satisfying $\omega^{i+1} y = 0$.

Let us show that ω^i is injective on $\text{Coker } \varphi_{P_{-i}/P_{-i-k}}$ for all $i > 0$. By the induction assumption we have a short exact sequence

$$0 \rightarrow \text{Coker } \varphi_{P_{-i-1}/P_{-i-k}} \rightarrow \text{Coker } \varphi_{P_{-i}/P_{-i-k}} \rightarrow \text{Coker } \varphi_{P_{-i}/P_{-i-1}} \rightarrow 0,$$

and the same short exact sequence for i instead of $-i$. By snake lemma applied to the maps induced by ω^i , we have a short exact sequence

$$0 \rightarrow \text{Ker}(\text{Coker } \varphi_{P_{-i-1}/P_{-i-k}} \rightarrow \text{Coker } \varphi_{P_{-i-1}/P_{i-k}}) \rightarrow$$

$$\text{Ker}(\text{Coker } \varphi_{P_{-i}/P_{-i-k}} \rightarrow \text{Coker } \varphi_{P_i/P_{i-k}}) \rightarrow \text{Ker}(\text{Coker } \varphi_{P_{-i}/P_{-i-1}} \rightarrow \text{Coker } \varphi_{P_i/P_{i-1}}).$$

Since $\text{Coker } \text{Gr } \varphi$ is a representation of \mathfrak{sl}_2 , the map $\omega^i : \text{Coker } \varphi_{P_{-i}/P_{-i-1}} \rightarrow \text{Coker } \varphi_{P_i/P_{i-1}}$ is an isomorphism, and therefore its kernel vanishes. We know that ω^{i+1} on $\text{Coker } \varphi_{P_{-i-1}/P_{-i-k}}$ is injective, since this was already proved when treating the $k-1$ case. Hence ω^i on it is injective, and therefore $\text{Ker}(\text{Coker } \varphi_{P_{-i-1}/P_{-i-k}} \rightarrow \text{Coker } \varphi_{P_{i-1}/P_{i-k}})$ vanishes, and we conclude that the remaining kernel also has to vanish.

Coming back to the element $y \in \text{Coker } \varphi_{P_{-i-1}/P_{-i-1-k}}$, we see that $\omega^{i+1}y = 0$ implies $y = 0$, and therefore the connecting map vanishes on all primitive elements, and therefore on all elements. \square

Having established the vanishing of connecting map for all k , taking k sufficiently large we see that for all i the map

$$\text{Ker } \varphi \cap P_i U \rightarrow \text{Ker } \varphi_{P_i/P_{i-1}},$$

is surjective, which implies that the natural map

$$\text{Ker } \varphi \cap P_i U / \text{Ker } \varphi \cap P_{i-1} U \rightarrow \text{Ker } \varphi_{P_i/P_{i-1}}$$

is an isomorphism; equivalently, $\text{Gr Ker } \varphi \rightarrow \text{Ker Gr } \varphi$ is an isomorphism. Thus $\text{Ker } \varphi$ is a Lefschetz structure. Similarly, we obtain that $\text{Coker Gr } \varphi \rightarrow \text{Gr Coker } \varphi$ is an isomorphism, and so $\text{Coker } \varphi$ is a Lefschetz structure as well. This implies that the category of Lefschetz structures has kernels and cokernels given by the usual kernels and cokernels together with the induced filtrations. To see that it is abelian, we use the commutative square

$$\begin{array}{ccc} \text{Coker Ker Gr } \varphi & \xrightarrow{\sim} & \text{Ker Coker Gr } \varphi \\ \wr \downarrow & & \wr \uparrow \\ \text{Gr Coker Ker } \varphi & \longrightarrow & \text{Gr Ker Coker } \varphi \end{array}$$

to see that $\text{Gr Coker Ker } \varphi \rightarrow \text{Gr Ker Coker } \varphi$ is an isomorphism. This implies that the two filtrations on $\text{Im } \varphi$ coincide. We have shown that the category of Lefschetz structures is abelian. We have also shown that kernels and cokernels commute with Gr , which means that Gr is exact. Finally, it is faithful because $\text{Gr } V = 0$ and finiteness of the filtration implies $V = 0$. \square

Definition 3.5. For a nilpotent operator N acting on a vector space the *canonical filtration* or *weight filtration* is the increasing filtration W_\bullet defined by

$$W_k = \sum_{i \geq \max(0, k)} \text{Ker } N^{i+1} \cap \text{Im } N^{i-k}$$

We refer to e.g. [dCM05] for the definition of the perverse filtration associated to a projective morphism. We will only use the following properties of the perverse filtration, proved by de Cataldo and Migliorini:

Theorem 3.6 (See [dCM05], Theorems 2.1.1, 2.1.4, 2.3.3 and Sections 4.5, 4.6). *Let $f : X \rightarrow Y$ be a projective map of algebraic varieties. Suppose that X is smooth of dimension n . Let $\omega \in H^2(X)$ be a relative ample class, i.e. a class whose positive multiple comes from the embedding $X \rightarrow Y \times \mathbb{P}^N$. Then the associated perverse filtration $P_\bullet H^*(X)$ together with the operator of cup product by ω form a Lefschetz structure on $H^*(X)$. Moreover,*

- (i) *If $U \subset Y$ is open, then the restriction $H^*(X) \rightarrow H^*(f^{-1}(U))$ is a map of Lefschetz structures,*
- (ii) *If Y is projective and L is the pullback of an ample class on Y , then $P_i H^*(X) = \bigoplus_{b \in \mathbb{Z}} W_{i-b} H^{n+b}(X)$, where W_\bullet is the canonical filtration induced by the nilpotent operator of cup product by L .*

3.2. Functoriality. In this section we analyze the effect of pullback and Gysin maps on the perverse filtration. We assume that a quasi-projective base $A \subset \mathbb{P}^M$ is fixed. Let \bar{A} be the closure of A . Any projective map $\pi : X \rightarrow A$ induces an embedding $X \subset \mathbb{P}^N \times \mathbb{P}^M$ and resolving the singularities of the closure we obtain a smooth projective compactification $\iota : X \rightarrow \bar{X}$ together with a map $\bar{X} \rightarrow \mathbb{P}^N \times \mathbb{P}^M$. Let ω be an ample class of \bar{X} . By pullback we obtain a map of Lefschetz structures

$$\iota^* : H^*(\bar{X}) \rightarrow H^*(X),$$

which induces a Lefschetz structure on $H_{\text{pure}}^*(X)$. In particular, we have

$$P_k H_{\text{pure}}^*(X) = P_k H^*(X) \cap H_{\text{pure}}^*(X) = \iota^* P_k H^*(\bar{X}).$$

Any $\alpha \in H^i(\bar{X})$ induces the cup-product map $H^*(\bar{X}) \rightarrow H^{*+i}(\bar{X})$, which commutes with L , which is the pullback of the hyperplane class from \mathbb{P}^M . So α preserves the canonical filtration associated with L and therefore we have

$$\alpha P_k H^*(\bar{X}) \subset P_{k+i} H^{*+i}(\bar{X}).$$

Restricting to $H^*(X)$, we obtain the following:

Proposition 3.7. *For any $\alpha \in H_{\text{pure}}^i(X)$ we have $\alpha P_k H_{\text{pure}}^*(X) \subset P_{k+i} H_{\text{pure}}^{*+i}(X)$. □*

Next, suppose we have a commutative diagram

$$(3.1) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \pi_X & \swarrow \pi_Y & \\ A & & \end{array} ,$$

where π_X and π_Y are projective and X, Y are smooth of dimensions d_X, d_Y . Note that the map f is automatically projective. We pick smooth compactifications $\iota_X : X \rightarrow \overline{X}$, $\iota_Y : Y \rightarrow \overline{Y}$ and a commutative diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\ \downarrow \pi_{\overline{X}} & \swarrow \pi_{\overline{Y}} & \\ A & & \end{array},$$

which produces (3.1) when restricted to A . The pullback $H^*(\overline{Y}) \rightarrow H^*(\overline{X})$ and the Gysin map $H^*(\overline{X}) \rightarrow H^{*+2d_Y-2d_X}(\overline{Y})$ commute with the multiplication by L and therefore preserve the canonical filtration associated to it. So we obtain

Proposition 3.8. *For a commutative diagram of the form (3.1) with π_X, π_Y projective and X, Y smooth of dimensions d_X, d_Y the pullback map and the Gysin map satisfy*

$$f^* P_k H_{\text{pure}}^*(Y) \subset P_{k+d_Y-d_X} H_{\text{pure}}^*(X), \quad f_* P_k H_{\text{pure}}^*(X) \subset P_{k+d_Y-d_X} H_{\text{pure}}^{*+2d_Y-2d_X}(Y).$$

More generally, suppose that we have a commutative diagram

$$(3.2) \quad \begin{array}{ccccc} X & \xleftarrow{\pi_1} & Z & \xrightarrow{\pi_2} & Y \\ & \searrow \pi_X & \downarrow \pi_Z & \swarrow \pi_Y & \\ & & A & & \end{array},$$

where X, Y, Z are smooth and π_X, π_Y, π_Z are projective. Let us also assume that Z carries a virtual fundamental class $[Z]^{\text{vir}} \in H_*(Z)$, see Section 2.4. Consider the action of the correspondence

$$(3.3) \quad \alpha \rightarrow \pi_{2*}(\pi_1^* \alpha \cap [Z]^{\text{vir}}).$$

By (2.1), we can assume that Z is smooth. Suppose the dimensions of X, Y, Z are d_X, d_Y, d_Z respectively. Then Proposition 3.8 implies

Proposition 3.9. *The correspondence (3.3) sends $P_i H^*(X)$ to $P_{i+d_X+d_Y-2d_Z+2k} H^{*+2d_Y-2d_Z+2k}(Y)$.*

Notice that d_Z and k enter the statement via the difference $d_Z - k$, which corresponds to the difference between the (naive) dimension and the virtual dimension of Z .

Remark 3.10. (i) When $2d_Z - 2k = d_X + d_Y$, e.g. for $k = 0$, X and Y symplectic, and Z a Lagrangian in $X \times Y$, the correspondence preserves the perverse filtration.

(ii) When $d_X = d_Y$, a correspondence of cohomological degree j sends P_i to P_{i+j} .

4. HECKE CORRESPONDENCES

4.1. Notations. Let S be a smooth projective surface, and set $H = H^* = H^*(S, \mathbb{Q})$. Let $\delta \in H^4$ be the class of a point; by abuse of notation we write $\Delta \in \bigoplus_i H^i \otimes H^{4-i}$ for the class of the diagonal. Denote the Chern roots of the tangent bundle of S by t_1, t_2 . We then have

$$\mathrm{Td}_S^{-1} = \frac{(1 - e^{-t_1})(1 - e^{-t_2})}{t_1 t_2} = 1 - \frac{t_1 + t_2}{2} + \frac{2t_1^2 + 2t_2^2 - 3t_1 t_2}{12}.$$

In this section we consider actions of correspondences on moduli stacks of coherent sheaves on S induced by pointwise modifications. Such actions are best studied using CoHAs, as in [MMSV23], [Neg19]. However we will not need the full generality of *loc.cit.* and thus only cite the relevant results. In particular, as we will consider correspondences over *smooth* stacks only, we will freely pass between cohomology and Borel-Moore homology.

Fix a non-zero $\alpha_0 \in H^{\mathrm{ev}} := H^0 \oplus H^2 \oplus H^4$. For any $\alpha = \alpha_0 + n\delta$ with $n \in \mathbb{N}$ we fix a smooth open substack \mathcal{M}_α of the (classical) moduli stack of coherent sheaves on S of Chern character α . We denote by Coh_δ the stack of length 1 sheaves on S ; it is explicitly given by $\mathrm{Coh}_\delta = S \times B\mathbb{G}_m$. We denote by \mathcal{F}_α the universal coherent sheaf on $\mathcal{M}_\alpha \times S$.

We make the following two running assumptions:

- (A1) For any α , \mathcal{M}_α parametrizes coherent sheaves on S with no finite length subsheaves,
- (A2) For any $n > 0$, a coherent sheaf $\mathcal{F} \in \mathcal{M}_{\alpha_0+n\delta}$ and a colength one subsheaf $\mathcal{G} \subset \mathcal{F}$ we have $\mathcal{G} \in \mathcal{M}_{\alpha_0+(n-1)\delta}$.

Such assumptions are verified in several examples of interest, e.g. for stable Higgs bundles over the elliptic locus, see Section 7.

4.2. Length one Hecke correspondences. By assumption (A1), the universal sheaf \mathcal{F}_α over \mathcal{M}_α admits a length two resolution $\mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F}_\alpha$ by vector bundles. We may thus consider the projectivization $\mathcal{Z}_\alpha := \mathbb{P}(\mathcal{F}_\alpha)$, which may be explicitly realized as the zero set of the canonical section of the vector bundle $\mathcal{E}_1^*(1)$ on the projective bundle $\mathbb{P}(\mathcal{E}_0)$. The stack \mathcal{Z}_α parameterizes colength 1 subsheaves of \mathcal{F}_α , c.f. [Jia22, Sec. 8], or equivalently flags $F' \subset F$ of sheaves on S with $\mathrm{ch} F = \alpha$, and quotient F/F' of length 1.

By assumption (A2), we have a natural map $\mathcal{Z}_\alpha \rightarrow \mathcal{M}_{\alpha-\delta}$. Consider the following *Hecke correspondence*:

$$(4.1) \quad \begin{array}{ccc} & \mathcal{Z}_\alpha & \\ \pi_1 \times \pi_2 \swarrow & & \searrow \pi_3 \\ \text{Coh}_\delta \times \mathcal{M}_{\alpha-\delta} & & \mathcal{M}_\alpha \end{array} \quad \begin{array}{l} \pi_1 \times \pi_2 : (F' \subset F) \mapsto (F/F', F'), \\ \pi_3 : (F' \subset F) \mapsto F. \end{array}$$

The map π_3 is proper by construction. Therefore, as explained in Section 2, we obtain the following map, which we denote by T :

$$\begin{aligned} H^*(\text{Coh}_\delta) \otimes H^*(\mathcal{M}_{\alpha-\delta}) &\rightarrow H^*(\mathcal{M}_\alpha), \\ \xi \otimes c &\mapsto T(\xi)(\eta) = \pi_{3*}([Z_\alpha]^{\text{vir}} \cap \pi_1^*(\xi) \cap \pi_2^*(c)). \end{aligned}$$

4.3. The tautological classes. Denote by Λ the Macdonald ring of symmetric functions in infinitely many variables. We have $\Lambda = \mathbb{C}[p_1, p_2, \dots]$, where p_k is the power sum function of degree k . For each α there is a ring homomorphism $\Lambda \rightarrow H^*(\mathcal{M}_\alpha \times S)$, which sends p_k to the element $p_k(\mathcal{F}_\alpha) \in H^{2k}(\mathcal{M}_\alpha \times S)$ defined via the generating series:

$$\text{ch}(\mathcal{F}_\alpha) = \text{rk } \mathcal{F}_\alpha + \sum_{n=1}^{\infty} \frac{p_n(\mathcal{F}_\alpha)}{n!}.$$

It is convenient to denote $p_0(\mathcal{F}_\alpha) = \text{rk } \mathcal{F}_\alpha$. For any symmetric function f of degree k and any $\gamma \in H^m$ we set

$$f(\gamma) = \pi_{\mathcal{M}_\alpha^*}(f(\mathcal{F}_\alpha) \cup \pi_S^* \gamma) \in H^{2k+m-4}(\mathcal{M}_\alpha),$$

where $\pi_{\mathcal{M}_\alpha} : \mathcal{M}_\alpha \times S \rightarrow \mathcal{M}_\alpha$, $\pi_S : \mathcal{M}_\alpha \times S \rightarrow S$ are the projections.

The classes $f(\gamma)$ generate a subring $H_{\text{taut}}^*(\mathcal{M}_\alpha) \subset H_{\text{pure}}^*(\mathcal{M}_\alpha)$, which we call the *tautological ring*.

Let us write $\Delta\gamma := \Delta(\gamma \otimes 1) = \Delta(1 \otimes \gamma)$. The product of symmetric functions is related to the product in $H_{\text{taut}}^*(\mathcal{M}_\alpha)$ by

$$(4.2) \quad (fg)(\gamma) = (f \otimes g)(\Delta\gamma),$$

where the right hand side is defined via $(f \otimes g)(\gamma_1 \otimes \gamma_2) = f(\gamma_1)g(\gamma_2)$.

Recall the tautological sheaf \mathcal{F}_δ on $\text{Coh}_\delta \times S$. We have $\mathcal{F}_\delta = \mathcal{O}_\Delta(1)$, where \mathcal{O}_Δ is the structure sheaf of the diagonal in $S \times S$ and (1) means tensoring with the weight 1 character of \mathbb{G}_m . The Chern character of \mathcal{O}_Δ is ΔTd_S^{-1} , therefore

$$\text{ch}(\mathcal{F}_\delta) = \text{ch}(\mathcal{O}_\delta) \text{ch}(\mathcal{O}(1)) = \Delta \text{Td}_S^{-1} e^u,$$

where $u = c_1(\mathcal{O}(1))$ is the equivariant parameter. We deduce that

$$p_n(\mathcal{F}_\delta)(\eta) = \frac{u^n - (u - t_1)^n - (u - t_2)^n + (u - t_1 - t_2)^n}{t_1 t_2} \eta.$$

4.4. The action of the Hecke operators on 1. Let us explicitly compute the action of Hecke correspondences $T(\xi u^n)$ on the subspace $\bigoplus_n H_{\text{taut}}^*(\mathcal{M}_{\alpha_0+n\delta})$. Consider the space $\mathcal{Z}_\alpha \times S$ equipped with the maps $\pi_{i4} = \pi_i \times \pi_S$, ($i = 1, 2, 3$). The projection formula together with the identity $[\pi_{14}^* \mathcal{F}_\delta] + [\pi_{24}^* \mathcal{F}_{\alpha-\delta}] = [\pi_{34}^* \mathcal{F}_\alpha]$ in $K_0(\mathcal{Z}_\alpha \times S)$ imply the following identity of operators $H^*(\mathcal{M}_{\alpha-\delta}) \rightarrow H^*(\mathcal{M}_\alpha)$:

$$(4.3) \quad [p_n(\eta), T(\xi)] = T \left(\frac{u^n - (u - t_1)^n - (u - t_2)^n + (u - t_1 - t_2)^n}{t_1 t_2} \eta \xi \right)$$

for any $\eta \in H$, $\xi \in H^*(\text{Coh}_\delta)$ and $n \geq 0$. The commutator on the left hand side is *graded*, i.e. if both η and ξ are odd we take the sum instead of the difference.

Thanks to the commutation relations (4.3), in order to fully determine the action of the Hecke operators on the tautological ring $H_{\text{taut}}^*(\mathcal{M}_\alpha)$ we only need to compute the action of each Hecke operator on 1.

The map $\mathcal{Z}_\alpha \rightarrow \mathcal{M}_\delta = S \times B\mathbb{G}_m$ has two components. The component corresponding to S comes from the projection $\mathbb{P}(\mathcal{E}_0) \rightarrow S$. The component corresponding to $B\mathbb{G}_m$ is given by the tautological line bundle L on $\mathbb{P}(\mathcal{E}_0)$. The class $c_1(L)$ thus gets identified with the pullback of u . We will denote $c_1(L) = u$ by abuse of notation.

Recall that \mathcal{Z}_α is presented as the zero set of a section of the vector bundle $\mathcal{E}_1^*(1)$ on $\mathbb{P}(\mathcal{E}_0)$. The projection π_3 factors as the composition of proper maps

$$\mathcal{Z}_\alpha \rightarrow \mathbb{P}(\mathcal{E}_0) \rightarrow \mathcal{M}_\alpha \times S \rightarrow \mathcal{M}_\alpha.$$

We have

$$H^*(\mathbb{P}(\mathcal{E}_0)) = H^*(\mathcal{M}_\alpha \times S)[u] / (u^m - c_1(\mathcal{E}_0)u^{m-1} + \cdots + (-1)^m c_m(\mathcal{E}_0)),$$

where m is the rank of \mathcal{E}_0 . The Gysin map $H^*(\mathbb{P}(\mathcal{E}_0)) \rightarrow H^*(\mathcal{M}_\alpha \times S)$ applied to a polynomial $p(u)$ can be computed as follows: expand

$$\frac{p(u)}{u^m - c_1(\mathcal{E}_0)u^{m-1} + \cdots \pm c_m(\mathcal{E}_0)}$$

as a power series in u^{-1} , and take the coefficient of u^{-1} . On the other hand, the Euler class of the bundle $\mathcal{E}_1^*(1)$ is given by the expression

$$u^{m'} - c_1(\mathcal{E}_1)u^{m'-1} + \cdots + (-1)^m c_{m'}(\mathcal{E}_1),$$

where m' is the rank of \mathcal{E}_1 . So the result of taking an element u^n , multiplying it by the Euler class of $\mathcal{E}_1^*(1)$ and applying the Gysin map to $H^*(\mathcal{M}_\alpha \times S)$ is the coefficient of u^{-n-1} in the expansion of

$$\frac{u^{m'} - c_1(\mathcal{E}_1)u^{m'-1} + \cdots \pm c_{m'}(\mathcal{E}_1)}{u^m - c_1(\mathcal{E}_0)u^{m-1} + \cdots \pm c_m(\mathcal{E}_0)} = \sum_i u^{-\text{rk } \mathcal{F}_\alpha - i} s_i(\mathcal{F}_\alpha),$$

which is the generating series of the Segre classes of $\mathcal{F}_\alpha = \mathcal{E}_0/\mathcal{E}_1^1$. In our notation, the Segre classes correspond to the complete homogeneous symmetric functions h_n . Hence we obtain

$$(4.4) \quad T(\xi u^n)(1) = h_{n+1-\text{rk } \mathcal{F}_\alpha}(\xi), \quad \xi \in H, \quad n \geq 0.$$

4.5. The action of the Hecke operators on tautological classes. Equations (4.3) and (4.4) may be used to write the action of Hecke operators in a closed form. For this, let us introduce the super-commutative ring

$$(4.5) \quad \Lambda_S := \mathbb{Q}[p_k(\eta) ; k \geq 0, \eta \in H]/R$$

where $\deg(p_k(\eta)) = 2k - 4 + \deg \eta$, and the ideal R is generated by the following relations:

$$(4.6) \quad \begin{aligned} p_k(\eta + \lambda) &= p_k(\eta) + p_k(\lambda), & p_k(a\eta) &= ap_k(\eta), \\ p_0([\text{pt}]) &= \text{rk } \mathcal{F}_\alpha, & p_k(\lambda) &= 0 \text{ if } 2k + \deg \lambda < 4 \end{aligned}$$

for any $k \geq 0$, $\eta, \lambda \in H$ and $a \in \mathbb{Q}$. The ring Λ_S may be interpreted as the universal ring of tautological classes, that is, for any α there is a canonical evaluation morphism $\Lambda_S \rightarrow H^*(\mathcal{M}_\alpha)$ sending $p_k(\lambda)$ to $p_k(\lambda)$. Set $\mathbf{V}_S = \bigoplus_{\alpha \in \alpha_0 + \mathbb{N}\delta} H^*(\mathcal{M}_\alpha)$ and consider the linear map

$$\begin{aligned} \Phi: \Lambda_S \otimes H[u] &\rightarrow \text{End}(\mathbf{V}_S), \\ f \otimes \xi u^n &\rightarrow fT(\xi u^n) \end{aligned}$$

for $f \in \Lambda_S$, $\xi \in H$, $n \geq 0$. Note that Φ is *not* a ring homomorphism, but a morphism of Λ_S -modules for the action of Λ_S on $\Lambda_S \otimes H[u]$ by left by multiplication. Next, let us define a ring homomorphism

$$\begin{aligned} R: \Lambda_S &\rightarrow \Lambda_S \otimes H[u], \\ p_n(\eta) &\rightarrow p_n(\eta) - \frac{u^n - (u - t_1)^n - (u - t_2)^n + (u - t_1 - t_2)^n}{t_1 t_2} \eta, \end{aligned}$$

¹This also follows from the general theory of Segre classes, see [Ful98].

and a linear map

$$Q : \Lambda_S \otimes H[u] \rightarrow \Lambda_S, \quad f \otimes \xi u^n \mapsto fh_n(\xi).$$

In view of (4.3) and (4.4), we have

$$(4.7) \quad T(\xi u^n)(f \cdot 1) = \Phi(\xi u^n \cdot R(f))(1) = Q(\xi u^{n-\text{rk } \mathcal{F}_\alpha+1} R(f))$$

for any $f \in \Lambda_S, \xi \in H$ and $n \geq 0$. This completely describes the Hecke action on $\mathbf{V}_S^{\text{taut}} := \bigoplus_{\alpha \in \alpha_0 + \mathbb{N}\delta} H_{\text{taut}}^*(\mathcal{M}_\alpha)$.

4.6. The case of an open surface. So far, we have only considered the case of a projective surface S . All of the results above naturally extend to the quasiprojective case, see [MMSV23]. More precisely, let $S_0 \subsetneq S$ be an open subset and assume that \mathcal{M}_α parameterizes sheaves with (proper) support in S_0 . This automatically implies that such sheaves are of rank zero. In this situation, one can still define Hecke correspondences

$$H^*(\text{Coh}(S_0)_\delta) \otimes H^*(\mathcal{F}_\alpha) \rightarrow H^*(\mathcal{F}_{\alpha+\delta})$$

via the diagram (4.1), provided that the assumptions (A1), (A2) hold. The tautological class $f(\gamma) \in H^*(\mathcal{M}_\alpha)$ only depends the restriction of γ to $H_{\text{pure}}^*(S_0)$; indeed, if $\gamma \in \text{Ker}(H^*(S) \rightarrow H^*(S_0))$ then the long exact sequence in Borel-Moore homology shows that $\pi_S^* \gamma$ comes from $H_*(\mathcal{M}_\alpha \times (S \setminus S_0))$, on which the cap product by $\text{ch } \mathcal{F}_\alpha$ vanishes. This allows us to consider a ring of tautological classes Λ_{S_0} defined as in (4.5) by replacing $H^*(S)$ with $H_{\text{pure}}^*(S_0)$, as well as a family of Hecke operators $T(\xi u^n)$ for $\xi \in H_{\text{pure}}^*(S_0)$ and $n \geq 0$ acting on \mathbf{V}_S . In that setting, the formula (4.7) remains valid by an identical argument.

Remark 4.1. One caveat to keep in mind is that the value $h_0(\gamma) = 1(\gamma)$ is undefined, but because $\text{rk } \mathcal{F}_\alpha = 0$, the argument of Q in (4.7) has only strictly positive powers of u , so that $h_0(\gamma)$ never occurs.

5. W-ALGEBRAS AND THE FOCK SPACE REPRESENTATION

5.1. The Fock space. In order to study the relations between the Hecke operators $T(\xi u^n)$ acting on \mathbf{V}_S , it is useful to recast the results of the previous section in a more general, purely algebraic context. We do this by studying an algebra of operators defined by (4.4) on a polynomial ring Λ_S .

Let us summarize the data upon which our constructions depend.

- A base ring \mathbf{k} , which is graded super-commutative;²
- A graded super-commutative ring H over \mathbf{k} ;
- An augmentation map $\epsilon : H \rightarrow \mathbf{k}$, which is a \mathbf{k} -linear map of degree -4 ;
- An element $\Delta \in H \otimes_{\mathbf{k}} H$ of degree 4 satisfying $(\epsilon \otimes \text{Id})(\Delta) = (\text{Id} \otimes \epsilon)(\Delta) = 1$ and $(\xi \otimes 1)\Delta = (1 \otimes \xi)\Delta$ for all $\xi \in H$;
- Fixed elements $c_1 \in H^2$, $c_2 \in H^4$, satisfying $\Delta^2 = t_1 t_2 \Delta$. We formally write $c_1 = t_1 + t_2$ and $c_2 = t_1 t_2$ and identify symmetric functions in t_1, t_2 with polynomials in c_1, c_2 . Geometrically, they correspond to the Chern classes of the tangent bundle of S .

In the above setting, we consider the graded super-commutative algebra $\Lambda_H = \bigoplus_{n \geq 0} \Lambda_H^n$ generated over \mathbf{k} by elements $p_n(\xi)$, $n \geq 0$, $\xi \in H$ subject to the relations (4.6). The degree of $p_n(\xi)$ is $2n - 4 + i$ for $\xi \in H^i$. We extend the notation $f(\xi)$ to any symmetric function f by requiring

$$1(\xi) = \epsilon(\xi), \quad (fg)(\xi) = (f \otimes g)(\Delta\xi).$$

Finally, we define the Fock space

$$\mathbf{F}_H := \Lambda_H[s] \simeq \bigoplus_{n \geq 0} \Lambda_H s^n.$$

This is a \mathbb{N}^2 -graded vector space with $\mathbf{F}_H[a, b] = \Lambda_H^a s^b$. We call the first, resp. second grading the *vertical*, resp. *horizontal* grading.

5.2. The operators. Consider the ring morphism $R: \Lambda_H \rightarrow \Lambda_H \otimes_{\mathbf{k}} H[u]$ defined by

$$R(p_n(\eta)) = p_n(\eta) - \frac{u^n - (u - t_1)^n - (u - t_2)^n + (u - t_1 - t_2)^n}{t_1 t_2} \eta, \quad n \geq 0, \eta \in H,$$

and a Λ_H -linear map

$$Q: \Lambda_H \otimes_{\mathbf{k}} H[u] \rightarrow \Lambda_H, \quad 1 \otimes \eta u^n \mapsto h_n(\eta), \quad n \geq 0, \eta \in H.$$

For each $\xi \in H$, $n \in \mathbb{N}$ we define $T_n(\xi) : \Lambda_H \rightarrow \Lambda_H$ to be the operator

$$(5.1) \quad T_n(\xi)(f) = Q(\xi u^n R(f)).$$

²In the classical setup we have $\mathbf{k} = \mathbb{Q}$, but when a group G acts on S preserving \mathcal{M}_α we take $\mathbf{k} = H^*(BG)$. Another possibility is a relative situation when S is replaced by a family of surfaces and \mathbf{k} is the cohomology ring of the base.

Define also for each $\xi \in H$, $n \geq 0$ an element $\psi_n(\xi) \in \Lambda_H$ by the relation

$$(5.2) \quad \psi_n(\xi) = \sum_{1 \leq m \leq n+1} \frac{n!}{m!} p_m(\text{Td}_{n+1-m} \xi),$$

where $\text{Td}_k \in H^{2k}$ are the coefficients of the (abstract) Todd class:

$$\sum_{k \geq 0} \text{Td}_k x^k = \frac{t_1 t_2 x^2}{(1 - e^{-t_1 x})(1 - e^{-t_2 x})} \in H[[x]].$$

We will regard $\psi_n(\xi)$ as an operator on Λ_H by left multiplication.

Remark 5.1. When $H = H^*(S)$, the ψ_n 's correspond to the coefficients of $\text{Td}_S \text{ch}(\mathcal{F}_\alpha)$, and the generating series of ψ is

$$\sum_{n \geq 0} \frac{x^n \psi_n}{n!} = \left(\sum_{n \geq 1} \frac{x^n p_n}{n!} \right) \frac{t_1 t_2 x}{(1 - e^{-t_1 x})(1 - e^{-t_2 x})},$$

Observe that any p_n for $n \geq 1$ is a linear combination of ψ_m 's. The remaining element p_0 will not play a role in this work (but see [MMSV23, 1.6] for its geometric meaning).

We regard the operators $T_n(\xi), \psi_m(\xi)$ as acting on \mathbf{F}_H in the following way:

$$T_n(\xi) : \mathbf{F}_H[a, b] \rightarrow \mathbf{F}_H[a + 2n + \xi, b + 1],$$

$$\psi_m(\xi) : \mathbf{F}_H[a, b] \rightarrow \mathbf{F}_H[a + 2m - 2 + \xi, b],$$

i.e. $T_n(\xi), \psi_m(\xi)$ are of respective horizontal degrees 1 and 0.

Theorem 5.2. *Set $s_2 = c_1^2 - c_2$. The action of the operators $\psi_m(\xi), T_n(\xi)$, $m, n \geq 0$, $\xi \in H$ on \mathbf{F}_H satisfies the following relations:*

$$(5.3) \quad [\psi_m(\xi), \psi_n(\eta)] = 0, \quad \psi_m(a\xi + b\eta) = a\psi_m(\xi) + b\psi_m(\eta),$$

$$(5.4) \quad [\psi_m(\eta), T_n(\xi)] = mT_{m+n-1}(\eta\xi),$$

$$(5.5) \quad [T_m(\xi\xi'), T_n(\xi'')] = [T_m(\xi), T_n(\xi'\xi'')],$$

$$(5.6) \quad [T_m(\xi), T_{n+3}(\xi')] - 3[T_{m+1}(\xi), T_{n+2}(\xi')] + 3[T_{m+2}(\xi), T_{n+1}(\xi')] - [T_{m+3}(\xi), T_n(\xi')] \\ - [T_m(\xi), T_{n+1}(s_2\xi')] + [T_{m+1}(\xi), T_n(s_2\xi')] + \{T_m, T_n\}(c_1 \Delta \xi \xi') = 0,$$

$$(5.7) \quad \sum_{\pi \in S_3} \pi [T_{m_3}(\xi_3), [T_{m_2}(\xi_2), T_{m_1+1}(\xi_1)]] = 0.$$

Proof. This results from a direct computation, which is performed in [MMSV23, Sec. 3] in the case of $H = H^*(S)$. The general case is identical. \square

Remark 5.3. All the commutators above a super-commutators. In (5.6), $\{T_m, T_n\}(c_1\Delta\xi\xi')$ stands for the super-anticommutator of the operators T_m, T_n whose arguments are taken from the tensor $s_1\Delta\xi\xi' \in H \otimes H$. In (5.7), π permutes both the indices m_i and elements ξ_i . Moreover, when we permute ξ_i the sign changes according to the usual rules depending on the parity of ξ_i .

5.3. Surface W -algebra. Let us now return to the setting of a smooth quasiprojective surface S . Let $H = H_{\text{pure}}^*(S)$, and write $\mathbf{F}_S, \Lambda_S, \dots$ instead of $\mathbf{F}_H, \Lambda_H, \dots$. Note that there is a canonical evaluation morphism $\text{ev}: \mathbf{F}_S \rightarrow \mathbf{V}_S$.

Definition 5.4. The *surface W -algebra*³ W_S is the algebra generated by elements $\psi_m(\pi), T_m(\pi)$, where $m \geq 0$ and π runs over a basis of H , modulo relations (5.3)–(5.7).

Note that W_S is naturally \mathbb{N} -graded, where we put $\psi_m(\xi)$ and $T_n(\xi)$ in degrees 0 and 1 respectively. From the results of Section 4, we have

Corollary 5.5. *The canonical evaluation morphism $\text{ev}: \mathbf{F}_S \rightarrow \mathbf{V}_S$, together with the assignment*

$$\psi_n(\xi) \mapsto \psi_n(\xi), \quad T_m(\xi) \mapsto T(\xi u^{m-1+\text{rk } \mathcal{F}_\alpha})$$

intertwines the action of W_S on \mathbf{F}_S and of the corresponding tautological and Hecke operators on \mathbf{V}_S .

The setup of Section 5.1 only applies to projective surfaces, since otherwise we do not have an augmentation map ϵ . However, in the quasiprojective case one may fix a compactification of S and argue as in Section 4.6. As the relations (5.3)–(5.7) are invariant under index shifts in the generators T_m , the assignment

$$\psi_n(\xi) \mapsto \psi_n(\xi), \quad T_m(\xi) \mapsto T(\xi u^m)$$

still yields a representation of W_S on \mathbf{V}_S . From now on, we consider this renormalized action. Beware that with this renormalization, formula (5.1) only holds up to an index shift of $(1 - \text{rk } \mathcal{F}_\alpha)$.

³There is a notational clash with [MMSV23], where W_S is denoted W_S^+ , as it corresponds to *positive* punctual modifications.

5.4. **The structure of W_S .** Let us begin with the simpler case:

Theorem 5.6 ([MMSV23, Thm. 3.5]). *Suppose S is such that both $s_2 \in H^4(S)$ and $s_1\Delta \in H^6(S \times S)$ vanish. Then W_S is isomorphic to the universal enveloping algebra of the Lie algebra with basis $D_{m,n}(\pi)$, where $m, n \in \mathbb{Z}_{\geq 0}$ and π runs over a basis of $H^*(S)$, and Lie bracket given by*

$$(5.8) \quad [D_{m,n}(\xi), D_{m',n'}(\eta)] = (nm' - mn')D_{m+m',n+n'-1}(\xi\eta).$$

The generators are related by $D_{0,n}(\xi) = \psi_n(\xi)$, $D_{1,n}(\xi) = T_n(\xi)$. More generally, for any $\xi \in H$ and $n, m \geq 0$ we have

$$D_{m,n}(\xi) = \frac{n!}{(m+n)!} (-\text{Ad}_{T_0(1)})^m \psi_{m+n}(\xi).$$

Remark 5.7. The assumptions of the theorem are satisfied in the following cases:

- S is an abelian surface,
- S is a K3 surface without a point,
- S is the total space of a line bundle over a curve.

The same description works for an arbitrary S , but only after passing to the associated graded with respect to a suitable filtration. We write $W_S = \bigoplus_m W_S[m]$ for the decomposition into graded pieces for the horizontal grading. Let F_\bullet be the smallest filtration on W_S such that for all $m, n \in \mathbb{Z}$ and $\xi \in H$ we have

$$\psi_n(\xi), T_n(\xi) \in F_n, \quad F_m F_n \subset F_{m+n}, \quad [F_m, F_n] \subset F_{m+n-1}.$$

Theorem 5.8 ([MMSV23, Sec. 3]). *There exist elements $D_{m,n}(\xi) \in W_S[m]$, $m, n \geq 0$, $\xi \in H$ such that*

- (i) $D_{0,n}(\xi) = \psi_n(\xi)$, $D_{1,n}(\xi) = T_n(\xi)$;
- (ii) $F_{-1} = 0$ and F_N is spanned by products $D_{m_1, n_1}(\xi_1) \cdots D_{m_k, n_k}(\xi_k)$ satisfying $n_1 + \cdots + n_k \leq N$;
- (iii) $[D_{m,n}(\xi), D_{m',n'}(\eta)] = (nm' - mn')D_{m+m',n+n'-1}(\xi\eta)$ modulo $F_{n+n'-3}$.

The following elements will play an important role in the next section:

$$q_m(\xi) := D_{m,0}(\xi), \quad L_m(\xi) := D_{m,1}(\xi), \quad \mathfrak{d} := \frac{1}{2}\psi_2(1); \quad m \geq 0, \xi \in H.$$

Although we will not be using this, we quote for completeness a simple corollary of Theorem 5.6, which explains the link of W_S with (a half of) the Virasoro algebra and the element $W_3(0)$ of the Zamolodchikov algebra.

Corollary 5.9. *The elements $q_m(\xi)$, $L_m(\xi)$, $m \geq 0$, $\xi \in H$ and $\mathfrak{d} = \frac{1}{2}\psi_2(1)$ satisfy relations*

$$\begin{aligned} [q_m(\xi), q_n(\eta)] &= 0, & [L_m(\xi), q_n(\eta)] &= nq_{m+n}(\xi\eta), & [L_m(\xi), L_n(\eta)] &= (n-m)L_{m+n}(\xi\eta), \\ [\mathfrak{d}, q_m(\xi)] &= mL_m(\xi). \end{aligned}$$

6. DEGENERATION AND THE \mathfrak{sl}_2

We keep the geometric setup of Section 4; namely, a smooth quasiprojective surface S with a smooth compactification \bar{S} . We will write classes in H^{ev} as $\alpha = \alpha^{(0)} + \alpha^{(1)} + \alpha^{(2)}$, where $\alpha^{(i)} \in H^{2i}(\bar{S})$. Note that only $\alpha^{(2)}$ changes when we replace α by $\alpha + \delta$. From now on, we will fix some $\alpha_0 = \alpha_0^{(1)} + \alpha_0^{(2)}$ and consider only classes $\alpha \in \alpha_0 + \mathbb{Z}\delta$. In particular, for any moduli space \mathcal{M}_α we have $\text{rk } \mathcal{F}_\alpha = 0$, and $\alpha^{(1)}$ is a class of an effective divisor. For any $\eta \in H^2$ we have $\psi_0(\eta) = c_1(\mathcal{F}_\alpha) \cdot \eta = \alpha^{(1)} \cdot \eta \in \mathbb{Q}$, and so there exists an effective divisor class $\eta \in H^2$ such that $\psi_0(\eta) > 0$. We fix such η and set

$$r := \psi_0(\eta) = \alpha^{(1)} \cdot \eta > 0.$$

We make the following further assumptions:

(A3) For any α there is an isomorphism $\phi : M_\alpha \times B\mathbb{G}_m \simeq \mathcal{M}_\alpha$ for some smooth scheme M_α .

(A4) Tensoring by $\mathcal{O}(\eta)$ induces isomorphisms $\mathcal{M}_\alpha \simeq \mathcal{M}_{\alpha+r\delta}$.

It follows by [Hei10, Lemma 3.10] from assumption (A3) that there exists a sheaf F_α on $M_\alpha \times S$ such that $(\phi \times \text{Id}_S)^* \mathcal{F}_\alpha \cong F_\alpha(1)$, where (1) denotes tensoring with the weight one character of \mathbb{G}_m . Note that F_α depends on the choice of isomorphism ϕ and is therefore not canonical. We have an isomorphism

$$(6.1) \quad \phi^* : H^*(\mathcal{M}_\alpha) \rightarrow H^*(M_\alpha \times B\mathbb{G}_m) = H^*(M_\alpha)[v].$$

Using the sheaf F_α , we can define the tautological classes $f(\gamma) \in H^*(M_\alpha)$ for any symmetric function f and $\gamma \in H$ as in Section 4.3. We denote the subring generated by these classes by $H_{\text{taut}}^*(M_\alpha) \subset H^*(M_\alpha)$.

Lemma 6.1. *Under the assumption (A3), we have $H_{\text{taut}}^*(\mathcal{M}_\alpha) \simeq H_{\text{taut}}^*(M_\alpha)[v]$.*

Proof. From the definition of F_α , we have $\text{ch}(\mathcal{F}_\alpha) = \text{ch}(F_\alpha)e^v$; in particular, we have an inclusion $H_{\text{taut}}^*(\mathcal{M}_\alpha) \subset H_{\text{taut}}^*(M_\alpha)[v]$. For the opposite direction, note that

$$\psi_0([\text{pt}]) = \int_{\bar{S}} c_1(\mathcal{F}_\alpha) \cdot [\text{pt}] = v + \int_{\bar{S}} c_1(F_\alpha) \cdot [\text{pt}] = v.$$

Thus $v \in H_{\text{taut}}^*(\mathcal{M}_\alpha)$, and we conclude by the equality $\text{ch}(F_\alpha) = \text{ch}(\mathcal{F}_\alpha)e^{-v}$. \square

For the future use, we note the following expressions of some tautological classes of low degree, in terms of the identification (6.1).

$$\psi_0(\eta) = r, \quad \psi_1(\eta) = \frac{1}{2}p_2(\eta) + \int_{\overline{S}} c_1(F_\alpha) \cup (t_1 + t_2) \cup \eta, \quad \frac{1}{2}p_2(\eta) \in rv + H^2(M_\alpha).$$

6.1. Periodicity. Observe that by assumption (A4), tensoring with $\mathcal{O}(\eta)$ induces an isomorphism $H^*(\mathcal{M}_\alpha) \cong H^*(\mathcal{M}_{\alpha+r\delta})$ preserving the tautological rings. The following result provides a canonical identification between the cohomology spaces $H_{\text{taut}}^*(\mathcal{M}_\alpha)$ for all $\alpha \in \alpha_0 + \mathbb{N}\delta$.

Proposition 6.2. *The Hecke operator $T_0(\eta)$ induces an isomorphism of vector spaces $H_{\text{taut}}^*(\mathcal{M}_\alpha) \cong H_{\text{taut}}^*(\mathcal{M}_{\alpha+\delta})$.*

Proof. Let us show that $T_0(\eta)$ is surjective; this will imply the chain of inequalities

$$\dim H_{\text{taut}}^i(\mathcal{M}_\alpha) \geq \dim H_{\text{taut}}^i(\mathcal{M}_{\alpha+\delta}) \geq \dots \geq \dim H_{\text{taut}}^i(\mathcal{M}_{\alpha+r\delta}),$$

and we will deduce that $T_0(\eta)$ is an isomorphism from (A4).

Let f be a polynomial in the generators $\psi_i(\xi)$. By Theorem 5.8 we know that some power n of $\text{Ad}_{T_0(\eta)}$ annihilates f . We show by induction on n that $f(1)$ is in the image of $T_0(\eta)$. The induction base is not needed because the case $f = 0$ is trivial. Now suppose that we have proven the claim for $[T_0(\eta), f]$, so that we have

$$[T_0(\eta), f](1) = T_0(\eta)(g)$$

for some $g \in H_{\text{taut}}^*(\mathcal{M}_\alpha)$. Then we have

$$T_0(\eta)(f(1) - g) = fT_0(\eta)(1) = (\eta \cdot \alpha^{(1)})f(1) = rf(1),$$

hence $f(1)$ is in the image of $T_0(\eta)$. \square

Combining Proposition 6.2, Lemma 6.1 and (6.1) we obtain an isomorphism $\mathbf{V}_S^{\text{taut}} \simeq H_{\text{taut}}^*(M_\alpha)[v, s]$, valid for any $\alpha \in \alpha_0 + \mathbb{N}\delta$. Our next goal is to extract from the action of W_S on $\mathbf{V}_S^{\text{taut}}$ an action of a smaller algebra W_M on $H_{\text{taut}}^*(M_\alpha)$. We will achieve this in two steps. In Sections 6.2-6.4 we construct an algebra \widetilde{W}_M acting on $H_{\text{taut}}^*(\mathcal{M}_\alpha) \simeq H_{\text{taut}}^*(M_\alpha)[v]$. The construction of W_M is then given in Section 6.5, by reduction with respect to a suitable action of a Weyl algebra.

6.2. Polynomiality of $\psi_k(\xi)$.

Lemma 6.3. *Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a graded connected finite dimensional \mathbb{Q} -algebra, $B = A[v]$ with v of degree 2, and $\{b_i \in B_{2i}\}_{i \in \mathbb{N}}$ a collection of elements. Write $y = v + a$, where $a \in A$ is of degree 2. Then there exists a finite collection of elements $\{Q_i \in A_{2i}\}_{i \in \mathbb{N}}$ such that for all n*

$$b_n = \sum_{i=0}^n \binom{n}{i} Q_i y^{n-i},$$

and Q_i 's are polynomials in b_i 's and a .

Proof. Consider the sum $b = \sum_{i \in \mathbb{N}} b_i$, and expand it in the powers of v . Consider also the Laurent series $P_0(x) = 1/(1-x)$, and let $P_k = k^{-1} \frac{\partial}{\partial x} P_{k-1}$ for $k > 0$. Define $Q_k \in A^{2k}$ inductively to be the constant term of $v^{-(k-1)} \left(b - \sum_{i=0}^{k-1} Q_i P_i(y) \right)$. Since A is finite dimensional, $Q_i = 0$ for $i > N$ big enough, and so

$$\sum_n b_n = \sum_{i=0}^N Q_i P_i(y) = \sum_n \left(\sum_{i=0}^n \binom{n}{i} Q_i y^{n-i} \right). \quad \square$$

Let us set

$$y = \frac{\psi_1(\eta)}{r}.$$

Using the above Lemma with $A = H_{\text{taut}}^*(M_\alpha \times S)$ and applying $(\pi_{M_\alpha})_*(- \cup \xi)$ yields the following result.

Corollary 6.4. *There exists a family of elements $Q_i \in H_{\text{taut}}^{2i}(M_\alpha \times S)$, $i \geq 0$ such that for any $n \geq 0$, we have the following equality in $H_{\text{taut}}^*(\mathcal{M}_\alpha)$:*

$$\psi_n(\xi) = \sum_{i=0}^n \binom{n}{i} Q_i(\xi) y^{n-i}, \quad Q_i(\xi) := (\pi_{M_\alpha})_*(Q_i \cup \xi).$$

Moreover, $Q_n(\xi)$ is zero for n large enough.

6.3. Polynomiality of $D_{m,n}(\xi)$. Let us now set

$$X = \frac{T_0(\eta)}{r} = \frac{q_1(\eta)}{r}.$$

By Proposition 6.2, the operator X identifies all the spaces $H_{\text{taut}}^*(\mathcal{M}_\alpha)$ for $\alpha \in \alpha_0 + \mathbb{Z}\delta$. Let us furthermore define

$$D_{m,n}(\xi) := \frac{n!}{(m+n)!} (-\text{Ad}_{q_1(1)})^m \psi_{m+n}(\xi);$$

compare this with (5.6). These elements satisfy the properties listed in Theorem 5.8.

Proposition 6.5. *For any $\xi \in H$, $n \geq 0$ and any $\alpha \in \alpha_0 + \mathbb{Z}\delta$, the sequence of operators $X^{-m}D_{m,n}(\xi) \in \text{End}(H_{\text{taut}}^*(\mathcal{M}_\alpha))$ depends polynomially on m . In the case $n = 0$ the non-constant coefficients of the polynomial are nilpotent.*

Proof. Using Corollary 6.4 we may write

$$\psi_{m+n}(\xi) = \sum_{i=0}^N \binom{m+n}{i} y^{m+n-i} Q_i(\xi),$$

where $N = \dim(M_\alpha) + 2$. Let us denote $A := -\text{Ad}_{q_1(1)}$. Applying the Leibniz rule, we obtain

$$(6.2) \quad D_{m,n}(\xi) = \frac{n!}{(m+n)!} \sum_{i,j} \binom{m+n}{i} \binom{m}{j} (A^{m-j} y^{m+n-i}) (A^j Q_i(\xi)).$$

Observe that $Q_i(\xi)$, viewed as a tautological class and hence as an element of W_S , belongs to F_i . By Theorem 5.8, the summands of (6.2) vanish for $j > i$. The sum is therefore finite, and we only need to check that for each fixed triple (i, j, n) the sequence

$$X^{j-m} \frac{n!}{(m+n)!} \binom{m+n}{i} \binom{m}{j} A^{m-j} y^{m+n-i}$$

depends polynomially on m . Since X and $q_1(1)$ commute, and $\binom{m}{j}$ is manifestly polynomial in m , it is enough to show the polynomiality of

$$(6.3) \quad \frac{n!}{(m+n)!} \binom{m+n}{i} (X^{-1}A)^{m-j} y^{m+n-i}.$$

Using Corollary 5.9, we have $[y, q_1(1)] = r^{-1}[\psi_1(\eta), q_1(1)] = r^{-1}q_1(\eta) = X$ and similarly $[y, X] = r^{-2}q_1(\eta^2)$, which commutes with X and y . We deduce that for any $k \geq 0$

$$\begin{aligned} -X^{-1} \text{Ad}_{q_1(1)} y^k &= X^{-1} \sum_{i=0}^{k-1} y^i X y^{k-1-i} = k y^{k-1} + \binom{k}{2} X^{-1} [y, X] y^{k-2} \\ &= \left(\frac{\partial}{\partial y} + \frac{1}{2} X^{-1} [y, X] \frac{\partial^2}{\partial^2 y} \right) y^k. \end{aligned}$$

Write $z = \frac{1}{2} X^{-1} [y, X]$. The binomial formula gives

$$(-X^{-1} \text{Ad}_{q_1(1)})^{m-j} y^{m+n-i} = \sum_{k=0}^{\min(m-j, n-i+j)} \binom{m-j}{k} z^k \frac{(m+n-i)!}{(n-i+j-k)!} y^{n-i+j-k}.$$

The upper bound of the summation can be replaced by $n-i+j$. As a result we obtain that (6.3) is a linear combination of operators of the form $z^k y^{n-i+j-k}$, and each such operator

enters with a coefficient equal to

$$\frac{n!}{(m+n)!} \binom{m+n}{i} \binom{m-j}{k} \frac{(m+n-i)!}{(n-i+j-k)!} = \frac{n!}{i!(n-i+j-k)!} \binom{m-j}{k},$$

which is a polynomial in m .

Let us now consider the special case $n = 0$. The inequalities $j \leq i$ and $0 \leq k \leq n - i + j$ imply $k = 0$ and $i = j$. So the result is a linear combination of terms of the form

$$\frac{1}{i!} \binom{m}{i} ((-\text{Ad}_{q_1(1)})^i Q_i(\xi)).$$

Only the terms with $i \geq 1$ contribute to the non-constant coefficients of the polynomial. If $Q_i(\xi)$ has positive cohomological degree, then it is nilpotent, say $Q_i(\xi)^M = 0$. Applying $(-\text{Ad}_{q_1(1)})^{Mi}$ and using $(\text{Ad}_{q_1(1)})^{i+1} Q_i(\xi) = 0$ we obtain $((-\text{Ad}_{q_1(1)})^i Q_i(\xi))^M = 0$. If $Q_i(\xi)$ has cohomological degree zero (which only happens for $i = 1$, $\xi \in H^0$), then $(-\text{Ad}_{q_1(1)})^i Q_i(\xi)$ has negative cohomological degree as an operator and is therefore nilpotent. \square

6.4. Construction of $\tilde{D}_{m,n}(\xi)$ and relations. Proposition 6.5 gives rise to operators on each individual $H_{\text{taut}}^*(\mathcal{M}_\alpha)$, by taking the coefficients in the polynomial expansion of $X^{-m} D_{m,n}(\xi)$ in m . We will need a slight renormalization. Let θ be the linear term in the polynomial expansion of $X^{-k} q_k(\eta)$. By Proposition 6.5 θ is nilpotent, and thus $e^{k\theta}$, $e^{-k\theta}$ are polynomials in k . So Proposition 6.5 remains valid when X^{-k} is replaced by u^{-k} , where

$$u = X e^{\theta/q_0(\eta)} = X e^{\theta/r}.$$

This choice of u ensures that the linear term in the expansion of $u^{-k} q_k(\eta)$ vanishes. Let us now define operators $\tilde{D}_{i,n}(\xi)$ for all $n, m \in \mathbb{N}$ and $\xi \in H$ by

$$(6.4) \quad D_{m,n}(\xi) = u^m \sum_i \frac{m^i}{i!} \tilde{D}_{i,n}(\xi).$$

In particular, note that $\psi_n(\xi) = D_{0,n}(\xi) = \tilde{D}_{0,n}(\xi)$. As in Section 5.4, we set

$$\tilde{q}_i(\xi) = \tilde{D}_{i,0}(\xi), \quad \tilde{L}_i(\xi) = \tilde{D}_{i,1}(\xi)$$

for $i \geq 0$; observe that by construction $\tilde{q}_1(\eta) = 0$ and $\tilde{q}_i(\xi) = \tilde{L}_i(\xi) = 0$ for $i \gg 0$. Define $\tilde{W}_{\mathcal{M}}$ as the subalgebra of $\text{End}(H_{\text{taut}}^*(\mathcal{M}_\alpha))$ generated by all the operators $\tilde{D}_{m,n}(\xi)$. Finally, denote by \tilde{F}_n the filtration of \tilde{W} obtained by placing $\tilde{D}_{m,n}(\xi)$ in weight n .

Proposition 6.6. *The following relation holds modulo $\tilde{F}_{n+n'-3}$:*

$$\begin{aligned} [\tilde{D}_{m,n}(\xi), \tilde{D}_{m',n'}(\xi')] &= (nm' - mn')\tilde{D}_{m+m'-1, n+n'-1}(\xi\xi') \\ &\quad - r^{-1}nm'\tilde{D}_{m,n-1}(\xi\eta)\tilde{D}_{m'-1, n'}(\xi') \pm r^{-1}mn'\tilde{D}_{m', n'-1}(\xi'\eta)\tilde{D}_{m-1, n}(\xi) \\ &\quad - 2r^{-2}\binom{n}{2}\binom{m'}{2}\tilde{D}_{m, n-2}(\xi\eta^2)\tilde{D}_{m'-2, n'}(\xi') \pm 2r^{-2}\binom{m}{2}\binom{n'}{2}\tilde{D}_{m', n'-2}(\xi'\eta^2)\tilde{D}_{m-2, n}(\xi), \end{aligned}$$

where the sign is $-$ if both ξ and ξ' are odd. When $s_2 = c_1\Delta_S = 0$, the relations hold on the nose.

Proof. Let us first find commutation relations involving u . From $[\psi_1(\xi), q_k(\eta)] = kq_k(\xi\eta)$ and (6.4) we obtain

$$ku^{k-1}[\psi_1(\xi), u] \sum_i \tilde{q}_i(\eta) \frac{k^i}{i!} + u^k \sum_i [\psi_1(\xi), \tilde{q}_i(\eta)] \frac{k^i}{i!} = ku^k \sum_i \tilde{q}_i(\xi\eta) \frac{k^i}{i!}.$$

Dividing by u^k and taking the linear terms in k we obtain

$$q_0(\eta)u^{-1}[\psi_1(\xi), u] + [\psi_1(\xi), \tilde{q}_1(\eta)] = q_0(\xi\eta).$$

As $\tilde{q}_1(\eta) = 0$, this yields

$$(6.5) \quad [\psi_1(\xi), u] = u \frac{q_0(\xi\eta)}{r} = u \frac{\psi_0(\xi\eta)}{r}.$$

Recall the operator $\mathfrak{d} = \frac{D_{0,2(1)}}{2}$. Since $\text{Ad}_{q_i(\xi)}$ lowers the F -filtration degree by 1, we have $\text{Ad}_{q_i(\xi)}^3 \mathfrak{d} = 0$, and so in particular

$$[\mathfrak{d}, u^k] = ku^{k-1}[\mathfrak{d}, u] + \binom{k}{2}u^{k-2}[[\mathfrak{d}, u], u].$$

Expanding q in terms of \tilde{q} in $[\mathfrak{d}, q_k(\eta)] = kL_k(\eta)$ and applying the equation above, we obtain

$$\left(ku^{k-1}[\mathfrak{d}, u] + \binom{k}{2}u^{k-2}[[\mathfrak{d}, u], u] \right) \sum_i \tilde{q}_i(\eta) \frac{k^i}{i!} + u^k \sum_i [\mathfrak{d}, \tilde{q}_i(\eta)] \frac{k^i}{i!} = ku^k \sum_i \tilde{L}_i(\eta) \frac{k^i}{i!}.$$

Dividing by u^k , using the relations $\tilde{q}_0(\eta) = r$, $\tilde{L}_0(\eta) = L_0(\eta)$ and collecting the coefficient of k^1 produces

$$u^{-1}[\mathfrak{d}, u] - \frac{1}{2}u^{-2}[[\mathfrak{d}, u], u] = \frac{L_0(\eta)}{r} = \frac{\psi_1(\eta)}{r}.$$

Applying $-\text{Ad}_u$ and using (6.5) we thus obtain

$$u^{-2}[[\mathfrak{d}, u], u] = \frac{\psi_0(\eta^2)}{r^2} \quad \Rightarrow \quad u^{-1}[\mathfrak{d}, u] = \frac{\psi_1(\eta)}{r} + \frac{1}{2} \frac{\psi_0(\eta^2)}{r^2}.$$

Using $[\mathfrak{d}, D_{m,n}(\xi)] = mD_{m,n+1}(\xi) \pmod{F_{n-1}}$ we obtain by induction on n

$$[D_{m,n}(\xi), u] = u \left(n \frac{D_{m,n-1}(\xi\eta)}{r} + \binom{n}{2} \frac{D_{m,n-2}(\xi\eta^2)}{r^2} \right) \pmod{F_{n-3}},$$

and then by induction on k

$$[D_{m,n}(\xi), u^k] = u^k \left(kn \frac{D_{m,n-1}(\xi\eta)}{r} + k^2 \binom{n}{2} \frac{D_{m,n-2}(\xi\eta^2)}{r^2} \right) \pmod{F_{n-3}}.$$

Since this holds for all $m \geq 1$, by linearity the same identity is true with \tilde{D} in place of D .

Finally we expand the identity (5.8) modulo $F_{n+n'-3}$:

$$(m'n - mn')u^{m+m'} \sum_k \tilde{D}_{k,n+n'-1}(\xi) \frac{(m+m')^k}{k!} = \sum_{i,j} \frac{m^i}{i!} \frac{m'^j}{j!} \left[u^m \tilde{D}_{i,n}(\xi), u^{m'} \tilde{D}_{j,n'}(\xi') \right].$$

The commutator expands as follows:

$$\begin{aligned} u^{m+m'} \left([\tilde{D}_{i,n}(\xi), \tilde{D}_{j,n'}(\xi')] + \left(\frac{m'n}{r} \tilde{D}_{i,n-1}(\xi\eta) + \frac{m'^2}{r^2} \binom{n}{2} \tilde{D}_{i,n-2}(\xi\eta^2) \right) \tilde{D}_{j,n'}(\xi') \right. \\ \left. \mp \left(\frac{mn'}{r} \tilde{D}_{j,n'-1}(\xi'\eta) + \frac{m^2}{r^2} \binom{n'}{2} \tilde{D}_{j,n'-2}(\xi'\eta^2) \right) \tilde{D}_{i,n}(\xi) \right), \end{aligned}$$

where the sign is $+$ if both ξ and ξ' are odd. Collecting monomials in m, m' yields the first claim. Finally, if $s_2 = c_1 \Delta_S = 0$, the identity (5.8) holds on the nose by Theorem 5.6, which proves the second claim. \square

Corollary 6.7. *The algebra $\widetilde{W}_{\mathcal{M}}$ is generated by $\{\psi_2(1), \tilde{q}_k(\xi) \mid k \geq 0, \xi \in H\}$.* \square

6.5. Reduction with respect to a Weyl subalgebra. We now come to the second reduction step. Observe that we have

$$[\psi_1(\eta), \tilde{q}_1(1)] = [\tilde{D}_{0,1}(\eta), \tilde{D}_{1,0}(1)] = D_{0,0}(\eta) - r^{-1} D_{0,0}(\eta^2) D_{0,0}(1) = r$$

(recall that $D_{0,0}(1) = \psi_0(1) = 0$ for degree reasons). Thus the pair of elements

$$y = \frac{\psi_1(\eta)}{r}, \quad \partial_y := -\tilde{q}_1(1)$$

generate a Weyl subalgebra of $\widetilde{W}_{\mathcal{M}}$, i.e. we have $[\partial_y, y] = 1$.

Lemma 6.8. *The operator ∂_y acts locally nilpotently on $H_{\text{taut}}^*(\mathcal{M}_\alpha)$. The operators Ad_y and Ad_{∂_y} act locally nilpotently on $\widetilde{W}_{\mathcal{M}}$.*

Proof. The first statement is a consequence of the fact that ∂_y is of cohomological degree -2 . Next, because $\partial_y \in \widetilde{F}_0$, the operator Ad_{∂_y} acts locally nilpotently on \widetilde{W}_S . Finally,

from the relations in Proposition 6.6 one sees that Ad_y acts nilpotently on $\widetilde{F}_m/\widetilde{F}_{m-2}$, and hence locally nilpotently on \widetilde{W}_S . \square

Consider the following general setup. Let A be an associative algebra defined over a field \mathbb{K} of characteristic 0, acting on a \mathbb{K} -vector space V . Assume that the Weyl algebra $B = \mathbb{K}y \oplus \mathbb{K} \oplus \mathbb{K}\partial_y$ is a Lie subalgebra of A , that Ad_y and Ad_{∂_y} both act locally nilpotently on A and ∂_y acts locally nilpotently on V .

Lemma 6.9. *Let A, B, V be as above. Then*

- (i) *Multiplication induces an isomorphism of vector spaces $m : \mathbb{K}[y] \otimes A_{\text{red}} \otimes \mathbb{K}[\partial_y] \xrightarrow{\sim} A$, where A_{red} is the subalgebra of A of elements commuting with both y and ∂_y ,*
- (ii) *The action map defines an isomorphism of vector spaces $V_{\text{red}}[y] \xrightarrow{\sim} V$, where $V_{\text{red}} = \text{Ker}(\partial_y)$,*
- (iii) *The action of A on V restricts to an action of A_{red} on V_{red} .*
- (iv) *The natural projections $A \rightarrow A_{\text{red}}$, $f = \sum_{i,j} f_{i,j} y^i \partial_y^j \mapsto f_{\text{red}} := f_{0,0}$ and $V \rightarrow V_{\text{red}}$, $v = \sum_i y^i v_i \mapsto v_{\text{red}} := v_0$ are given by the following formulas*

$$(6.6) \quad f_{\text{red}} = \sum_{i,j} \frac{1}{i!j!} y^i (\text{Ad}_y^j (-\text{Ad}_{\partial_y})^i f) \partial_y^j, \quad v_{\text{red}} = \sum_i \frac{y^i (-\partial_y)^i}{i!} v.$$

Proof. We will first show that the map $m : \mathbb{K}[y] \otimes A_{\text{red}} \otimes \mathbb{K}[\partial_y] \rightarrow A$ is both injective and surjective. To begin, a standard argument via applying Ad_y shows that the multiplication map $m_1 : A_{\text{red}} \otimes \mathbb{K}[\partial_y] \rightarrow \text{Ker}(\text{Ad}_{\partial_y})$ is injective. To see that it is also surjective we argue by induction on n to prove that $\text{Ker}(\text{Ad}_{\partial_y}) \cap \text{Ker}(\text{Ad}_y^n) \subset \text{Im}(m_1)$, using the observation that for $f \in \text{Ker}(\text{Ad}_y^n) \setminus \text{Ker}(\text{Ad}_y^{n-1})$ and $u = \frac{1}{(n-1)!} \text{Ad}_y^{n-1} f$ we have $\text{Ad}_y^{n-1}(u\partial_y^{n-1} - f) = 0$. We conclude using the fact that Ad_y is locally nilpotent. The same argument with y and ∂_y swapped implies that m is an isomorphism. Statement (ii) is classical, and shown in the same fashion; statement (iii) immediately follows. For $v = y^a u$ with $u \in V_{\text{red}}$, we compute

$$\sum_i \frac{y^i (-\partial_y)^i}{i!} v = \sum_{i=0}^a (-1)^i \frac{a!}{(a-i)!i!} y^a u = \sum_{i=0}^a (-1)^i \binom{a}{i} y^a u = \delta_{a,0} u,$$

proving the second formula of (iv). The first equality of (iv) is shown by a double application of the second one. \square

In our situation, Lemma 6.9 yield canonical decompositions

$$(6.7) \quad H_{\text{taut}}^*(\mathcal{M}_\alpha) = H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}}[y], \quad \widetilde{W}_M = W_M[y][\partial_y]$$

where $H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}} = \text{Ker } \partial_y$, and W_M consists of operators commuting with y and ∂_y . Moreover, W_M preserves $H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}}$. Observe that because $y - v \in H^2(M_\alpha)$ is nilpotent, there are identifications

$$H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}} \simeq H_{\text{taut}}^*(\mathcal{M}_\alpha)/yH_{\text{taut}}^*(\mathcal{M}_\alpha) \simeq H_{\text{taut}}^*(\mathcal{M}_\alpha)/vH_{\text{taut}}^*(\mathcal{M}_\alpha) \simeq H_{\text{taut}}^*(M_\alpha).$$

We have thus obtained an action of W_M on $H_{\text{taut}}^*(M_\alpha)$.

Remark 6.10. There are two ways of viewing $H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}} \simeq H_{\text{taut}}^*(M_\alpha)$: as the subspace $H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}} = \text{Ker } \partial_y$ or as the quotient space $H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}} = \text{Coker } y$. Since $\psi_m(\xi)$ commutes with y , we have

$$\psi_m(\xi)_{\text{red}} = \sum_i \frac{y^i (-\text{Ad}_{\partial_y})^i}{i!} \psi_m(\xi),$$

and the action of $\psi_m(\xi)_{\text{red}}$ is compatible with that of $\psi_m(\xi)$ via the quotient identification. Because of this, we will omit subscript red for the tautological classes in $H^*(M_\alpha)$. On the other hand, $\tilde{q}_m(\xi)$ commutes with ∂_y , and so we have

$$\tilde{q}_m(\xi)_{\text{red}} = \sum_i (\text{Ad}_y^i \tilde{q}_m(\xi)) \frac{\partial_y^i}{i!}.$$

Since ∂_y annihilates $H_{\text{taut}}^*(\mathcal{M}_\alpha)_{\text{red}}$, the action coincides with the induced action of $\tilde{q}_m(\xi)$ with respect to the subspace identification. In general, for any operator f the induced action from $\text{Ker } \partial_y$ to $\text{Coker } y$ coincides with f_{red} .

6.6. \mathfrak{sl}_2 -triple. Let us extract an \mathfrak{sl}_2 -triple in $\text{End}(H_{\text{taut}}^*(M_\alpha))$, which will play a key role later, out of the algebra W_M . Set \mathfrak{h} to be the operator $-\tilde{D}_{1,1}(1)_{\text{red}}$. Explicitly, by (6.6) we have

$$\mathfrak{h} = -\tilde{D}_{1,1}(1) - (y - r^{-2}\psi_0(\eta^2)\psi_1(1))\partial_y.$$

Taking into account that $\tilde{D}_{1,0}(\eta) = \tilde{q}_1(\eta) = 0$, we obtain by Proposition 6.6 that

$$[\tilde{D}_{1,1}(1), \tilde{D}_{m,n}(\xi)] = (m - n)\tilde{D}_{m,n}(\xi) + r^{-1}n\tilde{D}_{m,n-1}(\eta\xi)\psi_1(1) \pmod{\tilde{F}_{n-2}}.$$

In particular, using commutation relations between $\psi_1(1)$ and $\tilde{q}_k(\xi)$ we get

$$[\mathfrak{h}, -\tilde{q}_m(\xi)] = -m\tilde{q}_m(\xi) - [y, \tilde{q}_m(\xi)]\partial_y,$$

and taking the constant terms

$$(6.8) \quad [\mathfrak{h}, \tilde{q}_m(\xi)_{\text{red}}] = -m\tilde{q}_m(\xi)_{\text{red}}.$$

As a consequence, the following holds.

Proposition 6.11. *The operator $\text{Ad}_{\mathfrak{h}}$ restricted to $\tilde{F}_0 \cap W_{\text{red}}$ is diagonalizable and its eigenvalues are non-positive integers. \square*

Consider

$$\mathfrak{d}_{\text{red}} = \frac{\tilde{D}_{0,2}(1)_{\text{red}}}{2} = \frac{\psi_2(1)}{2} - y\psi_1(1).$$

As $\psi_1(1) \in H^0(\mathcal{M}_\alpha)$ is a scalar, it commutes with all operators. Since $\mathfrak{d}_{\text{red}}$ commutes with both y and ∂_y we have

$$[\mathfrak{h}, \mathfrak{d}_{\text{red}}] = - \left[\tilde{D}_{1,1}(1), \frac{\psi_2(1)}{2} - y\psi_1(1) \right] = \psi_2(1) - 2y\psi_1(1) - r^{-2}\psi_0(\eta^2)\psi_1(1)^2 = 2\mathfrak{d}_{\text{red}} \pmod{\tilde{F}_0}.$$

Write

$$[\mathfrak{h}, \mathfrak{d}_{\text{red}}] = 2\mathfrak{d}_{\text{red}} + \sum_{i \geq 0} f_i,$$

where each $f_i \in \tilde{F}_0 \cap W_M$ is an eigenvector of $\text{Ad}_{\mathfrak{h}}$ of eigenvalue $(-i)$, and set

$$(6.9) \quad \mathfrak{e} = \mathfrak{d}_{\text{red}} + \sum_i \frac{f_i}{i+2}.$$

Then we have

$$[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{d}_{\text{red}} + \sum_{i \geq 0} \left(1 - \frac{i}{i+2}\right) f_i = 2\mathfrak{e}.$$

Finally, set $\mathfrak{f} = -\frac{\tilde{q}_2(1)_{\text{red}}}{2}$. From Proposition 6.6, we have

$$[\mathfrak{e}, \tilde{q}_2(1)] = [\mathfrak{d}_{\text{red}}, \tilde{q}_2(1)] = 2\tilde{D}_{1,1}(1) - 2r^{-1}\tilde{D}_{0,1}(\eta)\tilde{D}_{1,0}(1) + 2r^{-2}\psi_0(\eta)^2\tilde{D}_{1,0}(1)\psi_1(1).$$

Taking the constant terms we obtain $[\mathfrak{e}, \tilde{q}_2(1)_{\text{red}}] = -2\mathfrak{h}$, and hence we have

$$[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h}.$$

We summarize the above discussion:

Theorem 6.12. (i) *The operators $(\mathfrak{e}, \mathfrak{h}, \mathfrak{f})$ form a \mathfrak{sl}_2 -triple in $\text{End}(H_{\text{taut}}^*(M_\alpha))$, for any $\alpha \in \alpha_0 + \mathbb{Z}\delta$.*

(ii) *Define a filtration P_\bullet of $H_{\text{taut}}^*(M_\alpha)$ by setting P_i to be the sum of the \mathfrak{h} -eigenspaces with eigenvalues $\leq i$. Then:*

(a) *the tautological classes satisfy $\psi_k(\xi)P_i \subset P_{i+k}$,*

(b) *the Hecke operators satisfy $\tilde{q}_k(\xi)_{\text{red}}P_i \subset P_{i-k}$,*

(c) *on the associated graded, the operator $\psi_2(1)$ coincides with $2\mathfrak{e}$ and therefore satisfies the Hard Lefschetz property, that is $\psi_2(1)^k : P_{-k}/P_{-k-1} \rightarrow P_k/P_{k-1}$ is an isomorphism.*

(iii) For any sequence of open subspaces $\mathcal{M}'_{\alpha_0+\mathbb{Z}\delta} \subset \mathcal{M}_{\alpha_0+\mathbb{Z}\delta}$ preserved by the Hecke correspondences the restriction maps commute with the \mathfrak{sl}_2 -action and are therefore strictly compatible with the filtrations P'_\bullet, P_\bullet .

Proof. All statements save for (ii.a-b) follow from our construction. The statement (ii.b) follows from the commutation relation (6.8). Finally, for (ii.a) we use Proposition 6.6. Since the relation $[D_{1,1}(1), \psi_k(\xi)]$ holds on the nose in W_S , the corresponding relation in $\widetilde{W}_{\mathcal{M}}$ holds on the nose as well:

$$[\widetilde{D}_{1,1}(1), \psi_k(\xi)] = -k\psi_k(\xi) + r^{-1}k\psi_{k-1}(\xi\eta)\psi_1(1).$$

Let f be an eigenvector of \mathfrak{h} with eigenvalue i . We have

$$\begin{aligned} \mathfrak{h}(\psi_k(\xi)f) &= \psi_k(\xi)\mathfrak{h}f + [\mathfrak{h}, \psi_k(\xi)]f = i(\psi_k(\xi)f) - [\widetilde{D}_{1,1}(1), \psi_k(\xi)]f \\ &= (i+k)\psi_k(\xi)f - (r^{-1}k\psi_1(1))\psi_{k-1}(\xi\eta)f. \end{aligned}$$

Reasoning by induction on k , the second term belongs to P_{i+k-1} , and so we conclude that $\psi_k(\xi)f \in P_{i+k}$. \square

For any \mathfrak{sl}_2 -triple acting on $H^*(M_\alpha)$ we will refer to the filtration P_\bullet defined in (ii) as the \mathfrak{h} -degree filtration.

7. HIGGS BUNDLES

From now on and until the end of the paper, we restrict to the following situation. Let C be a smooth projective curve of genus $g \geq 0$, and let D be a (possibly zero) effective simple divisor. We denote by $S = \mathbb{P}(\Omega(D) \oplus \mathcal{O})$ the projective completion of the line bundle $\Omega(D)$ over C , and let $\pi_C : S \rightarrow C$ be the projection. We also let $S^\circ \subset S$ be the total space of $\Omega(D)$. Recall that a D -twisted Higgs bundle on C consists of a pair (\mathcal{E}, θ) where \mathcal{E} is a vector bundle on C and $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega(D)$. By the BNR correspondence one may equivalently view (\mathcal{E}, θ) as a purely one-dimensional coherent sheaf \mathcal{F} on S whose support is disjoint from the boundary divisor $\partial S = S \setminus S^\circ$; the equivalence is given by $\mathcal{E} = \pi_*(\mathcal{F})$. The support of \mathcal{F} is called the *spectral curve* of (\mathcal{E}, θ) .

Let us fix a basis

$$\Pi = \{1, \gamma_1, \dots, \gamma_{2g}, \omega\},$$

of the cohomology ring $H = H^*(C)$ in which the product is given by

$$1 \cdot x = x \cdot 1 = x \quad (x \in \Pi), \quad \gamma_i \gamma_{i+g} = -\gamma_{i+g} \gamma_i = \omega \quad (1 \leq i \leq g),$$

and all other products of generators vanish. Note that we have canonical isomorphisms $H_{\text{pure}}^*(S^\circ) \simeq H^*(S^\circ) \simeq H^*(C) = H$.

Remark 7.1. We have $H^*(S) = H^*(C)[\xi]/\langle \xi^2 + (|D| + 2g - 2)\omega\xi \rangle$, where $\xi = c_1(\mathcal{O}(\partial S))$. If \mathcal{F} corresponds to a Higgs bundle (\mathcal{E}, θ) of rank r and degree d , then

$$c_1(\mathcal{F}) = r(\xi + (|D| + 2g - 2)\omega), \quad c_2(\mathcal{F}) = (r(r + 1)(1 - g) - d)\omega\xi.$$

It will be more convenient to work with (r, d) , so we will not use these formulas.

In this section, we specialize the results of Section 6 to twisted Higgs bundles and their parabolic versions.

7.1. Algebras in the twisted case. We apply the construction of Section 4 to the following moduli space. Fix $r > 0$. For each $d \in \mathbb{Z}$, let $\mathcal{M}_{r,d,D}^{\text{ell}}$ be the moduli stack of D -twisted Higgs bundles of rank r and degree d whose spectral curve is reduced and irreducible. We call it the *elliptic* moduli stack.

It is well-known that $\mathcal{M}_{r,d,D}^{\text{ell}}$ is of finite type and is a \mathbb{G}_m -gerbe over its coarse moduli space $M_{r,d,D}^{\text{ell}}$, which is smooth; more precisely, it is the relative Jacobian over the family of spectral curves. Note that Hecke modifications and tensoring by line bundles change the degree d but not the support of purely one-dimensional sheaves, hence assumptions (A1) and (A2) of Section 4.1 as well as (A3), (A4) of Section 6 are satisfied. By Theorems 5.2 and 5.6 we have

Corollary 7.2. *For fixed r and D , there is an action of the (super) Lie algebra with basis $D_{m,n}(\pi)$, $m, n \geq 0$, $\pi \in \Pi$ and Lie bracket*

$$[D_{m,n}(\pi), D_{m',n'}(\pi')] = (m'n - mn')D_{m+m',n+n'-1}(\pi\pi')$$

on the direct sum

$$\bigoplus_{d \in \mathbb{Z}} H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}}).$$

For $\pi \in H^i$, $D_{0,n}(\pi)$ is the operator of cup product by the tautological class $\psi_n(\pi) \in H^{2n-2+i}(\mathcal{M}_{r,d,D}^{\text{ell}})$ and

$$D_{1,n}(\pi) = T_n(\pi) : H^j(\mathcal{M}_{r,d-1,D}^{\text{ell}}) \rightarrow H^{j+2n-2+i}(\mathcal{M}_{r,d,D}^{\text{ell}})$$

is the Hecke operator.

Next, we make explicit the results of Section 6. Set $\eta = \omega$ so that $\alpha^{(1)} \cdot \eta = r$, i.e. the number r in Section 6 is precisely the rank of Higgs bundles under consideration. The

relations of the algebra $\widetilde{W}_{\mathcal{M}}$, which are given in Proposition 6.6, hold not only modulo $F_{n+n'-3}$, but on the nose by Remark 5.7.

Corollary 7.3. *For fixed r, d and D , there is an action of the algebra generated by $\widetilde{D}_{m,n}(\pi)$, $m, n \geq 0$, $\pi \in \Pi$ modulo relations*

$$\begin{aligned} [\widetilde{D}_{m,n}(\pi), \widetilde{D}_{m',n'}(\pi')] &= (nm' - mn')\widetilde{D}_{m+m'-1, n+n'-1}(\pi\pi') \\ &\quad - r^{-1}nm'\widetilde{D}_{m,n-1}(\pi\omega)\widetilde{D}_{m'-1, n'}(\pi') + r^{-1}mn'\widetilde{D}_{m', n'-1}(\pi'\omega)\widetilde{D}_{m-1, n}(\pi) \end{aligned}$$

on $H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})$, where for any $\pi \in H^i$, $\widetilde{D}_{0,n}(\pi) = D_{0,n}(\pi)$ is the operator of cup product $\psi_n(\pi)$. \square

We next move on to the reduced version W_M of $\widetilde{W}_{\mathcal{M}}$. Recall that the elements $y = \frac{\psi_1(\omega)}{r}$, $\partial_y = -\widetilde{D}_{1,0}(1)$ generate a Weyl algebra and induce a canonical decomposition

$$H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}}) = H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})_{\text{red}}[y],$$

where $H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})_{\text{red}}$ is the subring annihilated by ∂_y . From the relation

$$[\partial_y, \psi_n(\pi)] = [\widetilde{D}_{0,n}(\pi), \widetilde{D}_{1,0}(1)] = n\widetilde{D}_{0,n-1}(\pi) = n\psi_{n-1}(\pi)$$

it follows that coefficients of the generating series

$$\sum_{n=0}^{\infty} \frac{\psi_n(\pi)_{\text{red}}}{n!} = e^{-y} \sum_{n=0}^{\infty} \frac{\psi_n(\pi)}{n!}$$

generate $H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})_{\text{red}}$. Recall the identification

$$H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})_{\text{red}} = H_{\text{taut}}^*(M_{r,d,D}^{\text{ell}})$$

(see (6.7)). The subspace $H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})_{\text{red}}$ is preserved by the operators $\widetilde{D}_{m,n}(\pi)_{\text{red}}$ which commute with both ∂_y and y ; see Section 6.5.

Proposition 7.4. *For fixed r, d and D , the algebra generated by $\widetilde{D}_{m,n}(\pi)_{\text{red}}$, $m, n \geq 0$, $\pi \in \Pi$ modulo relations*

$$\begin{aligned} (7.1) \quad & [\widetilde{D}_{m,n}(\pi)_{\text{red}}, \widetilde{D}_{m',n'}(\pi')_{\text{red}}] = (nm' - mn')\widetilde{D}_{m+m'-1, n+n'-1}(\pi\pi')_{\text{red}} \\ & - r^{-1}nm' \left(\widetilde{D}_{m,n-1}(\pi\omega)_{\text{red}}\widetilde{D}_{m'-1, n'}(\pi')_{\text{red}} + \widetilde{D}_{m'-1, n'}(\pi'\omega)_{\text{red}}\widetilde{D}_{m,n-1}(\pi)_{\text{red}} \right) \\ & + r^{-1}mn' \left(\widetilde{D}_{m-1, n}(\pi\omega)_{\text{red}}\widetilde{D}_{m', n'-1}(\pi')_{\text{red}} + \widetilde{D}_{m', n'-1}(\pi'\omega)_{\text{red}}\widetilde{D}_{m-1, n}(\pi)_{\text{red}} \right). \end{aligned}$$

acts on $H_{\text{taut}}^*(M_{r,d,D}^{\text{ell}})$, where $\widetilde{D}_{0,n}(\pi) = \psi_n(\pi)_{\text{red}}$ for $\pi \in H^i$ is the operator of cup product by $\psi_n(\pi)_{\text{red}} \in H^{2n-2+i}(M_{r,d,D}^{\text{ell}})$.

Proof. Let f be an operator. We have the unique decomposition

$$f = \sum_{i,j} y^i f_{i,j} \partial_y^j,$$

where $f_{i,j}$ commutes with both y and ∂_y . Recall that $f_{\text{red}} = f_{0,0}$. Applying Ad_y , Ad_{∂_y} to the above identity we obtain

$$f_{i,j} = \left(\frac{1}{i!j!} \text{Ad}_{\partial_y}^i (-\text{Ad}_y)^j f \right)_{\text{red}}.$$

For any two operators f, g we write

$$fg = \sum_{i,j} y^i f_{i,j} \partial_y^j \sum_{i',j'} y^{i'} g_{i',j'} \partial_y^{j'}.$$

The terms with $i > 0$, $j' > 0$ or $i' \neq j'$ do not contribute to the constant term. The remaining terms with $i = j' = 0$ and $i' = j$ contribute $j! f_{0,j} g_{j,0}$. Thus we have

$$(fg)_{\text{red}} = \sum_i i! f_{0,i} g_{i,0} = \sum_i \frac{1}{i!} ((-\text{Ad}_y)^i f)_{\text{red}} (\text{Ad}_{\partial_y}^i g)_{\text{red}}.$$

Applying this identity to each term in Corollary 7.3 and using

$$[\partial_y, \tilde{D}_{m,n}(\pi)] = n \tilde{D}_{m,n-1}(\pi), \quad [y, \tilde{D}_{m,n}(\pi)] = \frac{m}{r} \tilde{D}_{m-1,n}(\pi\omega)$$

we obtain the required identities. □

We finish this section by relating the algebra W_M with cumbersome relations of Proposition 7.4 to the simpler algebra \mathcal{H}_2 of Hamiltonian vector fields on the plane. Although not directly used in the rest of the paper, it was one of our main motivations to study Hecke operators and their relation with tautological classes. The symmetry between m and n is evident in the relations (7.1) of W_M . Let us introduce a formal variable x and define the “unreduced” operators $\tilde{D}_{m,n}(\pi)_{\text{unred}}$ by

$$\tilde{D}_{m,n}(\pi)_{\text{unred}} = \sum_{i,j} x^i \binom{m}{i} \binom{n}{j} (-r)^{-j} \tilde{D}_{m-i,n-j}(\pi\omega^j) \partial_x^j.$$

This produces an action of the Lie algebra with bracket

$$[\tilde{D}_{m,n}(\pi)_{\text{unred}}, \tilde{D}_{m',n'}(\pi')_{\text{unred}}] = (m'n - mn') \tilde{D}_{m+m'-1,n+n'-1}(\pi\pi')_{\text{unred}}$$

on the space $H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})[x]$. By direct computation, the subalgebra spanned by elements $D_{m,n}(1)$ is isomorphic to the Lie algebra \mathcal{H}_2 of polynomial Hamiltonian vector fields on the plane.

Corollary 7.5. *The Lie algebra \mathcal{H}_2 acts on*

$$H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{ell}})[x] = H_{\text{taut}}^*(M_{r,d,D}^{\text{ell}})[x, y]$$

in such a way that $\psi_n(1)$ corresponds to the operator with Hamiltonian y^n . \square

7.2. Algebras in the parabolic case. Our argument to prove the $P = W$ conjecture treats simultaneously the classical and parabolic cases. Let us explain how to extend the constructions of Sections 4-6 to the parabolic setup. Let $\mathcal{M}_{r,d,D}^{\text{parell}}$ resp. $M_{r,d,D}^{\text{parell}}$ be the moduli stack resp. coarse moduli space parameterizing D -twisted Higgs bundles of rank r and degree d such that the spectral curve is reduced and irreducible, the residue of the Higgs field over each point of D has distinct eigenvalues, together with an ordering of the eigenvalues at each point. Equivalently, one can view this as a purely 1-dimensional sheaf \mathcal{F} on S° whose support $\text{supp}(\mathcal{F})$ is proper, reduced, irreducible, and intersects the fibers $\pi^{-1}(p)$ transversally for every $p \in D$ at r points, together with an ordering of $\pi^{-1}(p) \cap \text{supp}(\mathcal{F})$ for every $p \in D$. We call it the *parabolic elliptic stack*. There is an obvious forgetful map

$$\text{For} : \mathcal{M}_{r,d,D}^{\text{parell}} \rightarrow \mathcal{M}_{r,d,D}^{\text{ell}},$$

which is an unramified covering of degree $(r!)^{|D|}$ over its image. As Hecke operators do not change the support, the results of the previous section apply to these moduli spaces by base change; in particular we obtain actions of the respective algebras on $H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{parell}})$, $H_{\text{taut}}^*(M_{r,d,D}^{\text{parell}})$, which are the subrings generated by the tautological classes of the form $\psi_k(\pi)$, $\psi_k(\pi)_{\text{red}}$ respectively, for $\psi \in \Pi$. However, due to the presence of marked points we can introduce additional tautological classes and Hecke operators.

The tautological sheaf $\mathcal{F}_{r,d,D}^{\text{parell}}$ is isomorphic to the pullback $\text{For}^*(\mathcal{F}_{r,d,D})$ of the tautological sheaf on $\mathcal{M}_{r,d,D}$, so we will omit the superscript for simplicity. For each $p \in D$, $i = 1, \dots, r$, denote by $q_{p,i} : \mathcal{M}_{r,d,D}^{\text{parell}} \rightarrow S^\circ \subset S$ the map sending a parabolic Higgs bundle to the i -th point of $\text{supp}(\mathcal{F}) \cap \pi^{-1}(p)$. We have a tautological sheaf

$$\mathcal{U}_{p,i}^d := (\text{Id} \times q_{p,i})^*(\text{Id} \times q_{p,i})^* \mathcal{F}_{r,d,D}^{\text{parell}},$$

whose restriction to $\{\mathcal{F}\} \times S$, $\mathcal{F} \in \mathcal{M}_{r,d,D}^{\text{parell}}$ is the skyscraper sheaf $\delta_{q_{p,i}}$. Note that $(\text{Id} \times q_{p,i})^* \mathcal{F}_{r,d,D}^{\text{parell}}$ is a line bundle. In particular, we have

$$\text{Hom}(\mathcal{F}_{r,d,D}^{\text{parell}}, \mathcal{U}_{p,i}^d) = \text{Hom}((\text{Id} \times q_{p,i})^* \mathcal{F}_{r,d,D}^{\text{parell}}, (\text{Id} \times q_{p,i})^* \mathcal{F}_{r,d,D}^{\text{parell}}) = \mathbb{C}.$$

As a consequence, there is a canonical subsheaf $\mathcal{G} \subset \mathcal{F}_{r,d,D}^{\text{parell}}$ with $\mathcal{F}_{r,d,D}^{\text{parell}}/\mathcal{G} \simeq \mathcal{U}_{p,i}^d$. This gives rise to an isomorphism of stacks $\text{Mod}_{p,i} : \mathcal{M}_{r,d,D}^{\text{parell}} \rightarrow \mathcal{M}_{r,d-1,D}^{\text{parell}}$ and a canonical short

exact sequence

$$(7.2) \quad 0 \rightarrow \text{Mod}_{p,i}^*(\mathcal{F}_{r,d-1,D}) \rightarrow \mathcal{F}_{r,d,D} \rightarrow \text{Hom}_S(\mathcal{F}_{r,d,D}, \mathcal{U}_{p,i}^d)^\vee \otimes \mathcal{U}_{p,i}^d \rightarrow 0.$$

Observe that $\text{Mod}_{p,i}^*(\mathcal{U}_{p,i}^{d-1}) \simeq \mathcal{U}_{p,i}^d$. We set

$$X_{p,i} := \text{Mod}_{p,i}^* : H^*(\mathcal{M}_{r,d-1,D}^{\text{parell}}) \xrightarrow{\simeq} H^*(\mathcal{M}_{r,d,D}^{\text{parell}}).$$

Further, set $L_{p,i}^d := \text{Hom}_S(\mathcal{F}_{r,d,D}, \mathcal{U}_{p,i}^d)^\vee$, which is a line bundle on $\mathcal{M}_{r,d,D}^{\text{parell}}$. In terms of Higgs bundles, the fiber of $L_{p,i}^d$ at a pair (\mathcal{E}, θ) is the i -th eigenspace of the residue of the Higgs field θ at p . We set

$$y_{p,i} = c_1(L_{p,i}^d) \in H^2(\mathcal{M}_{r,d,D}^{\text{parell}})$$

and denote by the same symbol the operator of cup product with $c_1(L_{p,i}^d)$.

Proposition 7.6. *For each $p \in D$, $1 \leq i, i' \leq r$, $n \geq 0$, $\pi \in H^{>0}$ we have*

$$[X_{p,i}, y_{p',i'}] = 0, \quad [\psi_n(1), X_{p,i}] = ny_{p,i}^{n-1} X_{p,i}, \quad [\psi_n(\pi), X_{p,i}] = 0, \quad [y_{p,i}, T_0(1)] = X_{p,i}.$$

Proof. The first three relations follow easily from (7.2). The last relation is less straightforward. Recall that $T_0(1)$ is given by the Hecke correspondence \mathcal{Z} which parametrizes colength one inclusions $\mathcal{F}' \subset \mathcal{F}$ with $\mathcal{F} \in \mathcal{M}_{r,d,D}^{\text{parell}}$, $\mathcal{F}' \in \mathcal{M}_{r,d-1,D}^{\text{parell}}$. Denoting by π_d, π_{d-1} the maps to $\mathcal{M}_{r,d,D}^{\text{parell}}, \mathcal{M}_{r,d-1,D}^{\text{parell}}$ we have

$$T_0(1)(c) = \pi_{d*}([\mathcal{Z}]^{\text{vir}} \cap \pi_{d-1}^*(c)).$$

It follows that

$$[y_{p,i}, T_0(1)](c) = \pi_{d*}(c_1(\pi_d^*(L_{p,i}^d) - \pi_{d-1}^*(L_{p,i}^{d-1})) \cap \pi_{d-1}^*(c) \cap [\mathcal{Z}]^{\text{vir}}).$$

By construction, there is a canonical inclusion $\pi_{d-1}^*(L_{p,i}^{d-1}) \rightarrow \pi_d^*(L_{p,i}^d)$ whose cokernel is supported on the closed substack

$$\mathcal{Z}_{p,i} := \{\mathcal{F}' \subset \mathcal{F} \mid \text{supp}(\mathcal{F}/\mathcal{F}') = \{q_{p,i}\}\} \subset \mathcal{Z}.$$

Observe that π_d restricts to an isomorphism $\mathcal{Z}_{p,i} \simeq \mathcal{M}_{r,d,D}^{\text{parell}}$ under which π_{d-1} gets identified with $\text{Mod}_{p,i}$. Put $c_{p,i} = [y_{p,i}, T_0(1)](1) \in H^0(\mathcal{M}_{r,d,D}^{\text{parell}}) = \mathbb{C}$. By the projection formula we have $[y_{p,i}, T_0(1)] = c_{p,i} X_{p,i}$, hence it only remains to determine the constant $c_{p,i}$. Note that $c_{p,i}$ is independent of i by \mathfrak{S}_r -symmetry relabeling the points $\{q_{p,i}\}_i$. To determine this constant we note that $\sum_{i=1}^r y_{p,i}$ is the first Chern class of the rank r bundle

$$(7.3) \quad \mathcal{F}_{r,d,D}/\mathcal{F}_{r,d,D}(-\omega) = \bigoplus_i L_{p,i}^d$$

i.e. $\sum_{i=1}^r y_{p,i} = \psi_1(\omega)$. It follows that

$$(7.4) \quad \sum_{i=1}^p c_{p,i} X_{p,i} = [\psi_1(\omega), T_0(1)] = T_0(\omega).$$

Evaluating this at 1, we obtain $\sum_i c_{p,i} = r$ and thus $c_{p,i} = 1$ as desired. \square

Definition 7.7. Let $H_{r,D}$ be the ring extension of H obtained by adding generators $\tau_{p,i}$ of degree 2 where $p \in D$, $i = 1, \dots, r$ modulo the following relations:

$$\sum_i \tau_{p,i} = \omega, \quad \tau_{p,i} H^{>0} = 0, \quad \tau_{p,i} \tau_{q,j} = 0 \quad (p, q \in D, 1 \leq i, j \leq r).$$

Set for each $n \geq 0$

$$\psi_n(\tau_{p,i}) = y_{p,i}^n, \quad T_n(\tau_{p,i}) = y_{p,i}^n X_{p,i}.$$

This assignment is compatible with the relations $\sum_i \tau_{p,i} = \omega$ because by (7.3)

$$\sum_{i=1}^r y_{p,i}^n = \psi_n(\omega),$$

and

$$\sum_{i=1}^r y_{p,i}^n X_{p,i} = \sum_{i=1}^r \frac{1}{n+1} [y_{p,i}^{n+1}, T_0(1)] = \frac{1}{n+1} [\psi_{n+1}(\omega), T_0(1)] = T_n(\omega).$$

Now we have, for $\pi \in H$,

$$(7.5) \quad [\psi_{p,i}^n, T_{n'}(\pi)] = \frac{1}{n'+1} [y_{p,i}^n, [\psi_{n'+1}(\pi), T_0(1)]] = \frac{n}{n'+1} y_{p,i}^{n-1} [\psi_{n'+1}(\pi), X_{p,i}].$$

The r.h.s. of (7.5) is equal to $n y_{p,i}^{n+n'-1} X_{p,i} = n T_{n+n'-1}(\tau_{p,i})$ for $\pi = 1$ and vanishes for $\pi \in H^{>0}$. In addition, we have

$$[\psi_n(1), y_{p,i}^{n'} X_{p,i}] = n y_{p,i}^{n+n'-1} X_{p,i} = n T_{n+n'-1}(\tau_{p,i}).$$

All together, we see that the commutation relations

$$[\psi_n(\pi), T_{n'}(\pi')] = n T_{n+n'-1}(\pi \pi')$$

hold for all $\pi, \pi' \in H_{r,D}$. Finally, from the definition of $T_n(\tau_{p,i})$ we have

$$T_n(\tau_{p,i})(1) = y_{p,i}^n.$$

This is compatible with the computation carried out in Section 4.4. Note that the diagonal class, coming from the surface S° equals $\omega \otimes \omega$, and therefore annihilates $\tau_{p,i}$. This implies

that for any two symmetric functions f, g of positive degree we have $(fg)(\tau_{p,i}) = 0$. Hence

$$h_{n+1}(\tau_{p,i}) = \frac{p_{n+1}(\tau_{p,i})}{n+1} = \psi_n(\tau_{p,i}) = y_{p,i}^n.$$

We have shown that the action of T on the polynomial ring generated by $\psi_n(\pi)$, $\pi \in H_{r,D}$ fits into the framework of Section 5. In particular, the results of that section apply, such as the commutation relations between operators T 's, and Theorems 5.2 and 5.6. We summarize this as follows.

Theorem 7.8. *Corollaries 7.2, 7.3, 7.5 and Proposition 7.4 extend verbatim to the spaces $\mathcal{M}_{r,d,D}^{\text{parell}}$ and $M_{r,d,D}^{\text{parell}}$ when H is replaced by $H_{r,D}$ and the tautological ring is enlarged by adding the classes $y_{p,i}$.*

From now on, we denote by $H_{\text{taut}}^*(\mathcal{M}_{r,d,D}^{\text{parell}}) \subset H^*(\mathcal{M}_{r,d,D}^{\text{parell}})$, $H_{\text{taut}}^*(M_{r,d,D}^{\text{parell}}) \subset H^*(M_{r,d,D}^{\text{parell}})$ the subrings generated by $\psi_n(\pi)$, $\pi \in H_{r,D}$.

7.3. \mathfrak{sl}_2 -triples. In Section 6.6 we explicitly constructed an \mathfrak{sl}_2 -triple acting on $H_{\text{taut}}^*(M_\alpha)$ from the algebra W_M . As explained in Remark 5.7, the situation simplifies in the case of Higgs bundles on a curve. Let us spell out the form of the \mathfrak{sl}_2 -triple in our case. As this applies both to the parabolic and the classical cases, we write \mathcal{M} , resp. M for either $\mathcal{M}_{r,d,D}^{\text{parell}}$ or $\mathcal{M}_{r,d}$, resp. $M_{r,d,D}^{\text{parell}}$ or $M_{r,d}$.

Recall that by construction $\tilde{D}_{1,0}(\eta) = \tilde{D}_{1,0}(\omega)$ vanishes. Hence $\tilde{D}_{1,0}(\omega)_{\text{red}} = 0$. Likewise, we have $y_{\text{red}} = 0$ and $(\partial_y)_{\text{red}} = 0$. The element $D_{0,1}(1) = \psi_1(1) \in H^0(\mathcal{M})$ being the Euler characteristic of a vector bundle of rank r and degree d , we may fix $D_{0,1}(1) = 0$ by choosing d appropriately⁴. Thus,

$$\tilde{D}_{0,1}(1)_{\text{red}} = \tilde{D}_{1,0}(1)_{\text{red}} = \tilde{D}_{0,1}(\omega)_{\text{red}} = \tilde{D}_{1,0}(\omega)_{\text{red}} = 0$$

and $\tilde{D}_{0,2}(1) = \psi_2(1)$. This implies that all the extra terms in Proposition 7.4 vanish for the operators $\tilde{D}_{1,1}(1)_{\text{red}}$, $\tilde{D}_{0,2}(1)_{\text{red}}$, $\tilde{D}_{2,0}(1)_{\text{red}}$, so that we have

$$\begin{aligned} [\tilde{D}_{2,0}(1)_{\text{red}}, \tilde{D}_{m,n}(\pi)_{\text{red}}] &= -2n\tilde{D}_{m+1,n-1}(\pi)_{\text{red}}, \\ [\tilde{D}_{1,1}(1)_{\text{red}}, \tilde{D}_{m,n}(\pi)_{\text{red}}] &= (m-n)\tilde{D}_{m,n}(\pi)_{\text{red}}, \\ [\tilde{D}_{0,2}(1)_{\text{red}}, \tilde{D}_{m,n}(\pi)_{\text{red}}] &= 2m\tilde{D}_{m-1,n+1}(\pi)_{\text{red}}. \end{aligned}$$

⁴Recall that the operators $\tilde{D}_{m,n}(\xi)_{\text{red}}$ act on the cohomology $H^*(M_\alpha)$ for each $\alpha \in \alpha_0 + \mathbb{Z}\delta$, and that these have all been identified; in particular, we may freely choose the degree d . By Remark 7.1, our choice of d is a multiple of r .

Proposition 7.9. *The operators $\mathfrak{e} = \frac{1}{2}\widetilde{D}_{0,2}(1)_{\text{red}}$, $\mathfrak{h} = -\widetilde{D}_{1,1}(1)_{\text{red}}$, $\mathfrak{f} = -\frac{1}{2}\widetilde{D}_{2,0}(1)_{\text{red}}$ form an \mathfrak{sl}_2 -triple acting on $H_{\text{taut}}^*(M)$. \square*

We will need a slightly more general statement. The \mathfrak{sl}_2 -triple constructed above will be called the *original* \mathfrak{sl}_2 -triple. The space $H_{\text{taut}}^2(M)$ is spanned by $\psi_2(1)$, $\psi_1(\pi)$ for $\pi \in H_{r,D}^2$ and the products $\psi_1(\gamma_i)\psi_1(\gamma_j)$ for $1 \leq i, j \leq 2g$ (recall that γ_i 's form a basis of H^1).

Proposition 7.10. *Fix $\alpha \in H_{\text{taut}}^2(M)$ and write*

$$(7.6) \quad \alpha = A \left(\psi_2(1) + \frac{1}{r} \sum_{i=1}^g \psi_1(\gamma_i)\psi_1(\gamma_{i+g}) \right) + \sum_{i < j} B_{i,j} \psi_1(\gamma_i)\psi_1(\gamma_j) + \sum_{p,i} C_{p,i} \psi_1(\tau_{p,i}),$$

for some $A, B_{i,j}, C_{p,i} \in \mathbb{C}$, where the third term only occurs in the parabolic case. If $A \neq 0$ and the antisymmetric matrix with off-diagonal entries $B_{i,j}$ is non-degenerate, then there exists an \mathfrak{sl}_2 -triple $(\mathfrak{e}', \mathfrak{h}', \mathfrak{f}')$ acting on $H_{\text{taut}}^*(M)$ with $\mathfrak{e}' = \alpha$, whose associated \mathfrak{h}' -degree filtration coincides with the \mathfrak{h} -degree filtration of the original \mathfrak{sl}_2 -triple.

Proof. Consider the operators $y(\pi) = \psi_1(\pi)$ and $x(\pi) = \widetilde{D}_{1,0}(\pi)_{\text{red}}$ for $\pi \in H_{r,D}^{>0}$. By Theorem 6.12, the operators $y(\pi)$ increase the \mathfrak{h} -degree by 1, and the operators $x(\pi)$ decrease it by 1. In particular, they are both nilpotent as $H^*(M)$ is finite-dimensional. Proposition 7.4 yields

$$(7.7) \quad [y(\pi), x(\pi')] = \psi_0(\pi\pi').$$

Note that $\psi_0(\pi\pi') = r\pi \cdot \pi'$ if both π, π' are in H^1 (and the intersection product is taken in $H^*(C)$) while $\psi_0(\pi\pi') = 0$ if one of π, π' belongs to $H_{r,D}^{>1}$. In addition we have

$$[\mathfrak{e}, x(\pi)] = y(\pi).$$

For $\pi \in H_{r,D}^2$ let $X(\pi) = e^{x(\pi)}$. By the equation above it satisfies $[\mathfrak{e}, X(\pi)] = y(\pi)X(\pi)$, and so

$$X(\pi)^{-1}\mathfrak{e}X(\pi) = \mathfrak{e} + y(\pi).$$

Starting from the original \mathfrak{sl}_2 triple and conjugating by a product of complex powers of $X(\tau_{p,i})$ (as $x(\tau_{p,i})$ is nilpotent, the complex powers of $X(\tau_{p,i})$ are well-defined) we can obtain a new \mathfrak{sl}_2 -triple $(\mathfrak{e}', \mathfrak{h}', \mathfrak{f}')$ in which $\mathfrak{e}' = \mathfrak{e} + \sum_{p,i} C_{p,i} \psi_1(\tau_{p,i})$ for any given complex values of the coefficients $C_{p,i}$. Moreover, since $x(\pi)$ has negative \mathfrak{h} -degree, the operators $X(\tau_{p,i})$ preserve the \mathfrak{h} -degree filtration, i.e. the \mathfrak{h} -degree and \mathfrak{h}' -degree filtrations coincide.

In order to obtain the general form, we again argue in a similar fashion, this time using operators of the form $x(\gamma)y(\gamma')$ for $\gamma, \gamma' \in H^1$. Notice that these operators preserve the

\mathfrak{h} -degree and hence the \mathfrak{h} -degree (and \mathfrak{h}' -degree) filtration. A direct computation yields

$$[\mathfrak{e}, x(\gamma)y(\gamma')] = y(\gamma)y(\gamma'), \quad \left[\sum_i y(\gamma_i)y(\gamma_{i+g}), x(\gamma)y(\gamma') \right] = -ry(\gamma)y(\gamma').$$

As $[x(\gamma)y(\gamma'), y(\tau_{p,i})] = 0$ for any γ, γ', p, i , the element $\mathfrak{e}' + \frac{1}{r} \sum_i y(\gamma_i)y(\gamma_{i+g})$ commutes with any element of the form $x(\gamma)y(\gamma')$. Equation (7.7) implies that the elements of the form $x(\gamma)y(\gamma')$ generate a Lie algebra isomorphic to \mathfrak{gl}_{2g} . The adjoint action by this Lie algebra can be integrated to an action of $\mathrm{GL}_{2g}(\mathbb{C})$ on $H_{\mathrm{taut}}^2(M)$ which fixes elements of the form $\mathfrak{e}' + \frac{1}{r} \sum_i y(\gamma_i)y(\gamma_{i+g})$ and transforms elements of the form $y(\gamma)y(\gamma')$ according to the exterior square of the standard representation. Since any two non-degenerate antisymmetric forms are $\mathrm{GL}_{2g}(\mathbb{C})$ -equivalent, the orbit of $-\sum_{i \leq g} y(\gamma_i)y(\gamma_{i+g})$ contains all the nondegenerate elements. Writing $\mathfrak{e}'' = (\mathfrak{e}' + \frac{1}{r} \sum_i y(\gamma_i)y(\gamma_{i+g})) - \frac{1}{r} \sum_i y(\gamma_i)y(\gamma_{i+g})$ we see that we can transform the \mathfrak{sl}_2 -triple $(\mathfrak{e}', \mathfrak{h}', \mathfrak{f}')$ into another \mathfrak{sl}_2 -triple of the desired form. We conclude by noting that the $\mathrm{GL}_{2g}(\mathbb{C})$ -action preserves the \mathfrak{h} -degree. \square

Remark 7.11. It is clear from the proof that the symplectic group $\mathrm{Sp}_{2g}(\mathbb{C}) \subset \mathrm{GL}_{2g}(\mathbb{C})$ acts on $H_{\mathrm{taut}}^*(M)$ respecting the \mathfrak{sl}_2 -action, preserving the cohomological degree, and acting via the standard representation on the span of classes $\psi_1(\gamma_i)$.

Corollary 7.12. *For any $\alpha \in H_{\mathrm{taut}}^2(M)$ and for all but finitely many values $\lambda \in \mathbb{C}$ there exists an \mathfrak{sl}_2 -triple $(\mathfrak{e}, \mathfrak{h}, \mathfrak{f})$ with $\mathfrak{e} = \alpha + \lambda\psi_2(1)$ for which the \mathfrak{h} -degree filtration coincides with the \mathfrak{h} -degree filtration of the original \mathfrak{sl}_2 -triple.* \square

8. PROOF OF THE $P=W$ CONJECTURES

Throughout this section, we fix a smooth complex projective curve C of genus $g > 1$ and a positive integer r . We will consider Higgs bundles on C of rank r and degree d ; the latter will be allowed to vary. We also fix a (nontrivial) simple effective divisor D which we specialize in Section 8.5 to a divisor of degree one.

8.1. Strategy of the proof. Let us briefly explain the steps of our proof of the $P = W$ conjectures, in both classical and parabolic settings. First of all, thanks to a classical result of Markman ([Mar02], see [LL24] for the parabolic case), the cohomology of the moduli spaces of stable Higgs bundles is generated by the tautological classes. The $P = W$ conjectures then reduce, by work of Shende [She16] (see [Mel19] for the parabolic case) to the $P = C$ conjecture, see Introduction for the statement. Our proof of the $P = C$

conjecture proceeds in four steps, each one establishing a version of $P = C$ in a different setup.

Recall that a D -parabolic Higgs bundle⁵ is a triple $(\mathcal{E}, \theta, (F_p)_p)$ where \mathcal{E} is a vector bundle on C , $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega(D)$ and for each $p \in D$, $F_p : F_{p,1} \subset F_{p,2} \subset \cdots \subset F_{p,r} = \mathcal{E}|_p$ is a full flag in the fiber of \mathcal{E} at p which is preserved by $\text{res}_p(\theta)$. A *nilpotent* D -parabolic Higgs bundle⁶ is a triple $(\mathcal{E}, \theta, (F_p)_p)$ as above, for which $\text{res}_p(F_{p,i}) \subseteq F_{p,i-1}$ for all i . Finally, note that a parabolic Higgs bundle $(\mathcal{E}, \theta, (F_p)_p)$ with $\text{res}_p = 0$ for all p is simply a Higgs bundle (\mathcal{E}, θ) together with a collection of full flags in each of the fibers $\mathcal{E}|_p$.

We proceed as follows:

- (i) Identify the \mathfrak{h} -degree filtration on $H^*(M_{r,d,D}^{\text{parell}})$ from Section 7.3 with the perverse filtration P , and deduce $P = C$ for the parabolic elliptic moduli $M_{r,d,D}^{\text{parell}}$ from commutation relations in W_M (Section 8.2);
- (ii) Check that the restriction map $H^*(M_{r,d,D}^{\text{par}}) \rightarrow H^*(M_{r,d,D}^{\text{parell}})$ is injective and compatible with P , deducing $P = C$ for stable parabolic Higgs bundles (Section 8.3);
- (iii) Check that the restriction map $H^*(M_{r,d,D}^{\text{par}}) \rightarrow H^*(M_{r,d,D}^{\text{par},0})$ to the nilpotent parabolic Higgs bundles is an isomorphism compatible with P , hence $P = C$ for $M_{r,d,D}^{\text{par},0}$ (Section 8.4);
- (iv) Finally, for the classical moduli of Higgs bundles $M_{r,d}$, $\text{gcd}(r, d) = 1$ we realize $H^*(M_{r,d})$ as a direct summand of $H^*(M_{r,d,D}^{\text{par},0})$, deducing $P = C$ by another compatibility check (Section 8.5).

Remark 8.1. Note that we only use the W_S -action and its consequences in the first step. The reason is that our construction *a priori* does not give an action of W_S on $H^*(M_{r,d,D}^{\text{par}})$ and other moduli spaces.

8.2. Elliptic case. In an effort to unburden the notation, we will suppress the subscript D . Fix a generic stability parameter and let $M_{r,d}^{\text{par}}$ be the moduli space of stable D -parabolic Higgs bundles. Let $\chi^{\text{par}} : M_{r,d}^{\text{par}} \rightarrow \mathbb{A}^{\text{par}}$ be the Hitchin map; here \mathbb{A}^{par} is an affine space which only depends on r, D and C . There is a cartesian square

$$\begin{array}{ccc} M_{r,d}^{\text{parell}} & \xrightarrow{i} & M_{r,d}^{\text{par}} \\ \chi^{\text{parell}} \downarrow & & \downarrow \chi^{\text{par}} \\ \mathbb{A}^{\text{parell}} & \xrightarrow{i'} & \mathbb{A}^{\text{par}} \end{array}$$

⁵this is sometimes referred to as a *twisted* parabolic Higgs bundle.

⁶this is sometimes simply called a parabolic or a strongly parabolic Higgs bundle.

where $i' : \mathbb{A}^{\text{parell}} \rightarrow \mathbb{A}^{\text{par}}$ and i are open immersions ($\mathbb{A}^{\text{parell}}$ is the parabolic elliptic locus). The map χ^{parell} is projective and hence yields a perverse filtration P_\bullet on $H^*(M_{r,d}^{\text{parell}})$.

By [LL24, Thm. 1.2, Cor. 1.3], we have the following parabolic version of Markman's theorem:

$$(8.1) \quad H_{\text{taut}}^*(M_{r,d}^{\text{par}}) = H_{\text{pure}}^*(M_{r,d}^{\text{par}}),$$

from which we deduce the same result for $M_{r,d}^{\text{parell}}$ by restricting along i .

As Hecke correspondences do not change the support of purely 1-dimensional sheaves, the convolution diagram (4.1) is defined over \mathbb{A}^{par} . Applying Propositions 3.7 and 3.8 we deduce that for $\xi \in H^i(S)$ and $m \geq 0$

- $\psi_m(\xi)P_j \subseteq P_{j+2m-2+i}$;
- $T_m(\xi)P_j \subseteq P_{j+2m-2+i}$;
- More generally, the operator $D_{m,n}(\xi)$ (of vertical degree $2n - 2 + i$) satisfies $D_{m,n}(\xi)P_j \subseteq P_{j+2n-2+i}$.

In particular, the operator $q_1(\omega)$, which was used to identify the cohomologies of $M_{r,d}^{\text{parell}}$ for different values of d respects the filtration, i.e. $q_1(\omega)P_i H^*(M_{r,d}^{\text{parell}}) \subseteq P_i H^*(M_{r,d+1}^{\text{parell}})$. Tensoring by a line bundle of degree one yields an isomorphism $M_{r,d}^{\text{parell}} \simeq M_{r,d+r}^{\text{parell}}$ which is compatible with the Hitchin map; it follows that $H^*(M_{r,d}^{\text{parell}})$ and $H^*(M_{r,d+r}^{\text{parell}})$ have the same perverse filtrations; applying $q_1(\omega)^r$, we easily deduce that $q_1(\omega)P_i H^*(M_{r,d}^{\text{parell}}) = P_i H^*(M_{r,d+1}^{\text{parell}})$ for any d (hence the same holds for $q_1(\omega)^{-1}$). Since the operators $\tilde{D}_{m,n}(\xi)_{\text{red}}$ have the same vertical degree as $D_{m,n}(\xi)$, we obtain the following:

Proposition 8.2. *The operators $\mathfrak{e}, \mathfrak{h}, \mathfrak{f}$ of the original \mathfrak{sl}_2 -triple (see Proposition 7.9, as well as any of the triples constructed in Proposition 7.10) change perversity by $2, 0, -2$ respectively.*

This completely pins down the perverse filtration:

Proposition 8.3. *The perverse filtration on $H_{\text{pure}}^*(M_{r,d}^{\text{parell}})$ coincides with the \mathfrak{h} -degree filtration induced by the original \mathfrak{sl}_2 -triple.*

Proof. Choose a relatively ample class $\alpha \in H^2(M_{r,d}^{\text{parell}})$ and replace it by some linear combination $\alpha' = \alpha + \lambda\psi_2(1)$ with $\lambda \in \mathbb{Q}$ which is relatively ample and for which the operator \mathfrak{e} of cup product with α' is a part of an \mathfrak{sl}_2 -triple $(\mathfrak{e}, \mathfrak{h}', \mathfrak{f}')$ (see Corollary 7.12). By Proposition 8.2, $(\mathfrak{e}, \mathfrak{h}', \mathfrak{f}')$ descends to an \mathfrak{sl}_2 -triple of operators on $\text{Gr}_\bullet^P H^*(M_{r,d}^{\text{parell}})$, for which \mathfrak{h}' is of degree zero. By Theorem 3.6, one can complete the operator \mathfrak{e} to an \mathfrak{sl}_2 -triple

$(\mathfrak{e}, \mathfrak{h}, \mathfrak{f})$ acting on $\mathrm{Gr}_{\bullet}^P H^*(M_{r,d}^{\mathrm{parell}})$ for which $\mathfrak{h}|_{\mathrm{Gr}_i^P} = i \mathrm{Id}_{\mathrm{Gr}_i^P}$. In particular, $[\mathfrak{h}, \mathfrak{h}'] = 0$. The elementary lemma below allows us to conclude that $\mathfrak{h} = \mathfrak{h}'$, from which Proposition 8.3 readily follows. \square

Lemma 8.4. *Let V be a finite-dimensional vector space and let $(\mathfrak{e}, \mathfrak{h}, \mathfrak{f}), (\mathfrak{e}, \mathfrak{h}', \mathfrak{f}')$ be two \mathfrak{sl}_2 -triples in $\mathrm{End}(V)$. Assume that $[\mathfrak{h}, \mathfrak{h}'] = 0$. Then $\mathfrak{h} = \mathfrak{h}'$.*

Proof. By the Jacobson-Morozov theorem, there exists $g \in \mathrm{Stab}_{GL(V)}(\mathfrak{e})$ such that $\mathrm{Ad}_g \mathfrak{h} = \mathfrak{h}'$. Write $V = \bigoplus_i V[i]$ for the \mathfrak{h} -weight decomposition and $V^{\mathfrak{f}} = \bigoplus_i V^{\mathfrak{f}}[i]$ for the space of lowest weight vectors. Then g is determined by its action on each $V^{\mathfrak{f}}[i]$ and we have

$$g(V^{\mathfrak{f}}[i]) \subseteq (U(\mathfrak{e}) \cdot V^{\mathfrak{f}}[> i]) \oplus V^{\mathfrak{f}}[i].$$

This reflects the decomposition $\mathrm{Stab}_{GL(V)}(\mathfrak{e}) \simeq \prod_i GL(V^{\mathfrak{f}}[i]) \times \mathrm{Rad}(\mathrm{Stab}_{GL(V)}(\mathfrak{e}))$. As $[\mathfrak{h}, \mathfrak{h}'] = 0$, \mathfrak{h}' preserves each $V[i]$. But this is only possible if $g \in \prod_i GL(V^{\mathfrak{f}}[i])$, which implies that g commutes with \mathfrak{h} and thus $\mathfrak{h}' = \mathfrak{h}$. \square

The following statement is the version of $P = C$ for the parabolic elliptic locus:

Theorem 8.5. *The subspace $P_m H_{\mathrm{pure}}^*(M_{r,d}^{\mathrm{parell}})$ is the span of products $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \leq m + N$, where $N = (g-1)r^2 + |D|\binom{r}{2} + 1$.*

Proof. Observe that $1 \in H_{\mathrm{pure}}^0(M_{r,d}^{\mathrm{parell}})$ is of perverse degree equal to the dimension of the Hitchin map, which is equal to $(g-1)r^2 + |D|\binom{r}{2} + 1$. The statement is now an obvious consequence of the relation $[\mathfrak{h}, \psi_m(\xi)] = m\psi_m(\xi)$. \square

8.3. Parabolic case. Next, we consider the entire space $M_{r,d}^{\mathrm{par}}$. As the stability parameter considered is primitive, there are no strictly semistables and the Hitchin map χ^{par} is projective. The group \mathbb{C}^* acts on $M_{r,d}^{\mathrm{par}}$ by scaling the Higgs field, turning it into a semi-projective variety, see [HRV15]. The cohomology of $M_{r,d}^{\mathrm{par}}$ is pure by [HRV15, Corollary 1.3.2], hence (8.1) yields

$$H_{\mathrm{taut}}^*(M_{r,d}^{\mathrm{par}}) = H^*(M_{r,d}^{\mathrm{par}}).$$

There is a decomposition $\mathbb{A}^{\mathrm{par}} = \mathbb{A}_D \times \mathbb{A}^{\mathrm{par},0}$, and correspondingly $\chi^{\mathrm{par}} = \chi_D \times \chi'$, where $\mathbb{A}^{\mathrm{par},0}$ is the Hitchin base for the D -twisted Higgs bundles, $\mathbb{A}_D := \mathbb{C}^{r|D|-1}$ and

$$\chi_D : M_{r,d}^{\mathrm{par}} \rightarrow \mathbb{A}_D$$

is the map sending (\mathcal{E}, θ, F) to the ordered collection of the eigenvalues of $\mathrm{res}_p \theta$ for all $p \in D$. Note that the sum of all the eigenvalues is always zero, hence the rank of \mathbb{A}_D .

The map χ_D is smooth, i.e. it is a surjective submersion. In fact, $M_{r,d}^{\text{par}}$ is known to be a Poisson variety and the fibers of χ_D are precisely the symplectic leaves. Moreover, the image of the tangent map at every point is dual to the kernel of the Poisson tensor, so the dimension of the image of the tangent map is constant. From Corollary 1.3.3 in [HRV15] we deduce

Proposition 8.6. *For each $\boldsymbol{\mu} \in \mathbb{A}_D$ the restriction map $H^*(M_{r,d}^{\text{par}}) \rightarrow H^*(\chi_D^{-1}(\boldsymbol{\mu}))$ is an isomorphism.*

In particular, the cohomology of all the fibers of χ_D are pure.

Proposition 8.7. *The restriction map $i^* : H^*(M_{r,d}^{\text{par}}) \rightarrow H^*(M_{r,d}^{\text{parell}})$ is injective, i.e. $H^*(M_{r,d}^{\text{par}}) = H_{\text{pure}}^*(M_{r,d}^{\text{parell}})$.*

Proof. For a generic (in the sense of [HLRV11, Def. 2.1.1]) $\boldsymbol{\mu} \in \mathbb{A}_D$, the identity

$$\sum_p \text{res}_p \text{Tr}(\theta) = 0$$

implies that any parabolic Higgs bundle in $(\mathcal{E}, \theta, (F_p)_p) \in \chi_D^{-1}(\boldsymbol{\mu})$ is simple, which in turn implies that the spectral curve of (\mathcal{E}, θ) is reduced and irreducible, hence $\chi_D^{-1}(\boldsymbol{\mu}) \subset M_{r,d}^{\text{parell}}$. The composition

$$H^*(M_{r,d}^{\text{par}}) \xrightarrow{i^*} H^*(M_{r,d}^{\text{parell}}) \rightarrow H^*(\chi_D^{-1}(\boldsymbol{\mu}))$$

is an isomorphism by Proposition 8.6, and so i^* is injective. \square

We deduce from (8.1) that i^* induces an isomorphism of tautological rings $H^*(M_{r,d}^{\text{par}}) \simeq H_{\text{taut}}^*(M_{r,d}^{\text{parell}})$. By Theorem 3.6, i^* preserves the perverse filtrations and becomes a map of Lefschetz structures once a relatively ample class $\alpha \in H^2(M_{r,d}^{\text{par}})$ has been chosen. It therefore has to be an isomorphism of Lefschetz structures on $H^*(M_{r,d}^{\text{par}}) \simeq H_{\text{taut}}^*(M_{r,d}^{\text{parell}})$. Theorem 8.5 implies

Corollary 8.8. *The $P = C$ conjecture holds for $H^*(M_{r,d}^{\text{par}})$, i.e. $P_m H^*(M_{r,d}^{\text{par}})$ is the span of products of tautological classes $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \leq m + N$, where $N = (g - 1)r^2 + |D|\binom{r}{2} + 1$.*

8.4. Nilpotent parabolic case. Let us now denote by $M_{r,d}^{\text{par},0} := \chi_D^{-1}(0) \subset M_{r,d}^{\text{par}}$ the moduli space parameterizing stable nilpotent parabolic Higgs bundles. The restriction of the Hitchin morphism to $M_{r,d}^{\text{par},0}$ induces the perverse filtration on $H^*(M_{r,d}^{\text{par},0})$. Recall that by Proposition 8.6, the restriction map $H^*(M_{r,d}^{\text{par}}) \rightarrow H^*(M_{r,d}^{\text{par},0})$ is an isomorphism.

Proposition 8.9. *The restriction isomorphism $H^*(M_{r,d}^{\text{par}}) \xrightarrow{\sim} H^*(M_{r,d}^{\text{par},0})$ identifies the perverse filtrations on two sides.*

Proof. For simplicity, let us write $X = M_{r,d}^{\text{par}}$, $X_0 = M_{r,d}^{\text{par},0}$, $N = \dim X$, $N_0 = \dim X_0$, and $n = N - N_0$. We have $X_0 = \chi^{-1}(\mathbb{A}_0)$ for $\mathbb{A}_0 = \{0\} \times \mathbb{A}^{\text{par},0} \subset \mathbb{A}^{\text{par}}$ a linear subspace of codimension n .

Let \overline{X} be the projectivization with respect to the \mathbb{C}^* -action on X (see [Hau98]):

$$\overline{X} = (X \times \mathbb{C} \setminus \chi^{-1}(0) \times \{0\}) / \mathbb{C}^*,$$

and let $\partial X = \overline{X} \setminus X$. The spaces \overline{X} and ∂X are projective. They are not necessarily smooth, but the singularities are finite quotient singularities, so they are rationally smooth. In particular, their cohomology is pure, and may be identified with their Borel-Moore homology. We define \overline{X}_0 in an analogous fashion. The Hitchin map extends to a projective morphism $\overline{\chi} : \overline{X} \rightarrow \overline{\mathbb{A}}$, where $\overline{\mathbb{A}} := (\mathbb{A} \times \mathbb{C} \setminus \{0\}) / \mathbb{C}^*$ is the corresponding weighted projective space. This map restricts to $\overline{\chi}_0 : \overline{X}_0 \rightarrow \overline{\mathbb{A}}_0$, where $\overline{\mathbb{A}}_0 \subset \overline{\mathbb{A}}$ is the projective subspace associated to \mathbb{A}_0 . Let $L \in H^2(\overline{X})$, resp. $L_0 \in H^2(\overline{X}_0)$, be the pullback of the hyperplane class from $\overline{\mathbb{A}}$.

Lemma 8.10. *We have*

$$H^*(X) = H^*(\overline{X}) / LH^*(\overline{X}), \quad H^*(X_0) = H^*(\overline{X}_0) / L_0H^*(\overline{X}_0)$$

Proof. Let $U = \overline{X} \setminus \chi^{-1}(0)$. Since U is the total space of a line bundle over ∂X , its cohomology is pure, so the restriction map

$$H^*(\overline{X}) \rightarrow H^*(U) = H^*(\partial X)$$

is surjective. The kernel of this map is identified with the homology of $\chi^{-1}(0)$, and therefore $H_*(\chi^{-1}(0))$. Since χ is proper, $H^*(\chi^{-1}(0))$ is pure by duality. The restriction map $H^*(X) \rightarrow H^*(\chi^{-1}(0))$ is an isomorphism because the \mathbb{C}^* -action retracts X onto $\chi^{-1}(0)$. Hence $H^*(X)$ is pure, which implies the surjectivity of the restriction map $H^*(\overline{X}) \rightarrow H^*(X)$. By purity, the long exact sequence in Borel-Moore homology for $X \subset \overline{X}$ yields the collection of short exact sequences

$$0 \rightarrow H^{*-2}(\partial X) \rightarrow H^*(\overline{X}) \rightarrow H^*(X) \rightarrow 0.$$

The composition $H^*(\overline{X}) \rightarrow H^*(\partial X) \rightarrow H^{*+2}(\overline{X})$ is the multiplication by the fundamental class $[\partial X] = L$. We obtain a short exact sequence

$$H^{*-2}(\overline{X}) \xrightarrow{L} H^*(\overline{X}) \rightarrow H^*(X) \rightarrow 0,$$

as claimed. By dualizing this sequence and using the Poincaré duality we get a longer exact sequence:

$$0 \rightarrow H_{2N-*+2}(X) \rightarrow H^{*-2}(\overline{X}) \xrightarrow{L} H^*(\overline{X}) \rightarrow H^*(X) \rightarrow 0.$$

Finally, the same arguments apply for X_0, \overline{X}_0 . \square

Let $\iota : \overline{X}_0 \rightarrow \overline{X}$ be the closed embedding. The operators ι_*, ι^* both commute with the multiplication by L , i.e. $\iota^*L = L_0\iota^*$ and $\iota_*L_0 = L\iota_*$. The composition $\iota_*\iota^*$ is the operator of multiplication by $[\overline{X}_0] \in H^{2n}(\overline{X})$. On the other hand, since $\overline{X}_0 = \overline{\chi}^{-1}(\overline{\mathbb{A}}_0)$ and $\overline{\mathbb{A}}_0$ is a projective subspace, we have $L^n = c[\overline{X}_0]$ where $c > 0$ is the multiplicity. So we have

$$\iota_*\iota^* = c^{-1}L^n.$$

We claim that $\text{Im } \iota_* \subset \text{Im } L^n$. To see this, let $\alpha = \iota_*\beta$, where $\beta \in H^*(\overline{X}_0)$. Using the isomorphism

$$\iota^* : H^*(\overline{X})/LH^*(\overline{X}) \rightarrow H^*(\overline{X}_0)/L_0H^*(\overline{X}_0),$$

we can lift β and write $\beta = \iota^*\beta' + L_0\beta''$. This implies

$$\alpha = c^{-1}L^n\beta' + L\iota_*\beta''.$$

Let $\alpha' = \iota_*\beta''$. Repeating the argument for α' and continuing in this fashion n times we obtain $\alpha \in L^n H^*(\overline{X})$.

Next, we claim that for any $i \geq 0$ we have $(\iota^*)^{-1}(\text{Ker } L_0^i + \text{Im } L_0) \subset \text{Ker } L^{i+n} + \text{Im } L$. Indeed, suppose $\alpha \in H^*(\overline{X})$ is such that

$$\iota^*\alpha = \beta + L_0\beta', \quad L_0^i\beta = 0.$$

Applying ι_* and using the previous statement we obtain $c^{-1}L^n\alpha = \iota_*\beta + L\iota_*\beta'$ hence $c^{-1}L^n\alpha = \iota_*\beta + L^{n+1}\beta''$ and

$$L^{i+n}(\alpha - cL\beta'') = c\iota_*L^i\beta = 0,$$

which shows that $\alpha \in \text{Ker } L^{i+n} + \text{Im } L$.

Now suppose $\alpha \in P_i H^j(X_0)$. By Theorem 3.6 this means that α can be represented in $H^*(\overline{X}_0)$ by an element of $W_{i+N_0-j} \subset \text{Ker } L^{i+N_0-j+1} + \text{Im } L$. By the above, inside $H^*(\overline{X})$ the same element α can be represented by an element of $\text{Ker } L^{i+N-j+1}$, which implies $\alpha \in P_i H^j(X)$. Thus, the inverse of the pullback map $\iota^* : H^*(X_0) \rightarrow H^*(X)$ preserves the perverse filtration. Choosing a relatively ample line class $\omega \in H^2(X)$, we obtain a map of

Lefschetz structures, which is an isomorphism of vector spaces, hence also an isomorphism of Lefschetz structures. \square

Corollary 8.11. *The $P = C$ conjecture holds for $H^*(M_{r,d}^{\text{par},0})$, i.e. $P_m H^*(M_{r,d}^{\text{par},0})$ is the span of products of tautological classes $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \leq m + N$, where $N = (g-1)r^2 + |D|\binom{r}{2} + 1$.*

Remark 8.12. From $H^*(X) \cong H^*(\chi^{-1}(0)) \cong H^*(X_0)$ we obtain another proof of the fact that the restriction $H^*(M_{r,d}^{\text{par}}) \rightarrow H^*(M_{r,d}^{\text{par},0})$ is an isomorphism.

8.5. The classical case. We now make the assumption that $\gcd(r, d) = 1$. Let $M_{r,d}$ be the moduli space of stable Higgs bundles of rank r and degree d . Choose a point $p \in C$ and let $D = (p)$. Let $M_{r,d}^{\text{par},0}$ be the moduli space of stable nilpotent D -parabolic Higgs bundles $(\mathcal{E}, \theta, (F_p))$, with respect to a generic stability parameter. Let $\iota : \widetilde{M}_{r,d}^{\text{par},0} \hookrightarrow M_{r,d}^{\text{par},0}$ be the closed subvariety defined by the condition $\text{res}_p \theta = 0$. Then we have the forgetful morphism $\pi : \widetilde{M}_{r,d}^{\text{par},0} \rightarrow M_{r,d}$, which turns $\widetilde{M}_{r,d}^{\text{par},0}$ into a relative flag variety over $M_{r,d}$ (indeed, a parabolic bundle $(\mathcal{E}, \theta, (F_p))$ satisfying $\text{res}_p(\theta) = 0$ is stable if and only if the pair (\mathcal{E}, θ) is stable in the usual sense). The codimension of $\widetilde{M}_{r,d}^{\text{par},0}$ in $M_{r,d}^{\text{par}}$ equals to $\binom{r}{2}$. The relative dimension of $\widetilde{M}_{r,d}^{\text{par},0}$ over $M_{r,d}$ is also $\binom{r}{2}$. Thus we have maps

$$A = \pi_* \iota^* : H^*(M_{r,d}^{\text{par},0}) \rightarrow H^{*-2\binom{r}{2}}(M_{r,d}), \quad B = \iota_* \pi^* : H^*(M_{r,d}) \rightarrow H^{*+2\binom{r}{2}}(M_{r,d}^{\text{par},0}).$$

Let $\chi : M_{r,d} \rightarrow \mathbb{A}$, $\chi^{\text{par}} : M_{r,d}^{\text{par},0} \rightarrow \mathbb{A}^{\text{par},0}$ be the corresponding Hitchin maps, which are both projective. The space \mathbb{A} is identified with a linear subspace of $\mathbb{A}^{\text{par},0}$ so we may view both Hitchin morphisms as taking values in $\mathbb{A}^{\text{par},0}$. Applying Proposition 3.8, we see that both A and B preserve the perverse filtrations. Let

$$\Delta := \prod_{1 \leq i < j \leq r} (y_{p,i} - y_{p,j}) \in H^{2\binom{r}{2}}(M_{r,d}^{\text{par},0}).$$

We have $\pi_*(\Delta) = \pm r!$ because the class Δ is, up to a sign, the Euler class of the relative tangent bundle of $\widetilde{M}_{r,d}^{\text{par},0} \rightarrow M_{r,d}$. The composition $\iota^* \iota_*$ is the multiplication by the Euler class of the normal bundle of $\widetilde{M}_{r,d}^{\text{par},0}$ in $M_{r,d}^{\text{par},0}$, which is also given by Δ up to a sign. So we obtain

$$AB = \pi_* \iota^* \iota_* \pi^* = \pi_* \Delta \pi^* = \pm r!.$$

As $\widetilde{M}_{r,d}^{\text{par},0} \subset M_{r,d}^{\text{par},0}$ is the zero set of a vector bundle with Euler class $\pm \Delta$, Δ has to be a multiple of the fundamental class $[\widetilde{M}_{r,d}^{\text{par},0}] \in H^{2\binom{r}{2}}(M_{r,d}^{\text{par},0})$, and from $\iota^* \iota_* = \pm \Delta$ we obtain $[\widetilde{M}_{r,d}^{\text{par},0}] = \pm \Delta$.

Theorem 8.13. $P = C$ holds for $M_{r,d}$, i.e the subspace $P_m H^*(M_{r,d})$ is the span of products $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \leq m + N$, where $N = (g - 1)r^2 + 1$.

Proof. Let $-N'$ be the perversity of $1 \in H^0(M_{r,d}^{\text{par},0})$. We have $\pm r! = A(\Delta) \in P_{-N'+\binom{r}{2}} H^0(M_{r,d})$, so $-N \leq -N' + \binom{r}{2}$. Conversely, $\Delta = B(1) \in P_{-N}$. If $-N < -N' + \binom{r}{2}$, since Δ is a non-zero \mathfrak{h} -homogeneous element of weight $-N' + \binom{r}{2}$, we have

$$N' = N + \binom{r}{2}.$$

Consider a product of the form $f = \prod_i \psi_{m_i}(\xi_i) \in H^*(M_{r,d})$. Then

$$B(f) = \pm \Delta f \in P_{\sum_i m_i - N' + \binom{r}{2}} = P_{\sum_i m_i - N} \Rightarrow f = \pm \frac{1}{r!} ABf \in P_{\sum_i m_i - N},$$

so that $C \subset P$. Conversely, let $f \in P_m H^*(M_{r,d})$. By Markman's theorem, f can be explicitly written as a polynomial in tautological classes. Write

$$B(f) = \Delta f \in P_m H^*(M_{r,d}^{\text{par},0}).$$

By $P = C$ for $M_{r,d}^{\text{par},0}$ we can write

$$\Delta f = \sum_k \lambda_k g_k \quad (\lambda_k \in \mathbb{C}),$$

where each g_k is a monomial in tautological classes of the form $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \leq N' + m$. Using the identification of $H^*(M_{r,d}^{\text{par},0})$ with the pure part of the cohomology of the corresponding moduli space $M_{r,d}^{\text{parell}}$ we see that the symmetric group \mathfrak{S}_r acts on $H^*(M_{r,d}^{\text{par},0})$ permuting the generators $y_{p,r} = \psi_1(\tau_{p,r})$. Let $\text{ASym} : H^*(M_{r,d}^{\text{par},0}) \rightarrow H^*(M_{r,d}^{\text{par},0})$ be the corresponding antisymmetrization operator,

$$\text{ASym} = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} (-1)^{|\sigma|} \sigma.$$

We obtain

$$\Delta f = \sum_k \lambda_k \text{ASym}(g_k).$$

Each $\text{ASym}(g_k)$ is explicitly a product of Δ and a linear combination of monomials of the form $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \leq N + m$ not containing the generators $y_{p,i}$. Therefore we can lift each of these and write

$$B(f) = \Delta f = B \left(\sum_k \lambda'_k g'_k \right), \quad \lambda'_k \in \mathbb{C},$$

where each g_k is a monomial in tautological classes of the form $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \leq N + m$. Applying A we see that

$$f = \sum_k \lambda'_k g'_k,$$

so the converse inclusion $P \subset C$ is proved. \square

Remark 8.14. The number N , i.e. the negative of the perversity of 1 in the above statements is always the dimension of the fibers of the Hitchin map.

Remark 8.15. It is clear from the above proof that $H^*(M_{r,d})$ is the anti-invariant part of $H^*(\widetilde{M}_{r,d}^{\text{par},0})$. The corresponding statement on the Betti side was established in [Mel19]. The anti-invariant part is clearly preserved by the operators $\widetilde{D}_{m,n}(\xi)_{\text{red}}$ for $\xi \in H$; in particular, we have an action of \mathcal{H}_2 on $H^*(M_{r,d})[x, y]$ as in Corollary 7.5.

8.6. The weight filtration. Finally, let $X_{r,d}$ be the Betti moduli space, i.e. the character variety for a compact or non-compact curve C with generic local monodromies, see [HLRV11]. There is a decomposition turning $H^*(X_{r,d})$ into a graded ring

$$H^*(X_{r,d}) = \bigoplus_i W_{2i} H^*(X_{r,d}) \cap F^i H^*(X_{r,d}),$$

where W is the weight filtration and F is the Hodge filtration. The classes $f \in W_{2i} H^*(X_{r,d}) \cap F^i H^*(X_{r,d})$ are said to be of *pure weight* i . In the case of a compact curve it is explained in [She16] that the standard tautological classes are of pure weight. More precisely, let \mathcal{E} be the universal (topological) vector bundle on $X_{r,d} \times C$; then the Chern classes $e'_k(\xi) := \int_C c_k(\mathcal{E}) \cup \gamma$ are of pure weight k for all $\gamma \in H^*(C)$. The non-abelian Hodge theory provides a homeomorphism $X_{r,d} \simeq M_{r,d}$ which is compatible with the universal (topological) vector bundle. In particular, the tautological classes above can be seen as Chern characters of the universal vector bundle E on $M_{r,d} \times C$ instead.

Lemma 8.16. *We have $\psi_n(\gamma) = \frac{p'_n(\gamma)}{n+1} + \epsilon$, where ϵ is a linear combination of $p'_i(\gamma')$ with $i < n$, $\gamma' \in H^*(C)$.*

Proof. Let $S = \text{P}_C(\Omega(D) \oplus \mathcal{O})$, and $\pi : S \rightarrow C$ the natural projection. Recall that the tautological sheaf F on $M_{r,d} \times S$, is related to the tautological vector bundle E via $\pi_*(F) = E$. The relative Todd class of π is given by $\text{Td}_\pi = 1 + (\xi + (g-1)\omega)$, see Remark 7.1 for the notations. In particular, by Grothendieck-Riemann-Roch and projection formula

we have

$$\begin{aligned} p'_k(\gamma) &= \int_C \text{ch}_k(E) \cup \gamma = \int_S \text{ch}_{k+1}(F) \cup \pi^* \gamma + \int_S \text{ch}_k(F) \cup (\xi + (g-1)\omega) \cup \pi^* \gamma \\ &= p_{k+1}(\pi^* \gamma) + p_k((\xi + (g-1)\omega) \cup \pi^* \gamma). \end{aligned}$$

On the other hand, by the definition (5.2) of $\psi_n(\gamma)$ we have

$$\psi_n(\gamma) = \frac{p_{n+1}(\gamma)}{n+1} + \sum_{1 \leq i \leq n} \frac{n!}{i!} p_i(\text{Td}_{n+1-i} \gamma) = \frac{p'_n(\gamma)}{n+1} + \dots,$$

where dots stand for a linear combination of terms $p'_i(\gamma')$ with $i < n$, $\gamma' \in H^*(C)$. \square

Shende's result was extended to the parabolic character varieties in [Mel19]. Since the parabolic tautological classes are already expressed in terms of the vector bundle E , see Definition 7.7 and the discussion afterwards, the analogue of Lemma 8.16 (without the factor $\frac{1}{n+1}$) holds for $\gamma \in H_{r,D}$. Thanks to relations between Chern classes and Chern characters induced by (4.2), we see that

$$\begin{aligned} \text{Span} \left\{ \prod_i e'_{n_i}(\gamma_i) : \sum_i n_i \leq n \right\} &= \text{Span} \left\{ \prod_i p'_{n_i}(\gamma_i) : \sum_i n_i \leq n \right\} \\ &= \text{Span} \left\{ \prod_i \psi_{n_i}(\gamma_i) : \sum_i n_i \leq n \right\}. \end{aligned}$$

Altogether, we obtain

Proposition 8.17. *The subspace $W_{2m}H^*(X_{r,d})$ is the span of products $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \leq m$. \square*

Therefore the $P = C$ statements in Corollary 8.11 and Theorem 8.13 are equivalent to the claim $P_i = W_{2(i+N)}$.

Corollary 8.18. *The $P = W$ conjecture holds both in classical and parabolic setup. \square*

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