$P = W$ VIA $\mathcal{H}_2$

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ABSTRACT. Let $\mathcal{H}_2$ be the Lie algebra of polynomial Hamiltonian vector fields on the symplectic plane. Let X be the moduli space of stable Higgs bundles of fixed relatively prime rank and degree, or more generally the moduli space of stable parabolic Higgs bundles of arbitrary rank and degree for a generic stability condition. Let $H^*(X)$ be the cohomology with complex coefficients. Using the operations of cup-product by tautological classes and Hecke correspondences we construct an action of $\mathcal{H}_2$ on $H^*(X)[x, y]$, where x and y are formal variables. We show that the perverse filtration on $H^*(X)$ coincides with the filtration canonically associated to $\mathfrak{sl}_2 \subset \mathcal{H}_2$ and deduce the $P = W$ conjecture of de Cataldo-Hausel-Migliorini.

1. Introduction

1.1. $P = W$. Let $r > 0$ and d be relatively prime integers and let C be a smooth projective algebraic curve over $\mathbb{C}$. Let $M_{r,d}$ be the moduli space of stable Higgs bundles on C of rank r and degree d. Following Hitchin [Hit87] and Simpson [Sim90, Sim92], *Higgs bundles* are pairs $(\mathcal{E}, \theta)$ such that $\mathcal{E}$ is an algebraic vector bundle on C and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega$, where Ω denotes the line bundle of differential forms on C. A Higgs bundle is *stable* if for all proper subbundles $\mathcal{E}' \subset \mathcal{E}$ of rank r' and degree d' satisfying $\theta(\mathcal{E}') \subset \mathcal{E}' \otimes \Omega$ we have the inequality

$$\frac{d'}{r'} < \frac{d}{r}.$$

By the non-abelian Hodge correspondence ([Hit87, Sim91, Sim92]), the space $M_{r,d}$ is diffeomorphic to the moduli space $M_{r,d}^{\mathrm{Betti}}$ parameterizing certain twisted local systems on C. That latter space is an affine algebraic variety, but the complex structure is different from the one of $M_{r,d}$. By [Del71], the cohomology with complex coefficients $H^*(M_{r,d}^{\mathrm{Betti}})$ is equipped with a natural *weight filtration*. It is natural to ask for a description of the weight filtration on $H^*(M_{r,d}) = H^*(M_{r,d}^{\mathrm{Betti}})$ in terms of the algebraic geometry of the space $M_{r,d}$.

A conjectural answer to this question was suggested in [dCHM12]. The moduli space $M_{r,d}$ is endowed with the *Hitchin map*. This is the map $\chi : M_{r,d} \to \mathbb{C}^N$ which reads off the coefficients of the characteristic polynomial of θ. It turns out that χ is proper, and

Date: September 13, 2022.

applying the decomposition theorem [BBD82] one constructs the perverse filtration. The $P = W$ conjecture then claims that the weight filtration and the perverse filtration are essentially equal:

$$P_{i-N}H^*(M_{r,d}) = W_{2i}H^*(M_{r,d}^{\mathrm{Betti}}) \qquad (i = 0, 1, \ldots, 2N).$$

In the present paper we prove this conjecture.

By [Mar02] we know that the cohomology ring $H^*(M_{r,d})$ is generated by the tautological classes, which are the components of the Künneth decomposition of the Chern classes of the tautological sheaf on $M_{r,d} \times C$. By [She16] we know how to describe the weight filtration. Namely, each tautological class is assigned a weight so that the classes coming from the i-th Chern class have weight $2i$. Then W_m is spanned by products of tautological classes of total weight $\leq m$.

Our main result shows that the same kind of description holds for P:

Theorem 1.1. *The subspace $P_m H^*(M_{r,d})$ is the span of products of tautological classes of total weight $\leq 2(m + N)$.*

And therefore we have

Corollary 1.2. *The $P = W$ conjecture holds for the spaces $M_{r,d}$.*

In fact, we work in a more general context of moduli spaces of stable parabolic Higgs bundles of arbitrary rank and degree for a generic stability condition, and prove the corresponding result in that context.

The $P = W$ conjecture in the case of rank 2 was already proved in [dCHM12]. The case when C has genus 2 was established in [dCMS22]. While the present paper was in the final stages of preparation, we learned of a proof of $P = W$ by Maulik and Shen using a different approach, see [MS22].

1.2. $\mathcal{H}_2$**.** Let X be the moduli space $M_{r,d}$, or more generally the moduli space of parabolic Higgs bundles as above. Our proof of $P = W$ goes by constructing an action of an interesting algebra on $H^*(X)$.

The Lie algebra $\mathcal{H}_2$ of polynomial Hamiltonian vector fields on the plane with respect to the standard symplectic form has the following description. A basis is given by the fields $V_{m,n}$ with Hamiltonian $x^m y^n$. Explicitly, we have

$$V_{m,n} = n y^{n-1} x^m \frac{\partial}{\partial x} - m x^{m-1} y^n \frac{\partial}{\partial y}.$$

The Lie bracket is given as follows:

$$[V_{m,n}, V_{m',n'}] = (m'n - mn')V_{m+m'-1,n+n'-1}.$$

It is expected by physicists (Lev Rozansky, private communication) that $\mathcal{H}_2$ acts on many interesting geometric invariants. For instance, in [GHM21], in order to prove a Lefschetz property, certain operations were constructed acting on the so-called y-ified Khovanov-Rozansky homologies of links and it was speculated that these operations satisfy the relations of $\mathcal{H}_2$. For instance, in the situation of the n, n torus link, (a part of) the homology can be identified with the ideal J_n generated by the S_n-antiinvariant polynomials in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, and it is evident that the operators

$$\sum_{i=1}^{n} \left(n y_i^{n-1} x_i^m \frac{\partial}{\partial x_i} - m x_i^{m-1} y_i^n \frac{\partial}{\partial y_i} \right)$$

act on J and satisfy the relations of $\mathcal{H}_2$.

This and some other similarities with the situation of homologies of links gave us a hint that $\mathcal{H}_2$ may act on $H^*(X)$. It is easy to check however, that $\mathcal{H}_2$ does not have any non-zero finite-dimensional representation, so there is no hope that it acts directly on $H^*(X)$. It turns out, $\mathcal{H}_2$ acts on the infinite-dimensional space $H^*(X)[x, y]$, where x and y are formal variables satisfying the natural commutation relations with $V_{m,n}$.

Restricting to $H^*(X)$ we still obtain an action of interesting algebra, but the relations are much more cumbersome to write, see Proposition 7.3 and Corollary 7.4.

1.3. **The main construction.** In order to construct the action, we use the explicit presentation of a certain cohomological Hall algebra of zero-dimensional sheaves from [MMSV]. However, for reader's convenience, and for the purpose of making the paper self-contained, we use a trick to make things less technical, less general, but still sufficient for our purposes and reproduce the proofs here.

Namely, we consider the so-called *elliptic locus* X^{ell} in the moduli space. This is the locus where the spectral curve, i.e. the set of eigenvalues of θ is reduced and irreducible. Then the *Hecke operators* are defined via explicit correspondences using a certain natural virtual fundamental class, see Section 3. Then the Hecke operators and the commutation relations between them and the tautological classes are computed explicitly not on the whole ring $H^*(X^{\text{ell}})$, but on the subring $H^*_{\text{taut}}(X^{\text{ell}}) \subset H^*(X^{\text{ell}})$ generated by the tautological classes. See Theorems 4.1, 5.5 formulated in a little more general case of sheaves on a surface, and then Corollary 7.1 and Theorem 7.6 adapted to the situation of Higgs bundles.

After we know the commutation relations between the Hecke operators, we follow a certain degeneration procedure, analogous to the passage from the elliptic Cherednik algebra to the trigonometric Cherednik algebra, see Section 6. We arrive at operators satisfying relations which look almost like $\mathcal{H}_2$, see Corollary 7.2 and Proposition 7.3.

Then we show that certain operators form an $\mathfrak{sl}_2$-triple, see 7.7, and moreover, using one triple we create more triples applying certain symmetries, see 7.8. This allows us to obtain a triple $(\mathfrak{e}, \mathfrak{f}, \mathfrak{h})$ for which $\mathfrak{e}$ is ample and conclude that the perverse filtration matches the filtration coming from the $\mathfrak{sl}_2$, see Proposition 8.12. Thus we conclude that the $P = W$ holds for the pure part of the cohomology of the elliptic locus, Proposition 8.13.

In the rest of Section 8 we show how the results for the elliptic locus imply global results. The main idea is that in certain situations the global cohomology injects into the cohomology of the elliptic locus, and the image is the pure part.

1.4. **Some remarks.** The fact that the perverse filtration is controlled by the Hecke operators is reminiscent of the construction in [OY16]. There, the authors consider an action of trigonometric Cherednik algebra on the cohomology of affine Springer fibers by Hecke correspondences, and degenerate it to an action of rational Cherednik algebra by taking the associated graded with respect to the perverse filtration; the latter is inherited from a realization of certain affine Springer fibers as Hitchin fibers. In our situation, the algebra generated by the operators $D_{m,n}$ can be viewed as a global version of the (spherical) trigonometric Cherednik algebra, and the passage to the operators $\widetilde{D}_{m,n}$ in Section 6 can be understood as the rational degeneration.

Perhaps a more conceptual framework for the algebras in this paper is that of cohomological Hall algebras (CoHas) of curves and surfaces, as considered in [Min20, SS20, KV19]. We expect that using the powerful structural results of Davison and Kinjo (work in progress) together with the computation of the zero dimensional CoHa in [MMSV] one should be able to streamline our proof of $P = W$ and, for instance, avoid the usage of the elliptic locus and the parabolic Higgs bundles.

The Higgs bundles appearing here are sometimes called GL_r-Higgs bundles. We expect that our results, including the algebra action and $P = W$ translate to the PGL_r case with only slight modifications.

The original motivation for the $P = W$ conjecture came from the observation that the E-polynomials of character varieties and the conjectural mixed Hodge polynomials have an interesting symmetry, which would be geometrically explained by the so-called *curious*

Poincaré duality or *curious hard Lefschetz theorem*, see [HRV08], [HLRV11]. For the perverse filtration, the corresponding symmetry is explained by the relative hard Lefschetz theorem, see [dCM05], [dCHM12], and Theorem 8.5 here. The curious hard Lefschetz theorem was established in [Mel19] by working on the Betti side. The present proof of $P = W$ now implies another proof of the curious hard Lefschetz theorem.

2. Topological prerequisites

By a *space* we mean an algebraic variety or an algebraic stack X. All stacks appearing in this paper are in fact of the form $\mathcal{X} = X \times B\mathbb{G}_m$ where X is an algebraic variety. We work with the cohomology $H^*(X) = H^*(X, \mathbb{C})$, which is a ring, and the Borel-Moore homology $H_*(X) = H_*(X, \mathbb{C})$, which is a module over $H^*(X)$. The cohomology can be pulled back via an arbitrary map, and this is a ring homomorphism. The Borel-Moore homology can be pushed forward via a proper map and we have a projection formula. The Borel-Moore homology can also be pulled back via open embeddings and this is compatible with the cohomology action.

Pushforwards and pullbacks are functorial. The proper pushforward and the open pullback satisfy base change. There is a more general pullback via a complete intersection map which we do not need.

If X is of pure dimension n, we have the fundamental class $[X] \in H_{2n}(X)$ and therefore a map

$$H^i(X) \to H_{2n-i}(X), \qquad \alpha \to [X] \cap \alpha.$$

In the case when X is smooth, this is an isomorphism. If X is not necessarily smooth, consider a resolution of singularities $\pi : \widetilde{X} \to X$. Then the map above factors as

$$H^i(X) \xrightarrow{\pi^*} H^i(\widetilde{X}) \cong H_{2n-i}(\widetilde{X}) \xrightarrow{\pi_*} H_{2n-i}(X).$$

If X, Y are smooth spaces of dimensions n, m respectively, and if $f : X \to Y$ is proper, then we have the Gysin map in cohomology

$$H^i(X) \cong H_{2n-i}(X) \xrightarrow{f_*} H_{2n-i}(Y) \cong H^{2m-2n+i}(Y).$$

We denote the Gysin map also by f_*. When dealing with smooth spaces we replace Borel-Moore homologies with cohomologies and pushforward maps with Gysin maps without changing anything.

2.1. **Correspondences.** Suppose a space Z is of pure dimension n together with maps $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ such that π_2 is proper. We call such Z a *correspondence* and consider the *induced map*

$$H^i(X) \to H_{2n-i}(Y), \qquad \alpha \to \pi_{2*}([Z] \cap \pi_1^* \alpha).$$

If Y is smooth of dimension m this gives a map $H^i(X) \to H^{2m-2n+i}(Y)$. Notice that Z can be replaced by a resolution of singularities without changing the induced map. If X is proper, we can decompose the induced map as follows:

$$H^i(X) \xrightarrow{\pi_X^*} H^i(X \times Y) \xrightarrow{(\pi_1 \times \pi_2)_*[Z] \cap} H_{2n-i}(X \times Y) \xrightarrow{\pi_{Y*}} H_{2n-i}(Y).$$

Notice that since π_2 is proper, $\pi_1 \times \pi_2$ is also proper. If moreover X and Y are smooth, the class $(\pi_1 \times \pi_2)_*[Z]$ corresponds to a class in $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$, which using the Poincaré duality on X and a choice of basis in $H^*(X), H^*(Y)$ can be identified with a matrix so that the induced map of the correspondence is just the action of this matrix.

2.2. **Purity.** Let X be a smooth space. Consider any smooth compactification $\overline{X} \supset X$. The *pure part* $H^*_{\text{pure}}(X) \subset H^*(X)$ is by definition the image of the restriction map $H^*(\overline{X}) \to H^*(X)$. Given any two smooth compactifications $\overline{X}$, $\overline{X}'$, let $\overline{X}''$ be a resolution of singularities of the closure of X in $\overline{X} \times \overline{X}'$. Then we have a diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{=} & X \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \pi_2 \iota} \\ \overline{X} & \xleftarrow[\pi_1]{} & \overline{X}'' & \xrightarrow{\pi_2} & \overline{X}' \end{array}$$

where the right half is cartesian and therefore by base change we have $\iota^* = (\pi_2 \iota)^* \pi_{2*}$. So we obtain

$$\operatorname{Im}(\pi_1 \iota)^* \subset \operatorname{Im} \iota^* = \operatorname{Im}(\pi_2 \iota)^* \pi_{2*} \subset \operatorname{Im}(\pi_2 \iota)^*,$$

which proves that the pure part does not depend on the choice of the compactification. Considering the situation when only one of $\overline{X}, \overline{X}'$ is smooth shows that for any compactification $\iota : X \to \overline{X}$ we have

$$\operatorname{Im} \iota^* \subset H^*_{\text{pure}}(X) \subset \operatorname{Im} \iota^*_{\text{BM}},$$

where we distinguish between the ordinary restriction map $\iota^* : H^*(\overline{X}) \to H^*(X)$ and the restriction in Borel-Moore homology $\iota^*_{\text{BM}} : H_*(\overline{X}) \to H_*(X) \cong H^*(X)$.

For any map of smooth spaces $f : X \to Y$ pick a smooth compactification of Y, a smooth compactification of X, and resolve singularities in the closure of X in the corresponding product of compactifications to obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\iota_X} & & \downarrow{\iota_Y} \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \end{array}$$

where $i_X : X \to \overline{X}$, $i_Y : Y \to \overline{Y}$ are smooth compactifications. Taking pullbacks implies that $f^* H^*_{\mathrm{pure}}(Y) \subset H^*_{\mathrm{pure}}(X)$.

Consider the canonical map $j : X \to \overline{X} \times_{\overline{Y}} Y$. This map is clearly an open embedding, which shows that any map is a composition of an open embedding and a proper map.

Suppose f is proper. Then j is proper, and since j has Zariski dense image j is surjective. So j is an isomorphism, and the base change implies $f_* H^*_{\mathrm{pure}}(X) \subset H^*_{\mathrm{pure}}(Y)$.

Since the induced map of a correspondence between smooth spaces is the composition of a pullback map and a Gysin map, we have

Proposition 2.1. *The induced map of a correspondence between smooth spaces preserves the pure part. In the case of a correspondence between a proper smooth space X and a smooth Y, the image of the induced map is contained in $H^*_{\mathrm{pure}}(Y)$.*

2.3. **Diagonal class.** Suppose X is smooth of pure dimension n and $\overline{X} \supset X$ is a smooth compactification. The identity map $X \to X$ factors via proper maps $X \to \overline{X} \times X \to X$. Using the projection formula we see that the restriction map $H^*(\overline{X}) \to H^*_{\mathrm{pure}}(X)$ can be realized as the composition

$$H^*(\overline{X}) \xrightarrow{\pi_1^*} H^*(\overline{X} \times X) \xrightarrow{[\Delta]\cap} H^*(\overline{X} \times X) \xrightarrow{\pi_{2*}} H^*_{\mathrm{pure}}(X),$$

where $\Delta \subset \overline{X} \times X$ is the diagonal. Thus the classes in $H^*_{\mathrm{pure}}(X)$ of any Künneth decomposition of $[\Delta]$ span $H^*_{\mathrm{pure}}(X)$.

2.4. **Virtual fundamental class.** Suppose X is smooth of dimension n and $\mathcal{E}$ is an algebraic vector bundle on X of rank r. Then we have the Thom class

$$\tau_{\mathcal{E}} \in H^{2r}(\mathrm{Tot}_{\mathcal{E}}, \mathrm{Tot}_{\mathcal{E}} \setminus X),$$

where $\mathrm{Tot}_{\mathcal{E}}$ is the total space of $\mathcal{E}$. If $s : X \to \mathrm{Tot}_{\mathcal{E}}$ is a section, by pullback we obtain a class in $H^{2r}(X, X \setminus Z) \cong H_{2n-2r}(Z)$, where $Z = X \cap s(X)$ is the zero set of s, which we call the virtual fundamental class and denote $[Z]^{\mathrm{vir}}$. This class of course depends on the realization of Z as the zero set of a vector bundle. In the case when Z has pure dimension $n - r$ and the intersection $X \cap s(X)$ is generically transversal over Z, then we have $[Z]^{\mathrm{vir}} = [Z]$. The image of $[Z]^{\mathrm{vir}}$ under the pushforward $H_*(Z) \to H_*(X)$ is $[X] \cap c_r(\mathcal{E})$.

If X is not necessarily smooth, we take a resolution of singularities $\pi : \widetilde{X} \to X$, pull everything back to $\widetilde{X}$, obtain a virtual class in $H_{2n-2r}(\pi^{-1}(Z))$ and push it forward to $H_{2n-2r}(Z)$ to define the virtual fundamental class whose image in $H_*(X)$ is $[X] \cap c_r(\mathcal{E})$.

3. Hecke correspondences

3.1. **Notations.** Let S be a smooth compact algebraic surface (over $\mathbb{C}$). Denote $H = H^* = H^*(S)$. Denote by $\delta \in H^4$ the class of a point and by $\Delta \in \bigoplus_i H^i \otimes H^{4-i}$ the class of the diagonal.

For any $\alpha \in H^{\mathrm{ev}} = H^0 \oplus H^2 \oplus H^4$ we denote by $\overline{\mathcal{M}}_\alpha$ the moduli stack of coherent sheaves on S of Chern character α, semistable with respect to some stability. Pick an open $\mathcal{M}_\alpha \subset \overline{\mathcal{M}}_\alpha$ and assume it is smooth. One example to keep in mind is the Hilbert scheme. Another example is the moduli space of sheaves supported on reduced and irreducible curves, as in Section 8. The stack $\overline{\mathcal{M}}_\delta$ of length 1 sheaves on S is explicitly given by $\overline{\mathcal{M}}_\delta = S \times B\mathbb{G}_m$, and we let $\mathcal{M}_\delta = \overline{\mathcal{M}}_\delta$. We denote by $\overline{\mathcal{F}}_\alpha$ the universal coherent sheaf on $\overline{\mathcal{M}}_\alpha \times S$ and by $\mathcal{F}_\alpha$ its restriction to $\mathcal{M}_\alpha \times S$.

3.2. **The Hecke correspondence.** Denote by $\mathcal{Z}_\alpha = \mathbb{P}(\mathcal{F}_\alpha)$ the stack parameterizing colength 1 subsheaves of $\mathcal{F}_\alpha$. Assume the classifying map $\mathcal{Z}_\alpha \to \overline{\mathcal{M}}_{\alpha-\delta}$ has image contained in $\mathcal{M}_{\alpha-\delta}$. Then $\mathcal{Z}_\alpha$ admits maps π_1, π_2, π_3 to $\mathcal{M}_\delta$, $\mathcal{M}_{\alpha-\delta}$, $\mathcal{M}_\alpha$, and we have the following diagram:

$$\begin{array}{ccc} & \mathcal{Z}_\alpha & \\ {}^{\pi_1\times\pi_2}\swarrow & & \searrow^{\pi_3} \\ \mathcal{M}_\delta \times \mathcal{M}_{\alpha-\delta} & & \mathcal{M}_\alpha \end{array}$$

The map π_3 is the base change of the corresponding proper map $\overline{\mathcal{Z}}_\alpha \to \overline{\mathcal{M}}_\alpha$, and therefore is proper.

Suppose $\mathcal{M}_\alpha$ is sufficiently nice so that the sheaf $\mathcal{F}_\alpha$ has a resolution by two vector bundles $\mathcal{E}_1 \to \mathcal{E}_0$. Then $\mathcal{Z}_\alpha$ is the zero set of the canonical section of the vector bundle $\mathcal{E}_1^*(1)$ on the projective bundle $\mathbb{P}(\mathcal{E}_0)$, see e.g. [Neg19], Sect. 2.4. So we have the corresponding virtual fundamental class $[Z_\alpha]^{\mathrm{vir}}$.

We denote the action of the Hecke correspondence by T:

$$H^*(\mathcal{M}_\delta) \otimes H^*(\mathcal{M}_{\alpha-\delta}) \to H^*(\mathcal{M}_\alpha), \qquad \xi \otimes \eta \to T(\xi)(\eta) = \pi_{3*}([Z_\alpha]^{\mathrm{vir}} \cap \pi_1^*(\xi) \cap \pi_2^*(\eta)).$$

3.3. **The tautological classes.** Denote by Λ the ring of symmetric functions in infinitely many variables, abstractly we have $\Lambda = \mathbb{C}[p_1, p_2, \ldots]$. For each α we have a ring homomorphism $\Lambda \to H^*(\mathcal{M}_\alpha \times S)$ by sending p_k to $p_k(\mathcal{F}_\alpha) \in H^{2k}(\mathcal{M}_\alpha \times S)$ defined via the generating series identity

$$\mathrm{ch}(\mathcal{F}_\alpha) = \mathrm{rank}\,\mathcal{F}_\alpha + \sum_{n=1}^{\infty} \frac{p_n(\mathcal{F}_\alpha)}{n!}.$$

It is convenient to denote $p_0(\mathcal{F}_\alpha) = \mathrm{rank}\,\mathcal{F}_\alpha$. For any symmetric function f of degree k and any $\gamma \in H^m$ we set

$$f(\gamma) = \pi_{\mathcal{M}_\alpha *}(f(\mathcal{F}_\alpha) \cup \pi_S^* \gamma) \in H^{2k+m-4}(\mathcal{M}_\alpha),$$

where $\pi_{\mathcal{M}_\alpha} : \mathcal{M}_\alpha \times S \to \mathcal{M}_\alpha$, $\pi_S : \mathcal{M}_\alpha \times S \to S$ are the projections.

The classes $f(\gamma)$ generate a subring $H^*_{\mathrm{taut}}(\mathcal{M}_\alpha) \subset H^*_{\mathrm{pure}}(\mathcal{M}_\alpha)$, which we call the *tautological ring*. In many interesting examples we can apply Markman's argument [Mar02] to deduce $H^*_{\mathrm{taut}}(\mathcal{M}_\alpha) = H^*_{\mathrm{pure}}(\mathcal{M}_\alpha)$, see for example Section 8.3.

Notice that we have

$$\Delta(\gamma \otimes 1) = \Delta(1 \otimes \gamma),$$

so we write $\Delta\gamma$ for the both sides. The product of symmetric functions is related to the product in $H^*_{\mathrm{taut}}(\mathcal{M}_\alpha)$ as follows:

$$(fg)(\gamma) = (f \otimes g)(\Delta\gamma),$$

where the right hand side is defined via $(f \otimes g)(\gamma_1 \otimes \gamma_2) = f(\gamma_1)g(\gamma_2)$.

In the case $\alpha = \delta$ we have $\mathcal{F}_\delta = \mathcal{O}_\Delta(1)$, where $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal in $S \times S$ and (1) means twisting by the tautological line bundle on $B\mathbb{G}_m$. The Chern character of $\mathcal{O}_\Delta$ is $\Delta\,\mathrm{Td}_S^{-1}$, and therefore we have

$$\mathrm{ch}(\mathcal{F}_\delta) = \Delta\,\mathrm{Td}_S^{-1} \exp(u),$$

where $u = c_1(\mathcal{O}(1))$ is the equivariant parameter. Denote the Chern roots of the tangent bundle of S by t_1, t_2. Then we have

$$\mathrm{Td}_S^{-1} = \frac{(1 - e^{-t_1})(1 - e^{-t_2})}{t_1 t_2},$$

and

$$p_n(\mathcal{F}_\delta)(\eta) = \frac{u^n - (u - t_1)^n - (u - t_2)^n + (u - t_1 - t_2)^n}{t_1 t_2} \eta.$$

3.4. **The action of the Hecke operators.** Consider the space $\mathcal{Z}_\alpha \times S$ together with maps $\pi_{i4} = \pi_i \times \pi_S$ $(i = 1, 2, 3)$. The projection formula together with the identity $\pi_{14}^* \mathcal{F}_\delta + \pi_{24}^* \mathcal{F}_{\alpha-\delta} = \pi_{34}^* \mathcal{F}_\alpha$ in $K(\mathcal{Z}_\alpha \times S)$ imply the following identity of operators $H^*(\mathcal{M}_{\alpha-\delta}) \to H^*(\mathcal{M}_\alpha)$:

$$[p_n(\eta), T(\xi)] = T\left(\frac{u^n - (u-t_1)^n - (u-t_2)^n + (u-t_1-t_2)^n}{t_1 t_2}\eta\xi\right). \tag{3.1}$$

The commutator on the left hand side is graded, i.e. if both η and ξ are odd, we take the sum instead of the difference.

Having the commutation relations, in order to compute the action of the Hecke operators on the tautological ring $H^*_{\mathrm{taut}}(\mathcal{M}_\alpha)$ it remains to compute the action of any Hecke operator on 1. Recall that $\mathcal{Z}_\alpha$ was presented as the zero set of a section of a vector bundle on $\mathbb{P}(\mathcal{E}_0)$. So the projection π_3 is the composition of proper maps

$$\mathcal{Z}_\alpha \to \mathbb{P}(\mathcal{E}_0) \to \mathcal{M}_\alpha \times S \to \mathcal{M}_\alpha.$$

The map $\mathcal{Z}_\alpha \to \mathcal{M}_\delta = S \times B\mathbb{G}_m$ has two components. The component corresponding to S comes from the projection $\mathbb{P}(\mathcal{E}_0) \to S$. The component corresponding to $B\mathbb{G}_m$ is given by the tautological line bundle L on $\mathbb{P}(\mathcal{E}_0)$. The class $c_1(L)$ thus gets identified with the pullback of u. We will denote $c_1(L) = u$ by abuse of notation. We have

$$H^*(\mathbb{P}(\mathcal{E}_0)) = H^*(\mathcal{M}_\alpha \times S)[u] / \left(u^m - c_1(\mathcal{E}_0)u^{m-1} + \cdots \pm c_m(\mathcal{E}_0)\right),$$

where m is the rank of $\mathcal{E}_0$. The Gysin map $H^*(\mathbb{P}(\mathcal{E}_0)) \to H^*(\mathcal{M}_\alpha \times S)$ can be computed as follows[1]. Represent an element $H^*(\mathbb{P}(\mathcal{E}_0))$ by a polynomial $p(u)$, then expand

$$\frac{p(u)}{u^m - c_1(\mathcal{E}_0)u^{m-1} + \cdots \pm c_m(\mathcal{E}_0)}$$

as a power series in u^{-1}, and take the coefficient of u^{-1}. Indeed, this procedure sends $1, u, \ldots, u^{m-2}$ to zero, u^{m-1} to one, and any polynomial divisible by the defining relation of $H^*(\mathbb{P}(\mathcal{E}_0))$ to zero. The Euler class of the bundle $\mathcal{E}_1^*(1)$ is given by the expression

$$u^{m'} - c_1(\mathcal{E}_1)u^{m'-1} + \cdots \pm c_{m'}(\mathcal{E}_1).$$

So the result of taking an element u^n, multiplying it by the Euler class of $\mathcal{E}_1^*(1)$ and applying the Gysin map to $H^*(\mathcal{M}_\alpha \times S)$ is the coefficient of u^{-n-1} in the expansion of

$$\frac{u^{m'} - c_1(\mathcal{E}_1)u^{m'-1} + \cdots \pm c_{m'}(\mathcal{E}_1)}{u^m - c_1(\mathcal{E}_0)u^{m-1} + \cdots \pm c_m(\mathcal{E}_0)} = \sum_i u^{m'-m-i} s_i(\mathcal{F}_\alpha),$$

[1]See e.g. [Neg19], Prop. 5.19

the generating series of the Segre classes[2]. Taking into account that $m - m' = \operatorname{rank} \mathcal{F}_\alpha$, we obtain that the result is $s_{n+1-\operatorname{rank}\mathcal{F}_\alpha}(\mathcal{F}_\alpha)$. In our notation, the Segre classes correspond to the complete homogeneous symmetric functions. The final formula reads

$$T(\xi u^n)(1) = h_{n+1-\operatorname{rank}\mathcal{F}_\alpha}(\xi) \qquad (\xi \in H,\ n \geq 0). \tag{3.2}$$

Now we explain how (3.1) and (3.2) can be used to write the action of Hecke operators in closed form. Elements of the tautological ring are represented by polynomials in $p_k(\eta)$. Denote the ring of such polynomials by Λ_S. Elements of the tensor product $\Lambda_S \otimes H[u]$ represent operators via

$$f \otimes \xi u^n \to f T(\xi u^n) \qquad (f \in \Lambda_S,\ \xi \in H,\ n \geq 0).$$

The ring Λ_S acts on $\Lambda_S \otimes H[u]$ on the left and on the right, compatibly with the composition of operators. The left action is just the multiplication. To describe the right action, let R be the homomorphism $\Lambda_S \to \Lambda_S \otimes H[u]$ sending $p_n(\eta)$ to

$$p_n(\eta) - \frac{u^n - (u-t_1)^n - (u-t_2)^n + (u-t_1-t_2)^n}{t_1 t_2} \eta.$$

Then, in view of (3.1), the right action of any $f \in \Lambda_S$ is given by the left multiplication by $R(f)$.

Finally, denote the operation $\Lambda_S \otimes H[u] \to \Lambda_S$ sending $f \otimes \xi u^n$ to $f h_n(\xi)$ by Q. This is compatible, up to an index shift, with the application of operators to 1 by (3.2). By construction, we have

$$T(\xi u^n)(f) = Q\left(\xi u^{n-\operatorname{rank}\mathcal{F}_\alpha+1} R(f)\right). \tag{3.3}$$

So we have described the Hecke action in terms of the explicitly defined operators R and Q.

3.5. **The case of an open surface.** Suppose $S_0 \subsetneq S$ is an open subset and $\mathcal{M}_\alpha$ consists of sheaves with support in S_0. Notice that this condition automatically implies that such sheaves are of rank zero. If $\gamma \in \operatorname{Ker} H^*(S) \to H^*(S_0)$, via the long exact sequence in Borel-Moore homology we see that $\pi_S^*\gamma$ comes from $H_*(\mathcal{M}_\alpha \times (S \setminus S_0))$, on which the cap-product by $\operatorname{ch} \mathcal{F}_\alpha$ vanishes. Therefore $f(\gamma)$ depends only on the image of γ in $H^*_{\mathrm{pure}}(S_0)$. Also, the image of the map $\pi_1 : \mathcal{Z}_\alpha \to \mathcal{M}_\delta$ is contained in the open substack $S_0 \times B\mathbb{G}_m \subset \mathcal{M}_\delta$, and therefore the operator $T(\gamma u^n)$ depends only on the image of γ in $H^*_{\mathrm{pure}}(S_0)$. So in this case most of the statements of this section remain valid when the ring $H^*(S)$ is replaced by

[2]This also follows from the general theory of Segre classes, see [Ful98].

the smaller ring $H^*_{\mathrm{pure}}(S_0)$. One caveat to keep in mind is that the value $h_0(\gamma) = 1(\gamma)$ is undefined, but because $\operatorname{rank} \mathcal{F}_\alpha = 0$, the argument to Q in (3.3) has only strictly positive powers of u, so we never need to evaluate $h_0(\gamma)$.

4. Action on the Fock space

Since we have seen that the action of the Hecke operators is realized on the tautological ring by expressions which do not depend on the geometric setting, it makes sense to define Hecke operators abstractly as operators acting on the Fock space Λ_S. Let us summarize the data our constructions depend on.

- Let $\mathbf{k}$ be the base ring. In the classical setup we have $\mathbf{k} = \mathbb{C}$, but when a group G acts on S preserving $\mathcal{M}_\alpha$ we take $\mathbf{k} = H^*(BG)$. Another possibility is a relative situation when S is replaced by a family of surfaces and $\mathbf{k}$ is the cohomology ring of the base. In general we let $\mathbf{k}$ be a graded super-commutative ring.
- H is a graded super-commutative ring over $\mathbf{k}$.
- H is endowed with an augmentation map $\epsilon : H \to \mathbf{k}$, which is an $\mathbf{k}$-linear map of degree -4.
- $\Delta \in H \otimes_{\mathbf{k}} H$ is an element of degree 4 satisfying $(\varepsilon \otimes \mathrm{Id})(\Delta) = (\mathrm{Id} \otimes \varepsilon)(\Delta) = 1$ and $(\xi \otimes 1)\Delta = (1 \otimes \xi)\Delta$ for all $\xi \in H$.
- $c_1 \in H^2$, $c_2 \in H^4$ are fixed. In geometry they correspond to the Chern classes of the tangent bundle of S. We formally write $c_1 = t_1 + t_2$ and $c_2 = t_1 t_2$ and identify symmetric functions in t_1, t_2 with polynomials in c_1, c_2. We have the identity $\Delta^2 = t_1 t_2 \Delta$.

By definition, Λ_S is the polynomial ring over $\mathbf{k}$ generated by $p_n(\xi)$, where $n \geq 1$ and $\xi \in H$ modulo relations

$$p_n(\xi + \xi') = p_n(\xi) + p_n(\xi'), \quad p_n(\lambda \xi) = \lambda p_n(\xi) \qquad (\xi, \xi' \in H,\ \lambda \in \mathbf{k}).$$

The ring Λ_S is also graded super-commutative. The degree of $p_n(\xi)$ is $2n - 4 + i$ for $\xi \in H^i$. We extend the notation $f(\xi)$ to any symmetric function $f \in \Lambda_+$ by requiring

$$1(\xi) = \epsilon(\xi), \qquad (fg)(\xi) = (f \otimes g)(\Delta \xi) \qquad (f, g \in \Lambda).$$

4.1. **The operators.** The operator R is the ring homomorphism $\Lambda_S \to \Lambda_S \otimes_R H[u]$ sending $p_n(\eta)$ to

$$p_n(\eta) - \frac{u^n - (u - t_1)^n - (u - t_2)^n + (u - t_1 - t_2)^n}{t_1 t_2} \eta.$$

The operator $Q : \Lambda_S \otimes_{\mathbf{k}} H[u, u^{-1}] \to \Lambda_S$ is Λ_S-linear and sends ηu^n to $h_n(\eta)$ ($\eta \in H$) if $n \geq 0$ and to zero otherwise.

For each $\xi \in H$, $n \in \mathbb{Z}$ let $T_n(\xi) : \Lambda_S \to \Lambda_S$ be the operator

$$T_n(\xi)(f) = Q\left(\xi u^n R(f)\right).$$

Define also for each $\xi \in H$, $n \geq 0$ an operator $\psi_n(\xi)$ by

$$\psi_n(\xi) = \sum_{1 \leq m \leq n+1} \frac{n!}{m!} p_m(\mathrm{Td}_{n+1-m}\,\xi), \tag{4.1}$$

where $\mathrm{Td}_k \in H^{2k}$ are the coefficients of the Todd class:

$$\sum_{k \geq 0} \mathrm{Td}_k\, x^k = \frac{t_1 t_2 x^2}{(1 - e^{-t_1 x})(1 - e^{-t_2 x})}.$$

Geometrically ψ_n's correspond to the coefficients of $\mathrm{Td}_S\, \mathrm{ch}(\mathcal{F}_\alpha)$, and the generating series of ψ is

$$\sum_{n \geq 0} \frac{x^n \psi_n}{n!} = \left(\sum_{n \geq 1} \frac{x^n p_n}{n!} \right) \frac{t_1 t_2 x}{(1 - e^{-t_1 x})(1 - e^{-t_2 x})},$$

so the commutation relation between ψ and T is the relation (4.2) below. Any p_n for $n \geq 2$ is a linear combination of ψ_m's. The remaining element p_0 commutes with T_n.

The following theorem appears in [MMSV]; for the reader's convenience we sketch the proof here.

Theorem 4.1 ([MMSV])**.** *The action of the operators $\psi_m(\xi), T_n(\xi)$ ($m \geq 0$, $n \in \mathbb{Z}$) on Λ_S satisfies the following relations:*

$$[\psi_m(\eta), T_n(\xi)] = m T_{m+n-1}(\eta\xi), \tag{4.2}$$

$$[T_m(\xi\xi'), T_n(\xi'')] = [T_m(\xi), T_n(\xi'\xi'')], \tag{4.3}$$

$$\begin{aligned}[] [T_m(\xi), T_{n+3}(\xi')] - 3[T_{m+1}(\xi), T_{n+2}(\xi')] + 3[T_{m+2}(\xi), T_{n+1}(\xi')] - [T_{m+3}(\xi), T_n(\xi')] \\ - [T_m(\xi), T_{n+1}(s_2\xi')] + [T_{m+1}(\xi), T_n(s_2\xi')] + \{T_m, T_n\}(s_1 \Delta \xi\xi') = 0,\end{aligned} \tag{4.4}$$

$$\sum_{\pi \in S_3} \pi [T_{m_3}(\xi_3), [T_{m_2}(\xi_2), T_{m_1+1}(\xi_1)]] = 0. \tag{4.5}$$

In (4.4), we used the notation $s_1 = t_1 + t_2$, $s_2 = t_1^2 + t_1 t_2 + t_2^2$, and $\{T_m, T_n\}(s_1 \Delta \xi\xi')$ stands for the super-anticommutator of the operators T_m, T_n whose arguments are taken from the tensor $s_1\Delta\xi\xi' \in H\otimes H$. In (4.5), π permutes both the indices m_i and elements ξ_i. Moreover, when we permute ξ_i the sign changes according to the usual rules depending on the parity of ξ_i.

Having already proved (4.2), a proof of (4.3)–(4.5) is given below. Observe that by (4.2) each set of relations (4.3)–(4.5) is invariant under the adjoint action of all $\psi_m(\eta)$. So it is sufficient to establish each set of relations when applied to $1 \in \Lambda_S$.

4.2. **Explicit generating series.** Consider the generating series

$$\mathcal{T}_\xi(x) = \sum_{n\in\mathbb{Z}} x^n T_n(\xi).$$

Denote

$$\varphi(x) = \sum_{n\geq 0} x^n h_n.$$

We have

$$\mathcal{T}_\xi(x)(1) = \varphi(x)(\xi) = \sum_{n\geq 0} x^n h_n(\xi).$$

Let us evaluate $R\mathcal{T}_\xi(x)(1)$. The generating series of h_n is given by

$$\varphi(x) = \exp\left(\sum_{k\geq 1}\frac{x^k p_k}{k}\right).$$

So we have

$$R\mathcal{T}_\xi(x)(1) = \left(\exp\sum_{k\geq 1}\frac{x^k}{k}\left(1\otimes p_k - \Delta\frac{u^k-(u-t_1)^k-(u-t_2)^k+(u-t_1-t_2)^k}{t_1t_2}\right)\right)(\xi),$$

the expression is first evaluated in $H\otimes H\otimes\Lambda[u,x]$, and (ξ) here means that $\gamma_1\otimes\gamma_2\otimes f$ is sent to $\gamma_1 f(\gamma_2\xi)$. Let us compute

$$\exp\left(-\Delta\sum_{k\geq 1}\frac{x^k(u^k-(u-t_1)^k-(u-t_2)^k+(u-t_1-t_2)^k)}{kt_1t_2}\right). \tag{4.6}$$

We treat t_1, t_2 as formal variables, and since $\Delta^2 = t_1t_2\Delta$, in order to recover the result it is sufficient to set $\Delta = t_1t_2$. This gives

$$\frac{(1-xu)(1-x(u-t_1-t_2))}{(1-x(u-t_1))(1-x(u-t_2))} = 1 - \frac{x^2t_1t_2}{(1-x(u-t_1))(1-x(u-t_2))}.$$

So (4.6) equals to

$$1-\frac{x^2\Delta}{(1-x(u-t_1))(1-x(u-t_2))}.$$

For every $i \in \mathbb{Z}$ we have

$$\sum_{m\in\mathbb{Z}} Q(\xi' y^m u^{m+i}) = y^{-i} \sum_{m\geq 0} y^m h_m(\xi'),$$

so in order to compute $\mathcal{T}_{\xi'}(y)\mathcal{T}_{\xi}(x)(1)$ we simply need to substitute $u = y^{-1}$ and multiply by $\sum_{m\in\mathbb{Z}} y^m h_m(\xi')$. We obtain

$$\mathcal{T}_{\xi'}(y)\mathcal{T}_{\xi}(x)(1) = (\varphi(y)\otimes\varphi(x))\left((1-\Delta\Omega(x,y))\left(\xi'\otimes\xi\right)\right),$$

where

$$\Omega(x,y) = \frac{x^2}{(1-x(y^{-1}-t_1))(1-x(y^{-1}-t_2))} \in H[[x,y^{-1}]].$$

Inductively, we obtain a formula for the product of Hecke operators of arbitrary length

$$\mathcal{T}_{\xi_n}(x_n)\cdots\mathcal{T}_{\xi_1}(x_1)(1) = (\varphi(x_n)\otimes\cdots\otimes\varphi(x_1))\left(\prod_{i<j}(1-\Delta_{i,j}\Omega(x_i,x_j))\left(\xi_n\otimes\cdots\otimes\xi_1\right)\right),$$

where $\Delta_{i,j}$ denotes the corresponding diagonal class in $H^{\otimes n}$.

4.3. **Quadratic relations.** From

$$\mathcal{T}_{\xi'}(y)\mathcal{T}_{\xi}(x)(1) = (\varphi(y)\otimes\varphi(x))\left((1-\Delta\Omega(x,y))(\xi'\otimes\xi)\right)$$

we first deduce

$$[\mathcal{T}_{\xi'}(y),\mathcal{T}_{\xi}(x)](1) = -(\varphi(x)\varphi(y))\left((\Omega(x,y)-\Omega(y,x))\xi'\xi\right),$$

which implies (4.3). Write Ω as follows:

$$\Omega(x,y) = \frac{1}{(x^{-1}-y^{-1}-t_1)(x^{-1}-y^{-1}-t_2)}.$$

Denote $s_1 = t_1+t_2$, $s_2 = t_1^2+t_1t_2+t_2^2$. We notice that the following expression is anti-symmetric as an element of $H[x^{-1}]\otimes H[y^{-1}]$:

$$(1-\Delta\Omega(x,y))\left((x^{-1}-y^{-1})^3 - \frac{1}{2}(s_2\otimes 1 + 1\otimes s_2)(x^{-1}-y^{-1}) + s_1\Delta\right)$$
$$= (x^{-1}-y^{-1})^3 - \frac{1}{2}(s_2\otimes 1 + 1\otimes s_2 + 2\Delta)(x^{-1}-y^{-1}).$$

Therefore the following identity holds:

$$(x^{-1}-y^{-1})^3[\mathcal{T}_{\xi'}(y),\mathcal{T}_{\xi}(x)](1) - (x^{-1}-y^{-1})[\mathcal{T}_{\xi'}(y),\mathcal{T}_{s_2\xi}(x)](1) + \{\mathcal{T}_{\bullet}(y),\mathcal{T}_{\bullet}(x)\}(s_1\Delta\xi\xi')(1),$$

where in the last summand the components of $s_1\Delta$ are inserted into $\bullet$. Unpacking the generating series we obtain that (4.4) holds when applied to 1, and therefore holds in general.

4.4. **Cubic relations.** We have

$$\mathcal{T}_{\xi_3}(x_3)\mathcal{T}_{\xi_2}(x_2)\mathcal{T}_{\xi_1}(x_1)(1) = (\varphi(x_3)\otimes\varphi(x_2)\otimes\varphi(x_1))\Big(\\ (1-\Delta_{12}\Omega(x_1,x_2))(1-\Delta_{23}\Omega(x_2,x_3))(1-\Delta_{13}\Omega(x_1,x_3))(\xi_3\otimes\xi_2\otimes\xi_1)\Big).$$

Using the identity

$$[X_3,[X_2,X_1]] = X_3X_2X_1 - X_3X_1X_2 - X_2X_1X_3 + X_1X_2X_3 = (1-\sigma_{12})(1+\sigma_{13})X_3X_2X_1,$$

where $\sigma_{ij}\in S_3$ denotes the transposition permuting i and j, we have

$$[\mathcal{T}_{\xi_3}(x_3)[\mathcal{T}_{\xi_2}(x_2),\mathcal{T}_{\xi_1}(x_1)]](1) = (\varphi(x_3)\otimes\varphi(x_2)\otimes\varphi(x_1))\Big(K(x_1,x_2,x_3)(\xi_3\otimes\xi_2\otimes\xi_1)\Big),$$

where

$$K(x_1,x_2,x_3) = (1-\sigma_{12})(1+\sigma_{13})\Big((1-\Delta_{12}\Omega(x_1,x_2))(1-\Delta_{23}\Omega(x_2,x_3))(1-\Delta_{13}\Omega(x_1,x_3))\Big).$$

We expand the product and apply the permutations. The first term vanishes:

$$(1-\sigma_{12})(1+\sigma_{13})(1) = 0.$$

The following also vanishes

$$(1-\sigma_{12})(1+\sigma_{13})\left(\Delta_{12}\Omega(x_1,x_2)+\Delta_{13}\Omega(x_1,x_3)+\Delta_{23}\Omega(x_2,x_3)\right)\\ = (1-\sigma_{12})(1+\sigma_{13})(1+\sigma_{23}+\sigma_{12}\sigma_{23})\left(\Delta_{12}\Omega(x_1,x_2)\right),$$

because

$$(1+\sigma_{13})(1+\sigma_{23}+\sigma_{12}\sigma_{23}) = 1+\sigma_{23}+\sigma_{12}\sigma_{23}+\sigma_{13}+\sigma_{13}\sigma_{23}+\sigma_{12} = \sum_{\pi\in S_3}\pi.$$

So we are left with the pairwise products and the triple product. Denote $\Delta_{123}=\Delta_{12}\Delta_{23}=\Delta_{12}\Delta_{13}=\Delta_{13}\Delta_{23}$. We have

$$K(x_1,x_2,x_3) = \Delta_{123}(1-\sigma_{12})(1+\sigma_{13})\Big(\\ \Omega(x_1,x_2)\Omega(x_2,x_3)+\Omega(x_1,x_2)\Omega(x_1,x_3)+\Omega(x_2,x_3)\Omega(x_1,x_3)-t_1t_2\Omega(x_1,x_2)\Omega(x_2,x_3)\Omega(x_1,x_3)\Big).$$

We need to show that the following vanishes:

$$\sum_{\pi\in S_3}\pi[\mathcal{T}_{\xi_3}(x_3)[\mathcal{T}_{\xi_2}(x_2),x_1^{-1}\mathcal{T}_{\xi_1}(x_1)]](1)=(\varphi(x_3)\otimes\varphi(x_2)\otimes\varphi(x_1))\Big(K'(x_1,x_2,x_3)(\xi_3\otimes\xi_2\otimes\xi_1)\Big),$$

where $K'(x_1,x_2,x_3)=\sum_{\pi\in S_3}\pi\left(x_1^{-1}K(x_1,x_2,x_3)\right)$. We will show that K' vanishes. Note that we need to show vanishing of K' as a power series, which is stronger than vanishing of the corresponding rational function, because the Ω terms are series expansions in the region $|x_1|\ll|x_2|\ll|x_3|$ which is not invariant under permutations. We have the following operator identity:

$$\sum_{\pi\in S_3}\pi x_1^{-1}(1-\sigma_{12})(1+\sigma_{13})=\sum_{\pi\in S_3}\pi(x_1^{-1}-2x_2^{-1}+x_3^{-1}).$$

Thus we have

$$K'(x_1,x_2,x_3)=\sum_{\pi\in S_3}\pi\Big((x_1^{-1}-2x_2^{-1}+x_3^{-1})\Big(\Omega(x_1,x_2)\Omega(x_2,x_3)$$
$$+\Omega(x_1,x_2)\Omega(x_1,x_3)+\Omega(x_2,x_3)\Omega(x_1,x_3)-t_1t_2\Omega(x_1,x_2)\Omega(x_2,x_3)\Omega(x_1,x_3)\Big)\Big).$$

By a direct computation we verify that

$$\begin{aligned}&(x_1^{-1}-2x_2^{-1}+x_3^{-1})\left(\Omega(x_1,x_2)\Omega(x_2,x_3)-t_1t_2\Omega(x_1,x_2)\Omega(x_2,x_3)\Omega(x_1,x_3)\right)\\&=(x_3^{-1}-x_1^{-1})\left(\Omega(x_1,x_2)\Omega(x_1,x_3)-\Omega(x_2,x_3)\Omega(x_1,x_3)\right).\end{aligned}$$

Thus we have

$$K'(x_1,x_2,x_3)=\sum_{\pi\in S_3}\pi\Big(2(x_3^{-1}-x_2^{-1})\Omega(x_1,x_2)\Omega(x_1,x_3)+2(x_1^{-1}-x_2^{-1})\Omega(x_2,x_3)\Omega(x_1,x_3)\Big).$$

Now we notice that $\Omega(x_1,x_2)\Omega(x_1,x_3)$ is symmetric in x_2 and x_3 as a power series, and so its product with $(x_3^{-1}-x_2^{-1})$ is antisymmetric in x_2 and x_3 and therefore vanishes after applying $\sum_{\pi\in S_3}\pi$. Similarly, the second contribution vanishes because $\Omega(x_2,x_3)\Omega(x_1,x_3)$ is symmetric in x_1 and x_2 and so its product with $x_1^{-1}-x_2^{-1}$ is antisymmetric. So we have shown that $K'=0$ and therefore we have established (4.5). The proof of Theorem 4.1 is complete.

5. Surface W-algebra

Now suppose S is not necessarily compact, and let $H=H^*(S)$.

Definition 5.1. The *surface W algebra* is the algebra generated by $\psi_m(\pi)$, $T_m(\pi)$, where $m\geq 0$ and π runs over a basis of H, modulo relations (4.2)–(4.5) for $n,m,m_1,m_2,m_3\geq 0$

and the relations

$$[\psi_m(\xi), \psi_n(\eta)] = 0.$$

Corollary 5.2. *In the situation of Sections 3.1–3.2 (and 3.5 if S is not compact) the action of the Hecke correspondences on $\bigoplus_{i\in\mathbb{Z}} H^*_{\mathrm{taut}}(\mathcal{M}_{\alpha+i\delta})$ forms a representation of the surface W-algebra where ψ's act via multiplication operators* (4.1) *and $T_m(\xi) = T(u^m\xi)$.*

Remark 5.3. The operator $T(u_m\xi)$ acts on the tautological ring not via $T_m(\xi)$, but via $T_{m+1-\operatorname{rank}\mathcal{F}_\alpha}(\xi)$. The relations (4.2)–(4.5) are invariant under index shifts in T_m, so the statement holds nevertheless.

Remark 5.4. The corresponding equivariant version of Corollary 5.2 holds when an algebraic group G acts on S and the action extends to the compactification $\overline{S}$ preserving the moduli spaces $\mathcal{M}_\alpha$, so that the sheaves $\mathcal{F}_\alpha$ are G-equivariant.

5.1. **The undeformed case.**

Theorem 5.5 ([MMSV])**.** *Suppose S is such that both $s_2 \in H^4(S)$ and $s_1\Delta \in H^6(S\times S)$ vanish. Then the surface W-algebra is isomorphic to the universal enveloping algebra of the Lie algebra with basis $D_{m,n}(\pi)$, where $(m,n) \in \mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ and π runs over a basis of $H^*(S)$, the Lie bracket is given by*

$$[D_{m,n}(\xi), D_{m',n'}(\eta)] = (nm' - mn')D_{m+m',n+n'-1}(\xi\eta). \tag{5.1}$$

The generators are related as follows:

$$D_{0,n}(\xi) = \psi_n(\xi), \qquad D_{1,n}(\xi) = T_n(\xi).$$

Proof. Again, we provide the proof here for the reader's convenience. A similar statement in the context of deformed $W_{1+\infty}$ algebra was proved in [AS13] and [SV13], Appendix F, but the proof here is different.

It is straightforward to verify that the relations of the surface W-algebra hold in the constructed Lie algebra, which produces a homomorphism in one direction.

We abbreviate $T_m = T_m(1)$ and similarly for $\psi_m, D_{m,n}$. Under our assumptions the relation (4.4) implies

$$[T_{i+3}, T_j] - 3[T_{i+2}, T_{j+1}] + 3[T_{i+1}, T_{j+2}] - [T_i, T_{j+3}] = 0 \qquad (i,j\geq 0). \tag{5.2}$$

To obtain an inverse homomorphism, we set

$$D_{m,n}(\xi) = \frac{n!}{(m+n)!}(-\operatorname{Ad}_{T_0})^m \psi_{m+n}(\xi).$$

To complete the proof it is sufficient to show that these elements satisfy (5.1).

Note that (4.3) implies

$$[T_0, T_0(\xi)] = [T_0(\xi), T_0] = 0,$$

so the following case is clear:

$$[T_0(\xi), D_{m,n}(\eta)] = -nD_{m+1,n-1}(\xi\eta) \quad (n \geq 1). \tag{5.3}$$

Applying $\mathrm{Ad}_{\psi_1(\alpha)}$ we obtain, by induction on m

$$[\psi_1(\alpha), D_{m,n}(\beta)] = mD_{m,n}(\alpha\beta). \tag{5.4}$$

Fix $k \geq 2$ and consider the sequence of commutators $[T_i, T_{k-i}]$ for $i = 0, 1, \ldots, k$. The recursion (5.2) implies that dependence on i is quadratic, hence there exist unique expressions A, B, C such that

$$[T_i, T_j] = A + B(i-j) + Cij \qquad (i + j = k).$$

On the other hand, we have $[T_i, T_j] = -[T_j, T_i]$, which implies $A = 0$ and $C = 0$. Therefore we have

$$[T_i, T_j] = \frac{i-j}{i+j}[T_{i+j}, T_0] = (i-j)D_{2,i+j-1},$$

and since $k \geq 2$ was arbitrary, and since the relation clearly holds when $i + j < 2$, it holds for all $i, j \geq 0$. Applying $\mathrm{Ad}_{\psi_1(\xi)}$ and using (4.3) we obtain

$$2[T_i, T_j(\xi)] = [T_i(\xi), T_j] + [T_i, T_j(\xi)] = 2(i-j)D_{2,i+j-1}(\xi),$$

and more generally

$$[T_i(\xi), T_j(\eta)] = (i-j)D_{2,i+j-1}(\xi\eta).$$

This easily implies

$$[\psi_i(\xi), D_{2,j}(\eta)] = 2iD_{2,i+j-1}(\xi\eta).$$

From the relation $[D_{2,0}, T_0] = [[T_1, T_0], T_0] = 0$, which is a special case of (4.5), and using

$$[D_{2,0}(\xi), T_0(\eta)] = [[T_1(\xi), T_0], T_0(\eta)] = [[T_1(\xi), T_0(\eta)], T_0] = [D_{2,0}(\xi\eta), T_0] = [D_{2,0}, T_0(\xi\eta)]$$

we obtain more generally

$$[D_{2,0}(\xi), T_0(\eta)] = 0,$$

and by induction on m, having already established the base case $m = 0$, we have

$$[D_{2,0}(\xi), D_{m,n}(\eta)] = -2nD_{m+1,n-1}(\xi\eta) \quad (n \geq 1). \tag{5.5}$$

From this we deduce

$$[T_1(\xi), D_{m,n}(\eta)] = (m-n)D_{m+1,n}(\xi\eta),$$

and then

$$[\psi_2(\xi), D_{m,n}(\eta)] = 2mD_{m,n+1}(\xi\eta).$$

Now suppose we have shown that (5.1) holds for some fixed m, n and arbitrary m', n' under the assumption $n' \geq 1$. Applying Ad_{ψ_2} we see that it also holds for $m, n+1$. Similarly, if (5.1) holds for $n = 0$ and fixed m under the assumption $n' \geq 1$, then applying Ad_{T_1} we deduce it for $m+1$. Starting with $m = 1, n = 0$ we show the statement for $n = 0$, arbitrary $m \geq 1$, and then for arbitrary n. So (5.1) holds when $m \geq 1$ and $n' \geq 1$, and by symmetry whenever $m' \geq 1$ and $n \geq 1$. The case $m = m' = 0$ being clear, it remains to establish the case $n = n' = 0$.

Suppose we know that $[T_0, D_{m,0}(\xi)] = 0$. Applying Ad_{T_0} to the identity

$$[D_{m,0}(\xi), D_{m'-1,1}(\eta)] = -mD_{m+m'-1,0}(\xi\eta)$$

we obtain

$$-[D_{m,0}(\xi), D_{m',0}(\eta)] = -m[T_0, D_{m+m'-1,0}(\xi\eta)].$$

Suppose we also know that $[T_0, D_{m',0}(\eta)] = 0$. Applying Ad_{T_0} to the identity

$$[D_{m-1,1}(\xi), D_{m',0}(\eta)] = m'D_{m+m'-1,0}(\xi\eta)$$

we obtain

$$-[D_{m,0}(\xi), D_{m',0}(\eta)] = m'[T_0, D_{m+m'-1,0}(\xi\eta)].$$

Combining these identities we see that if the claim $[T_0, D_{n,0}(\xi)] = 0$ holds for $n = m$ and $n = m'$ $(m, m' \geq 1)$, then it also holds for $n = m + m' - 1$, and moreover we have

$$[D_{m,0}(\xi), D_{m',0}(\eta)] = 0.$$

Starting from the base cases $n = 1, 2$ we deduce by induction that $[T_0, D_{n,0}(\xi)] = 0$ holds for all n, and therefore the identity $[D_{m,0}(\xi), D_{m',0}(\eta)] = 0$ also holds for all m, m'. □

Remark 5.6. The assumptions of the theorem are satisfied in the following cases:

- S is an abelian surface,
- S is a K3 surface without a point,
- S is the total space of a line bundle over a curve.

5.2. **The general case.** Let now S be arbitrary and denote the corresponding algebra by W. Notice that W is graded by placing $T_n(\xi)$ in degree 1 and $\psi_n(\xi)$ in degree 0. We denote the degree m part by W_m. Let $F_\bullet$ be the smallest filtration on W such that for all $m, n \in \mathbb{Z}$ and $\xi \in H$ we have

- $\psi_n(\xi) \in F_n$,
- $T_n(\xi) \in F_n$,
- $F_m F_n \subset F_{m+n}$,
- $[F_m, F_n] \subset F_{m+n-1}$.

Explicitly, this filtration is defined as follows. By a *Lie word* we mean a combination of Lie brackets applied to the generators of W. By an *expression* we mean a product of Lie words. We assign *weight* to any Lie word by summing up the indices of the generators and subtracting the number of brackets. Then we assign weight to any expression by adding up the weights of the Lie words. Then F_n is the span of expressions of weight $\leq n$. Indeed, the filtration cannot be smaller than this, and the filtration clearly satisfies the first three requirements. The fourth requirement holds because of the non-commutative Leibnitz rule:

$$[f_1 f_2 \cdots f_k, g_1 g_2 \cdots g_{k'}] = \sum_{i,j} g_1 \cdots g_{j-1} f_1 \cdots f_{i-1} [f_i, g_j] f_{i+1} \cdots f_k g_{j+1} \cdots g_{k'}.$$

We denote by $F_{m,n}$ the degree m part of F_n. We have

Proposition 5.7. *$F_{0,-1} = 0$, and for any $m > 0$ $F_{m,-m} = 0$.*

Proof. We need to show that the weight of any non-zero expression of degree m is at least $-m+1$ if $m > 0$ and at least 0 if $m = 0$. We prove the claim by induction on the length of the expression. The generators have weight at least 0, so they clearly satisfy the claim. Let us prove the claim for Lie words. Take a Lie word of the form $[f, g]$ for two Lie words f, g of degrees m, m' and weights n, n'. The weight of $[f, g]$ is $n + n' - 1$. If $m, m' \geq 1$, then $n \geq -m + 1$, $n' \geq -m' + 1$ and therefore $n + n' - 1 \geq m + m - 1$. Now suppose one of m, m' is zero, say $m = 0$. Since the subalgebra of degree zero is super-commutative, f cannot be a Lie bracket, so we have $f = \psi_n(\xi)$ for some $\xi \in H$. Since $\psi_0(\xi)$ is central, we have $n \geq 1$, and therefore $n + n' - 1 \geq n'$. Since $m + m' = m'$ we are done showing the claim for Lie words. Now for a product of Lie words the degrees and the weights add up, so the statement is clear. □

Theorem 5.8. *There exist elements $D_{m,n}(\xi) \in W$ $(m, n \geq 0, \xi \in H)$ such that*

- *$D_{0,n}(\xi) = \psi_n(\xi)$, $D_{1,n}(\xi) = T_n(\xi)$,*
- *$F_{-1} = 0$ and F_N is spanned by products $D_{m_1,n_1}(\xi_1) \cdots D_{m_1,n_k}(\xi_k)$ satisfying $n_1 + \cdots + n_k \leq N$.*

- $[D_{m,n}(\xi), D_{m',n'}(\eta)] = (nm' - mn')D_{m+m',n+n'-1}(\xi\eta) \mod F_{n+n'-3}$

Proof. Consider the associated graded algebra

$$\operatorname{Gr} W = \bigoplus_{n\in\mathbb{Z}} F_n/F_{n-1}.$$

By construction, $\operatorname{Gr} W$ has two operations: a commutative product and a Lie bracket of degree -1. Also by construction $\operatorname{Gr} W$ is spanned by products of Lie words in the generators. Consider the Lie algebra generated by the generators. In it the relations (4.2), (4.3), (4.5) hold, and in place of relation (4.4) we have the undeformed relation

$$[T_m(\xi), T_{n+3}(\xi')] - 3[T_{m+1}(\xi), T_{n+2}(\xi')] + 3[T_{m+2}(\xi), T_{n+1}(\xi')] - [T_{m+3}(\xi), T_n(\xi')] = 0. \tag{5.6}$$

So this Lie algebra is isomorphic to the Lie subalgebra of the W algebra with vanishing $s_1\Delta$ and s_2. By Theorem 5.5 we see that the Lie algebra is spanned by $D_{m,n}(\xi)$ ($m, n \geq 0$, $\xi \in H$) and therefore $\operatorname{Gr} W$ is generated as a commutative algebra by $D_{m,n}(\xi)$. In particular, we see that $F_n/F_{n-1} = 0$ for $n < 0$. Together with Proposition 5.7 this implies $F_{-1} = 0$. We also have that F_N/F_{N-1} is spanned by products $D_{m_1,n_1}(\xi_1)\cdots D_{m_1,n_k}(\xi_k)$ satisfying $n_1 + \cdots + n_k = N$. This implies that if we arbitrarily lift $D_{m,n}(\xi)$ to W for $m \geq 2$, the first two claims of the Theorem are satisfied. At this point only a weaker version of the last claim is satisfied, namely the identity holds modulo $F_{n+n'-2}$.

In order to improve the identity, consider a finer associated graded algebra

$$\widetilde{\operatorname{Gr} W} = \bigoplus_{n\in\mathbb{Z}} F_n/F_{n-2}.$$

It is equipped with a Lie bracket of degree -1, and still the images of the generators satisfy (4.2), (4.3), (4.5) and the undeformed relation (5.6). So we can apply Theorem 5.5 to the Lie-subalgebra of $\widetilde{\operatorname{Gr} W}$ generated by the images of the generators to deduce the existence of $D_{m,n}(\xi) \in F_n/F_{n-2}$ satisfying the undeformed relations (5.1). Lifting these $D_{m,n}(\xi)$ to W we obtain elements satisfying all the required properties. □

Corollary 5.9. *The elements* $q_m(\xi) = D_{m,0}(\xi)$, $L_m(\xi) = D_{m,1}(\xi)$ *(*$m \geq 0$, $\xi \in H$*) and* $\mathfrak{d} = \frac{1}{2}\psi_2(1)$ *satisfy relations*

$$[q_m(\xi), q_n(\eta)] = 0, \quad [L_m(\xi), q_n(\eta)] = nq_{m+n}(\xi\eta), \quad [L_m(\xi), L_n(\eta)] = (n-m)L_{m+n}(\xi\eta),$$

$$[\mathfrak{d}, q_m(\xi)] = mL_m(\xi).$$

In the case of the Hilbert scheme, one can find these relations in [Leh99].

6. Degeneration and the $\mathfrak{sl}_2$

We return to the geometric setup of Section 3. Denote $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, $\alpha_i \in H^{2i}$. Suppose $\alpha_0 = 0$ so that $\operatorname{rank} \mathcal{F}_\alpha = 0$. The class α_1 has to be a class of an effective divisor, otherwise coherent sheaves with Chern class α do not exist. For any $\eta \in H^2$ we have $\psi_0(\eta) = c_1(\mathcal{F}_\alpha) \cdot \eta = \alpha_1 \cdot \eta$. In particular, there exists an effective divisor class $\eta \in H^2$ such that $\psi_0(\eta) > 0$. We fix such η and denote

$$r = \psi_0(\eta) = \alpha_1 \cdot \eta > 0.$$

6.1. **Polynomiality of $\psi_k(\xi)$.** Assume $\mathcal{M}_\alpha$ parameterizes sheaves whose endomorphism ring is $\mathbb{C}$. Assume there exists a tautological sheaf F_α on $M_\alpha \times S$ for the coarse moduli space M_α. By Lemma 3.10 in [Hei10] we have $\mathcal{M}_\alpha \cong M_\alpha \times B\mathbb{G}_m$.

The isomorphism is constructed as follows. Let $F_\alpha(1)$ be the corresponding sheaf on $M_\alpha \times B\mathbb{G}_m \times S$. This induces a classifying map $\phi : M_\alpha \times B\mathbb{G}_m \to \mathcal{M}_\alpha$ together with an isomorphism $(\phi \times \mathrm{Id}_S)^* \mathcal{F}_\alpha \cong F_\alpha(1)$. Consider the projection $\pi : \mathcal{M}_\alpha \to M_\alpha$. Since F_α is a tautological sheaf, we have a line bundle L on $\mathcal{M}_\alpha$ such that $\mathcal{F}_\alpha \cong \pi^*_{\mathcal{M}_\alpha} L \otimes \pi^* F_\alpha$. The line bundle L induces a classifying map $\ell : \mathcal{M}_\alpha \to B\mathbb{G}_m$, and so we obtain a map $\pi \times \ell : \mathcal{M}_\alpha \to M_\alpha \times B\mathbb{G}_m$ such that $(\pi \times \ell \times \mathrm{Id}_S)^* F_\alpha(1) \cong \mathcal{F}_\alpha$. The maps ϕ and $\pi \times \ell$ are mutually inverse isomorphisms. Note that the isomorphism depends on the choice of F_α and therefore is not canonical.

By pullback we have an isomorphism

$$\phi^* : H^*(\mathcal{M}_\alpha) \to H^*(M_\alpha \times B\mathbb{G}_m) = H^*(M_\alpha)[v].$$

By construction we have

$$e^{-v} (\phi \times \mathrm{Id}_S)^* \operatorname{ch}(\mathcal{F}_\alpha) \in H^*(M_\alpha \times S),$$

because it is the Chern character of the sheaf F_α. In particular, this implies

$$\frac{1}{2} \phi^* p_2(\eta) - vr \in H^2(M_\alpha)$$

and we have

$$(\phi \times \mathrm{Id}_S)^* e^{-\frac{p_2(\eta)}{2r}} \operatorname{ch}(\mathcal{F}_\alpha) \in H^*(M_\alpha \times S).$$

Let us write

$$e^{-\frac{p_2(\eta)}{2r}} \operatorname{ch}(\mathcal{F}_\alpha) \operatorname{Td}_S = \sum_{n=0}^{\infty} \frac{Q_n}{n!},$$

where $Q_n \in H^{2n+2}(\mathcal{M}_\alpha \times S)$. Then we have

$$\operatorname{ch}(\mathcal{F}_\alpha) \operatorname{Td}_S = \sum_{n=0}^{\infty} \frac{Q_n}{n!} e^{\frac{p_2(\eta)}{2r}},$$

and denoting $Q_n(\xi) = \pi_{\mathcal{M}_\alpha *}(Q_n \cup \xi)$ we obtain

$$\psi_n(\xi) = \sum_{i=0}^{n} \binom{n}{i} Q_i(\xi) \left(\frac{p_2(\eta)}{2r}\right)^{n-i}.$$

Since M_α is a finite dimensional algebraic variety, we have that $Q_n(\xi)$ is nilpotent when its cohomological degree is positive, and it vanishes for sufficiently large n. For $n = 0, 1$, $\xi = \eta$ we have

$$\psi_0(\eta) = Q_0(\eta),$$
$$\psi_1(\eta) = Q_1(\eta) + \frac{p_2(\eta)}{2r} Q_0(\eta),$$
$$\frac{\psi_1(\eta)}{\psi_0(\eta)} = \frac{p_2(\eta)}{2r} + \frac{Q_1(\eta)}{Q_0(\eta)}.$$

The element $\frac{Q_1(\eta)}{Q_0(\eta)} \in H^2(\mathcal{M}_\alpha)$ is nilpotent. Hence $\frac{p_2(\eta)}{2r}$ can be replaced by $\frac{\psi_1(\eta)}{r}$. Abusing notation, we denote the coefficients in the new expansion again by $Q_i(\xi)$ and arrive at the following statement:

Proposition 6.1. *Denote $y = \frac{\psi_1(\eta)}{r}$, where $r = \psi_0(\eta) = \eta \cdot \alpha_1$. Under the above assumptions there exist a family of elements $Q_n(\xi) \in H^*_{\mathrm{taut}}(\mathcal{M}_\alpha)$ ($n \geq 0$, $\xi \in H$) such that*

$$\psi_n(\xi) = \sum_{i=0}^{n} \binom{n}{i} Q_i(\xi) y^{n-i},$$

Moreover, $Q_n(\xi)$ is zero for large enough n and $Q_n(\xi)$ is nilpotent whenever its cohomological degree is positive.

6.2. **Identifying cohomologies of different moduli spaces.** Suppose $\eta \in H^2$ is the class of a divisor such that $\eta\alpha_1 > 0$. Write $\eta = c_1(L)$ for a line bundle L on S. Assume tensoring by L and L^{-1} preserves the moduli spaces $\mathcal{M}_\alpha$. Then tensoring by L induces an isomorphism $H^*(\mathcal{M}_\alpha) \cong H^*(\mathcal{M}_{\alpha(1+\eta)})$ preserving the tautological rings. Note that $\alpha(1+\eta) = \alpha + \delta(\eta \cdot \alpha_1)$. Under these assumptions we have

Proposition 6.2. *The Hecke operator $T_0(\eta)$ induces an isomorphism of tautological rings $H^*_\tau(\mathcal{M}_\alpha) \cong H^*_\tau(\mathcal{M}_{\alpha+\delta})$.*

Proof. We show that $T_0(\eta)$ is surjective. That will imply

$$\dim H^i_{\text{taut}}(\mathcal{M}_\alpha) \geq \dim H^i_{\text{taut}}(\mathcal{M}_{\alpha+\delta}) \geq \cdots \geq \dim H^i_{\text{taut}}(\mathcal{M}_{\alpha+(\eta\cdot\alpha_1)\delta}),$$

and from the isomorphism $H^i_{\text{taut}}(\mathcal{M}_\alpha) \cong H^i_{\text{taut}}(\mathcal{M}_{\alpha+(\eta\cdot\alpha_1)\delta})$ we deduce that all the inequalities are equalities, and so $T_0(\eta)$ is an isomorphism.

Let f be any polynomial in the generators $\psi_i(\xi)$, or more generally any polynomial in $D_{m,n}(\xi)$. By Theorem 5.8 we know that some power n of $\operatorname{Ad}_{T_0(\eta)}$ annihilates f. We show by induction on n that $f(1)$ is in the image of $T_0(\eta)$. The induction base is not needed because the case $f = 0$ is trivial. Now suppose we have shown the claim for $[T_0(\eta), f]$, so that we have

$$[T_0(\eta), f](1) = T_0(\eta)(g)$$

for some $g \in H^*_\tau(\mathcal{M}_\alpha)$. Then we have

$$T_0(\eta)(f(1) - g) = f T_0(\eta)(1) = (\eta \cdot \alpha_1) f(1),$$

and since $\eta \cdot \alpha_1 \neq 0$ we see that $f(1)$ is in the image of $T_0(\eta)$. □

6.3. **Polynomiality of $D_{m,n}(\xi)$.** Recall that

$$q_0(\eta) = \psi_0(\eta) = \eta \cdot \alpha_1 = r > 0, \qquad y = \frac{\psi_1(\eta)}{r} = \frac{L_0(\eta)}{r}.$$

Denote also

$$X = \frac{T_0(\eta)}{r} = \frac{q_1(\eta)}{r}.$$

Using the operator X we can identify all the spaces $H^*(\mathcal{M}_{\alpha+\mathbb{Z}\delta})$. Choose explicit lifts $D_{m,n}(\xi)$ by

$$D_{m,n}(\xi) = \frac{n!}{(m+n)!}(-\operatorname{Ad}_{q_1(1)})^m \psi_{m+n}(\xi).$$

Proposition 6.3. *For any $\xi \in H$ and $n \geq 0$ the sequence of operators $X^{-m} D_{m,n}(\xi)$ depends polynomially on m. In the case $n = 0$ the non-constant coefficients of the polynomial are nilpotent.*

Proof. Using Proposition 6.1 we write

$$\psi_{m+n}(\xi) = \sum_{i=0}^{N} \binom{m+n}{i} y^{m+n-i} Q_i(\xi),$$

where N is fixed. Applying the non-commutative Leibnitz rule we obtain

$$D_{m,n}(\xi) = \frac{n!}{(m+n)!}\sum_{i,j}\binom{m+n}{i}\binom{m}{j}\left((-\operatorname{Ad}_{q_1(1)})^{m-j}y^{m+n-i}\right)\left((-\operatorname{Ad}_{q_1})^{j}Q_i(\xi)\right).$$

Since we have $Q_i(\xi)\in F_i$, the summands for $j>i$ vanish. So the sum over i and j is finite and we only need to verify that for each i,j,n the sequence

$$X^{-m}\frac{n!}{(m+n)!}\binom{m+n}{i}\binom{m}{j}(-\operatorname{Ad}_{q_1(1)})^{m-j}y^{m+n-i}$$

depends polynomially on m. Multiplying by X^j and using the fact that $\operatorname{Ad}_{q_1(1)}$ and the operator of left multiplication by X commute we are reduced to showing polynomiality of

$$\frac{n!}{(m+n)!}\binom{m+n}{i}\binom{m}{j}\left(-X^{-1}\operatorname{Ad}_{q_1(1)}\right)^{m-j}y^{m+n-i}. \tag{6.1}$$

We have $[y,q_1(1)] = r^{-1}[L_0(\eta),q_1(1)] = r^{-1}q_1(\eta) = X$ and similarly $[y,X] = r^{-2}q_1(\eta^2)$, which commutes with X and y. For any $k\geq 0$ we have

$$\begin{aligned}-X^{-1}\operatorname{Ad}_{q_1(1)}y^k &= X^{-1}\sum_{i=0}^{k-1}y^iXy^{k-1-i} = ky^{k-1}+\binom{k}{2}X^{-1}[y,X]y^{k-2}\\ &= \left(\frac{\partial}{\partial y}+\frac{1}{2}X^{-1}[y,X]\frac{\partial^2}{\partial^2 y}\right)y^k.\end{aligned}$$

Denote $z=\frac{1}{2}X^{-1}[y,X]$. The binomial formula gives

$$(-X^{-1}\operatorname{Ad}_{q_1(1)})^{m-j}y^{m+n-i} = \sum_{k=0}^{\min(m-j,n-i+j)}\binom{m-j}{k}z^k\frac{(m+n-i)!}{(n-i+j-k)!}y^{n-i+j-k}.$$

The upper bound of the summation can be replaced by $n-i+j$. As a result we obtain that (6.1) is a linear combination of operators of the form $z^ky^{n-i+j-k}$, and each such operator enters with coefficient

$$\frac{n!}{(m+n)!}\binom{m+n}{i}\binom{m}{j}\binom{m-j}{k}\frac{(m+n-i)!}{(n-i+j-k)!} = \frac{n!}{i!(n-i+j-k)!}\binom{m}{j}\binom{m-j}{k},$$

which is clearly a polynomial in m.

Consider now the special case $n=0$. Then the inequalities $i\leq j$ and $0\leq k\leq n-i+j$ imply $k=0$ and $i=j$. So the result is a linear combination of terms of the form

$$\frac{n!}{i!}\binom{m}{i}\left((-\operatorname{Ad}_{q_1})^iQ_i(\xi)\right).$$

Only the terms with $i \geq 1$ contribute to the non-constant coefficients of the polynomial. If $Q_i(\xi)$ has positive cohomological degree, then it is nilpotent, say $Q_i(\xi)^M = 0$. Applying $(-\operatorname{Ad}_{q_1})^{Mi}$ we obtain $((-\operatorname{Ad}_{q_1})^i Q_i(\xi))^M = 0$. If $Q_i(\xi)$ has cohomological degree zero (which only happens when $i = 1$, $\xi \in H^0$), then $(-\operatorname{Ad}_{q_1})^i Q_i(\xi)$ has negative cohomological degree, and therefore is nilpotent. □

6.4. **Construction of $\widetilde{D}_{m,n}(\xi)$ and relations.** Let θ be the linear term in the polynomial expansion of $X^{-k} q_k(\eta)$. Then θ is nilpotent and $e^{k\theta}$, $e^{-k\theta}$ are polynomials in k. So Proposition 6.3 remains valid when X^{-k} is replaced by u^{-k}, where

$$u = X e^{\theta/q_0(\eta)}.$$

This choice of u ensures that the linear term in the expansion of $u^{-k} q_k(\eta)$ vanishes. Let us write

$$D_{m,n}(\xi) = u^m \sum_i \frac{m^i}{i!} \widetilde{D}_{i,n}(\xi).$$

We also denote $\widetilde{q}_i(\xi) = D_{m,0}(\xi)$. Denote by $\widetilde{W}$ the algebra generated by the operators $\widetilde{D}_{m,n}(\xi)$. Denote by $\widetilde{F}_n$ the filtration obtained by placing $\widetilde{D}_{m,n}(\xi)$ in weight n.

Proposition 6.4. *Denote $r = q_0(\eta)$. The following relation holds modulo $\widetilde{F}_{m+m'-3}$:*

$$[\widetilde{D}_{m,n}(\xi), \widetilde{D}_{m',n'}(\xi')] = (nm' - mn')\widetilde{D}_{m+m'-1,n+n'-1}(\xi\xi')$$
$$-r^{-1} nm' \widetilde{D}_{m,n-1}(\xi\eta)\widetilde{D}_{m'-1,n'}(\xi') \pm r^{-1} mn' \widetilde{D}_{m',n'-1}(\xi'\eta)\widetilde{D}_{m-1,n}(\xi),$$
$$-2r^{-2}\binom{n}{2}\binom{m'}{2}\widetilde{D}_{m,n-2}(\xi\eta^2)\widetilde{D}_{m'-2,n'}(\xi') \pm 2r^{-2}\binom{m}{2}\binom{n'}{2}\widetilde{D}_{m',n'-2}(\xi'\eta^2)\widetilde{D}_{m-2,n}(\xi),$$

where the sign is $-$ if ξ and ξ' are odd.

Proof. First we find commutation relations involving u. From $[\psi_1(\xi), q_k(\eta)] = k q_k(\xi\eta)$ we obtain

$$k u^{k-1}[\psi_1(\xi), u] \sum_i \widetilde{q}_i(\eta)\frac{k^i}{i!} + u^k \sum_i [\psi_1(\xi), \widetilde{q}_i(\eta)]\frac{k^i}{i!} = k u^k \sum_i \widetilde{q}_i(\xi\eta)\frac{k^i}{i!}.$$

Dividing by u^k and taking the linear terms in k we obtain

$$q_0(\eta) u^{-1}[\psi_1(\xi), u] + [\psi_1(\xi), \widetilde{q}_1(\eta)] = q_0(\xi\eta),$$

and since by construction we have $\widetilde{q}_1(\eta) = 0$ we obtain

$$[\psi_1(\xi), u] = u\frac{q_0(\xi\eta)}{r} = u\frac{\psi_0(\xi\eta)}{r}.$$

Since $\mathfrak{d} = \frac{D_{0,2}(1)}{2}$ is a differential operator of order two with respect to the algebra generated by $q_i(\xi)$, we have

$$[\mathfrak{d}, u^k] = ku^{k-1}[\mathfrak{d}, u] + \binom{k}{2} u^{k-2}[[\mathfrak{d}, u], u].$$

Applying this to $[\mathfrak{d}, q_k(\eta)] = kL_k(\eta)$ we obtain

$$\left(ku^{k-1}[\mathfrak{d}, u] + \binom{k}{2} u^{k-2}[[\mathfrak{d}, u], u]\right) \sum_i \widetilde{q}_i(\eta)\frac{k^i}{i!} + u^k \sum_i [\mathfrak{d}, \widetilde{q}_i(\eta)]\frac{k^i}{i!} = ku^k \sum_i \widetilde{L}_i(\eta)\frac{k^i}{i!}.$$

Dividing by u^k and collecting the coefficient of k produces

$$u^{-1}[\mathfrak{d}, u] - \frac{1}{2}u^{-2}[[\mathfrak{d}, u], u] = \frac{L_0(\eta)}{r} = \frac{\psi_1(\eta)}{r},$$

and applying $-\operatorname{Ad}_u$ we obtain

$$u^{-2}[[\mathfrak{d}, u], u] = \frac{\psi_0(\eta^2)}{r^2}, \qquad u^{-1}[\mathfrak{d}, u] = \frac{\psi_1(\eta)}{r} + \frac{1}{2}\frac{\psi_0(\eta^2)}{r^2}.$$

Using $[\mathfrak{d}, D_{m,n}(\xi)] = mD_{m,n+1}(\xi) \mod F_{n-1}$ we obtain, by induction on n

$$[D_{m,n}(\xi), u] = u\left(n\frac{D_{m,n-1}(\xi\eta)}{r} + \binom{n}{2}\frac{D_{m,n-2}(\xi\eta^2)}{r^2}\right) \mod F_{n-3},$$

and by induction on k

$$[D_{m,n}(\xi), u^k] = u^k\left(kn\frac{D_{m,n-1}(\xi\eta)}{r} + k^2\binom{n}{2}\frac{D_{m,n-2}(\xi\eta^2)}{r^2}\right) \mod F_{n-3}.$$

Since this holds for all $m \geq 1$, the same identity is true for $\widetilde{D}$ in place of D.

Finally we expand the identity (5.1) (modulo $F_{n+n'-3}$)

$$(m'n - mn')u^{m+m'} \sum_i \widetilde{D}_{i,n+n'-1}(\xi)\frac{(m+m')^i}{i!} = \sum_{i,j} \frac{m^i}{i!}\frac{m'^j}{j!}\left[u^m \widetilde{D}_{i,n}(\xi), u^{m'}\widetilde{D}_{j,n'}(\xi')\right].$$

The commutator expands as follows:

$$u^{m+m'}\Bigg([D_{i,n}(\xi), D_{j,n}(\xi')] + \left(\frac{m'n}{r}\widetilde{D}_{i,n-1}(\xi\eta) + \frac{m'^2}{r^2}\binom{n}{2}\widetilde{D}_{i,n-2}(\xi\eta^2)\right) D_{j,n'}(\xi')$$
$$\mp \left(\frac{mn'}{r}\widetilde{D}_{j,m-1}(\xi'\eta) + \frac{m^2}{r^2}\binom{n'}{2}\widetilde{D}_{j,n'-2}(\xi'\eta^2)\right) D_{i,n}(\xi)\Bigg),$$

where the sign is + is ξ and ξ' are odd. Collecting monomials in m, m' we obtain the claim. ∎

Corollary 6.5. *The operators* $\widetilde{q}_k(\xi)$ *together with the operator* $\psi_2(1)$ *generate the algebra* $\widetilde{W}$.

6.5. **Reduction with respect to a Weyl algebra.** We have

$$[\psi_1(\eta), \widetilde{q}_1(1)] = [\widetilde{D}_{0,1}(\eta), \widetilde{D}_{1,0}(1)] = D_{0,0}(\eta) - r^{-1} D_{0,0}(\eta^2) D_{0,0}(1) = r,$$

because $D_{0,0}(1) = \psi_0(1)$ vanishes by degree reasons. Thus the elements $y = \frac{\psi_1(\eta)}{r}$ and $\widetilde{q}_1(1)$ form a Weyl algebra. We denote $\partial_y = -\widetilde{q}_1(1)$ so that we have $[\partial_y, y] = 1$. The operator ∂_y has cohomological degree -2, and therefore acts locally nilpotently. This implies a canonical decomposition

$$H^*(\mathcal{M}_\alpha) = H^*(\mathcal{M}_\alpha)_{\mathrm{red}}[y],$$

where $H^*(\mathcal{M}_\alpha)_{\mathrm{red}} = \operatorname{Ker} \partial_y$. Because $\partial_y \in \widetilde{F}_0$, the operator Ad_{∂_y} acts locally nilpotently on operators, and therefore subalgebra $\widetilde{W}$ also has a canonical decomposition

$$\widetilde{W} = \widetilde{W}'[y],$$

where $\widetilde{W}'$ consists of operators commuting with ∂_y. We have

Proposition 6.6. *The operator* Ad_y *acts locally nilpotently on* $\widetilde{W}$.

Proof. In view of the relations, the operator acts locally nilpotently on the quotient $\widetilde{F}_m/\widetilde{F}_{m-2}$. This means that for any element $f \in F_m$ some power of Ad_y sends it to $\widetilde{F}_{m-2}$. So we obtain the claim by induction on m. □

So we can repeat the construction to obtain canonical decomposition

$$\widetilde{W}' = W_{\mathrm{red}}[\partial_y],$$

where W_{red} consists of operators commuting with y and ∂_y. The algebra W_{red} clearly preserves $H^*(\mathcal{M}_\alpha)_{\mathrm{red}}$.

For any element $f \in H^*(\mathcal{M}_\alpha)$ the constant term of the expansion $f = \sum_i y^i f_i$ will be denoted by f_{red}. Explicitly, we have

$$f_{\mathrm{red}} = \sum_i \frac{y^i (-\partial_y)^i}{i!} f.$$

Similarly, for $f \in \widetilde{W}$ the constant term will be denoted by f_{red} and we have

$$f_{\mathrm{red}} = \sum_i \frac{1}{i!j!} y^i \left(\mathrm{Ad}_y^j (-\mathrm{Ad}_{\partial_y})^i f\right) \partial_y^j.$$

These expansions can be computed with the help of the identities

$$[\partial_y, \widetilde{D}_{m,n}(\xi)] = n\widetilde{D}_{m,n-1}(\xi) \mod F_{n-3},$$
$$[y, \widetilde{D}_{m,n}(\xi)] = mr^{-1}\widetilde{D}_{m-1,n}(\xi\eta) - mr^{-2}\psi_0(\eta^2)\widetilde{D}_{m-1,n}(\xi) \mod F_{n-2}.$$

Remark 6.7. We have two ways of viewing $H^*(\mathcal{M}_\alpha)_{\mathrm{red}}$. One as the subspace $H^*(\mathcal{M}_\alpha)_{\mathrm{red}} = \operatorname{Ker}\partial_y$, another as the quotient space $H^*(\mathcal{M}_\alpha)_{\mathrm{red}} = \operatorname{Coker} y$. Since $\psi_m(\xi)$ commutes with y, we have

$$\psi_m(\xi)_{\mathrm{red}} = \sum_i \frac{y^i(-\operatorname{Ad}_{\partial_y})^i}{i!}\psi_m(\xi),$$

and the action of $\psi_m(\xi)_{\mathrm{red}}$ is compatible with that of $\psi_m(\xi)$ via the quotient identification. On the other hand, $\widetilde{q}_m(\xi)$ commutes with ∂_y, and so we have

$$\widetilde{q}_m(\xi)_{\mathrm{red}} = \sum_i \left(\operatorname{Ad}_y^i \widetilde{q}_m(\xi)\right)\frac{\partial_y^i}{i!}.$$

Since ∂_y annihilates $H^*(\mathcal{M}_\alpha)_{\mathrm{red}}$, the action coincides with the induced action of $\widetilde{q}_m(\xi)$ with respect to the subspace identification. For any operator f we have the induced action from $\operatorname{Ker}\partial_y$ to $\operatorname{Coker} y$, and this action coincides with f_{red}.

6.6. **$\mathfrak{sl}_2$-triple.** Denote by $\mathfrak{h}$ the operator $-\widetilde{D}_{1,1}(1)_{\mathrm{red}}$. Explicitly, we have

$$\mathfrak{h} = -\widetilde{D}_{1,1}(1) - (y - r^{-2}\psi_0(\eta^2)\psi_1(1))\partial_y.$$

Taking into account that $\widetilde{D}_{1,0}(\eta) = \widetilde{q}_1(\eta) = 0$, we obtain

$$[\widetilde{D}_{1,1}(1), \widetilde{D}_{m,n}(\xi)] = (m-n)\widetilde{D}_{m,n}(\xi) + r^{-1}n\widetilde{D}_{m,n-1}(\eta\xi)\psi_1(1) \mod \widetilde{F}_{n-2}.$$

In particular, we have

$$[\mathfrak{h}, -\widetilde{q}_m(\xi)] = -m\widetilde{q}_m(\xi) - [y, \widetilde{q}_m(\xi)]\partial_y,$$

and taking the constant terms we obtain

$$[\mathfrak{h}, \widetilde{q}_m(\xi)_{\mathrm{red}}] = -m\widetilde{q}_m(\xi)_{\mathrm{red}}.$$

So we see the following

Proposition 6.8. *The operator* $\operatorname{Ad}_{\mathfrak{h}}$ *restricted to* $\widetilde{F}_0 \cap W_{\mathrm{red}}$ *is diagonalizable and its eigenvalues are non-positive integers.*

Consider

$$\mathfrak{d}_{\mathrm{red}} = \frac{D_{0,2}(1)_{\mathrm{red}}}{2} = \frac{\psi_2(1)}{2} - y\psi_1(1).$$

Notice that $\psi_1(1) \in H^0(\mathcal{M}_\alpha)$ is a scalar, so it commutes with all operators. Since $\mathfrak{d}_{\mathrm{red}}$ commutes with y and ∂_y we have

$$[\mathfrak{h}, \mathfrak{d}_{\mathrm{red}}] = -\left[\widetilde{D}_{1,1}(1), \frac{\psi_2(1)}{2} - y\psi_1(1)\right] = \psi_2(1) - 2y\psi_1(1) - r^{-2}\psi_0(\eta^2)\psi_1(1)^2 = 2\mathfrak{d}_{\mathrm{red}} \mod \widetilde{F}_0.$$

Write

$$[\mathfrak{h}, \mathfrak{d}_{\mathrm{red}}] = 2\mathfrak{d}_{\mathrm{red}} + \sum_{i \geq 0} f_i,$$

for f_i satisfying $[\mathfrak{h}, f_i] = -if_i$ and set

$$\mathfrak{e} = \mathfrak{d}_{\mathrm{red}} + \sum_i \frac{f_i}{i+2}.$$

Then we have

$$[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{d}_{\mathrm{red}} + \sum_{i \geq 0} \left(1 - \frac{i}{i+2}\right) f_i = 2\mathfrak{e}.$$

Finally, let $\mathfrak{f} = -\frac{\widetilde{q}_2(1)_{\mathrm{red}}}{2}$. We already know that $[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e}$ holds. From

$$[\mathfrak{e}, q_2(1)] = [\mathfrak{d}_{\mathrm{red}}, q_2(1)] = 2\widetilde{D}_{1,1}(1) - 2r^{-1}\widetilde{D}_{0,1}(\eta)\widetilde{D}_{1,0}(1) + 2r^{-2}\psi_0(\eta)^2\widetilde{D}_{1,0}(1)\psi_1(1)$$

by taking the constant terms we obtain $[\mathfrak{e}, q_2(1)_{\mathrm{red}}] = -2\mathfrak{h}$, and hence we have

$$[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h}.$$

To summarize, we have shown the following:

Theorem 6.9. *The Lie algebra $\mathfrak{sl}_2$ acts on $H^*_{\mathrm{taut}}(\mathcal{M}_\alpha)_{\mathrm{red}} = H^*_{\mathrm{taut}}(\mathcal{M}_\alpha)/\psi_1(\eta)H^*_{\mathrm{taut}}(\mathcal{M}_\alpha) = H^*_{\mathrm{taut}}(M_\alpha)$ via operators $\mathfrak{e}, \mathfrak{f}, \mathfrak{h}$. Define a filtration by setting P_i to be the sum of the $\mathfrak{h}$-eigenspaces with eigenvalues $\leq i$. Then we have the following properties.*

(i) The tautological classes satisfy $\psi_k(\xi)P_i \subset P_{i+k}$.
(ii) The Hecke operators satisfy $q_k(\xi)P_i \subset P_i$.
(iii) On the associated graded, the operator $\psi_2(1)$ coincides with $2\mathfrak{e}$ and therefore satisfies the hard Lefschetz property: $\psi_2(1)^k : P_{-k}/P_{-k-1} \to P_k/P_{k-1}$ is an isomorphism.
(iv) For any sequence of open subspaces $\mathcal{M}'_{\alpha+\mathbb{Z}\delta} \subset \mathcal{M}_{\alpha+\mathbb{Z}\delta}$ preserved by the Hecke correspondences the restriction maps commute with the $\mathfrak{sl}_2$-action and therefore are strictly compatible with the filtration.

7. Higgs bundles

Let C be a smooth projective curve of genus $g \geq 0$ over $\mathbb{C}$. The cohomology ring $H = H^*(C, \mathbb{C})$ has basis

$$\Pi = \{1, \gamma_1, \dots, \gamma_{2g}, \omega\},$$

in which the product is given by

$$1 \cdot x = x \cdot 1 = x \quad (x \in \Pi), \qquad \gamma_i \gamma_{i+g} = -\gamma_{i+g}\gamma_i = \omega \quad (1 \leq i \leq g),$$

and all other products of generators vanish.

Fix an effective simple divisor D. A D-twisted Higgs bundle is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a vector bundle on C and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega(D)$. A Higgs bundle can also be viewed as a sheaf of modules over the sheaf of algebras $\bigoplus_{n \geq 0} \Omega(D)^{-n}$, whose spectrum is the total space of the line bundle $\Omega(D)$, denoted $\operatorname{Tot}_C \Omega(D)$, which is a smooth surface. Taking the projectivization we obtain a smooth compact surface $S \supset \operatorname{Tot}_C \Omega(D)$. So we view Higgs bundles as coherent sheaves on S. The functor sending $(\mathcal{E}, \theta)$ to $\mathcal{E}$ is geometrically realized as the pushforward along the projection map $\pi_C : S \to C$.

7.1. Algebras in the twisted case.

We apply the construction of Section 3 to the following moduli space. Fix $r > 0$. For each $d \in \mathbb{Z}$, let $\mathcal{M}^{\mathrm{ell}}_{r,d,D}$ be the moduli stack of D-twisted Higgs bundles of rank r and degree d whose spectral curve is reduced and irreducible. We call it the elliptic moduli stack. Denote by $M^{\mathrm{ell}}_{r,d,D}$ the corresponding coarse moduli space.

Note that we have $H^*_{\mathrm{pure}}(\operatorname{Tot}_C \Omega(D)) = H^*(\operatorname{Tot}_C \Omega(D)) = H^*(C) = H$. By Theorems 4.1 and 5.5 we have

Corollary 7.1. *For fixed r and D, the (super) Lie algebra with basis $D_{m,n}(\pi)$ $(m, n \geq 0$, $\pi \in \Pi)$ and the Lie bracket*

$$[D_{m,n}(\pi), D_{m',n'}(\pi')] = (m'n - mn') D_{m+m',n+n'-1}(\pi\pi')$$

acts on the direct sum

$$\bigoplus_{d \in \mathbb{Z}} H^*_{\mathrm{taut}}(\mathcal{M}^{\mathrm{ell}}_{r,d,D}),$$

where $D_{0,n}(\pi)$ for $\pi \in H^i$ is the operator of cup product by the tautological class $\psi_n(\pi) \in H^{2n-2+i}(\mathcal{M}^{\mathrm{ell}}_{r,d,D})$ and

$$D_{1,n}(\pi) = T_n(\pi) : H^j(\mathcal{M}^{\mathrm{ell}}_{r,d-1,D}) \to H^{j+2n-2+i}(\mathcal{M}^{\mathrm{ell}}_{r,d,D})$$

is the Hecke operator.

In Section 6 using this action we constructed an action of a different algebra acting on the cohomology of a single moduli stack, i.e. $H^*(\mathcal{M}^{\rm ell}_{r,d,D})$ for fixed d. The relations are given in Proposition 6.4. Note that because of Remark 5.6, we are in the undeformed case, i.e. the relations in Theorem 5.5 hold not only up to $F_{n+n'-3}$, but on the nose, and so the relations in Proposition 6.4 also hold on the nose. The constructions of Section 6 depended on a choice of η. We choose $\eta = \omega$ so that we have $\alpha_1 \cdot \eta = r$, i.e. the number r from Section 6 is precisely the rank. We summarize this in

Corollary 7.2. *For fixed r, d and D, the algebra generated by $\widetilde{D}_{m,n}(\pi)$ ($m, n \geq 0$, $\pi \in \Pi$) modulo relations*

$$[\widetilde{D}_{m,n}(\pi), \widetilde{D}_{m',n'}(\pi')] = (nm' - mn')\widetilde{D}_{m+m'-1,n+n'-1}(\pi\pi')$$
$$-r^{-1}nm'\widetilde{D}_{m,n-1}(\pi\omega)\widetilde{D}_{m'-1,n'}(\pi') + r^{-1}mn'\widetilde{D}_{m',n'-1}(\pi'\omega)\widetilde{D}_{m-1,n}(\pi)$$

acts on $H^*_{\rm taut}(\mathcal{M}^{\rm ell}_{r,d,D})$, where $\widetilde{D}_{0,n}(\pi) = D_{0,n}(\pi)$ for $\pi \in H^i$ is the operator of cup product by the tautological class $\psi_n(\pi) \in H^{2n-2+i}(\mathcal{M}^{\rm ell}_{r,d,D})$.

Notice that when π, π' are both odd the extra terms vanish automatically, so we do not need to worry about the sign change.

As the next step, we constructed the so-called reduction. We denoted $y = \frac{\psi_1(\omega)}{r}$ and $\partial_y = -\widetilde{D}_{1,0}(1)$ and observed that $[\partial_y, y] = 1$ holds. This implied a canonical decomposition

$$H^*_{\rm taut}(\mathcal{M}^{\rm ell}_{r,d,D}) = H^*_{\rm taut}(\mathcal{M}^{\rm ell}_{r,d,D})_{\rm red}[y],$$

where $H^*_{\rm taut}(\mathcal{M}^{\rm ell}_{r,d,D})_{\rm red}$ is the subring annihilated by ∂_y. Observe that

$$[\partial_y, \psi_n(\pi)] = [\widetilde{D}_{0,n}(\pi), \widetilde{D}_{1,0}(1)] = n\widetilde{D}_{0,n-1}(\pi) = n\psi_{n-1}(\pi).$$

So the coefficients of the generating series

$$\sum_{n=0}^{\infty} \frac{\psi_n(\pi)_{\rm red}}{n!} = e^{-y} \sum_{n=0}^{\infty} \frac{\psi_n(\pi)}{n!}$$

generate $H^*_{\rm taut}(\mathcal{M}^{\rm ell}_{r,d,D})_{\rm red}$. On the other hand, we see that the ring $H^*_{\rm taut}(\mathcal{M}^{\rm ell}_{r,d,D})_{\rm red}$ consists of those polynomials in the tautological classes which are invariant under the operation of twisting the tautological sheaf $\mathcal{F}$ by a line bundle. It is thus clear that we have

$$H^*_{\rm taut}(\mathcal{M}^{\rm ell}_{r,d,D})_{\rm red} = H^*_{\rm taut}(M^{\rm ell}_{r,d,D}).$$

The subspace $H^*_{\mathrm{taut}}(\mathcal{M}^{\mathrm{ell}}_{r,d,D})_{\mathrm{red}}$ is preserved by the operators $\widetilde{D}_{m,n}(\pi)_{\mathrm{red}}$ which commute with both ∂_y and y (see Section 6.5). We explicitly compute relations between these operators.

Proposition 7.3. *For fixed r, d and D, the algebra generated by $\widetilde{D}_{m,n}(\pi)_{\mathrm{red}}$ ($m,n \geq 0$, $\pi \in \Pi$) modulo relations*

$$[\widetilde{D}_{m,n}(\pi)_{\mathrm{red}}, \widetilde{D}_{m',n'}(\pi')_{\mathrm{red}}] = (nm' - mn')\widetilde{D}_{m+m'-1,n+n'-1}(\pi\pi')_{\mathrm{red}}$$
$$-r^{-1}nm'\left(\widetilde{D}_{m,n-1}(\pi\omega)_{\mathrm{red}}\widetilde{D}_{m'-1,n'}(\pi')_{\mathrm{red}} + \widetilde{D}_{m'-1,n'}(\pi'\omega)_{\mathrm{red}}\widetilde{D}_{m,n-1}(\pi)_{\mathrm{red}}\right)$$
$$+r^{-1}mn'\left(\widetilde{D}_{m-1,n}(\pi\omega)_{\mathrm{red}}\widetilde{D}_{m',n'-1}(\pi')_{\mathrm{red}} + \widetilde{D}_{m',n'-1}(\pi'\omega)_{\mathrm{red}}\widetilde{D}_{m-1,n}(\pi)_{\mathrm{red}}\right).$$

acts on $H^*_{\mathrm{taut}}(M^{\mathrm{ell}}_{r,d,D})$, where $\widetilde{D}_{0,n}(\pi) = \psi_n(\pi)_{\mathrm{red}}$ for $\pi \in H^i$ is the operator of cup product by the tautological class $\psi_n(\pi)_{\mathrm{red}} \in H^{2n-2+i}(M^{\mathrm{ell}}_{r,d,D})$.

Proof. Let f be an operator. We have a unique decomposition

$$f = \sum_{i,j} y^i f_{i,j} \partial_y^j,$$

where $f_{i,j}$ commutes with both y and ∂_y. Recall that by definition $f_{\mathrm{red}} = f_{0,0}$. Applying Ad_y, Ad_{∂_y} to the above identity we obtain

$$f_{i,j} = \left(\frac{1}{i!j!}\mathrm{Ad}^i_{\partial_y}(-\mathrm{Ad}_y)^j f\right)_{\mathrm{red}}.$$

Now for any two operators f, g we write

$$fg = \sum_{i,j} y^i f_{i,j}\partial_y^j \sum_{i',j'} y^{i'} g_{i',j'}\partial_y^{j'}.$$

The terms with $i > 0$, $j' > 0$ or $i' \neq j$ do not contribute to the constant term. The remaining terms with $i = j' = 0$ and $i' = j$ contribute $j! f_{0,j} g_{j,0}$. So we have

$$(fg)_{\mathrm{red}} = \sum_i i! f_{0,i} g_{i,0} = \sum_i \frac{1}{i!}\left((-\mathrm{Ad}_y)^i f\right)_{\mathrm{red}} (\mathrm{Ad}^i_{\partial_y} g)_{\mathrm{red}}.$$

Applying this identity to each term in Corollary 7.2 and using

$$[\partial_y, \widetilde{D}_{m,n}(\pi)] = n\widetilde{D}_{m,n-1}(\pi), \qquad [y, \widetilde{D}_{m,n}(\pi)] = \frac{m}{r}\widetilde{D}_{m-1,n}(\pi\omega)$$

we obtain the required identities. □

The relations in Proposition 7.3 look cumbersome, but the symmetry between m and n is evident. One can guess that in order to make relations simpler, one should "undo" the operation of reduction for the operators $\widetilde{D}_{m,n}$. Indeed, let us introduce a formal variable x and define the "unreduced" operators $\widetilde{D}_{m,n}(\pi)_{\mathrm{unred}}$ by

$$\widetilde{D}_{m,n}(\pi)_{\mathrm{unred}} = \sum_{i,j} x^i \binom{m}{i}\binom{n}{j}(-r)^{-j}\widetilde{D}_{m-i,n-j}(\pi\omega^j)\partial_x^j.$$

Then we have an action of the Lie algebra with bracket

$$[\widetilde{D}_{m,n}(\pi)_{\mathrm{unred}}, \widetilde{D}_{m',n'}(\pi')_{\mathrm{unred}}] = (m'n - mn')\widetilde{D}_{m+m'-1,n+n'-1}(\pi\pi')_{\mathrm{unred}}$$

on the space $H^*_{\mathrm{taut}}(\mathcal{M}^{\mathrm{ell}}_{r,d,D})[x]$. The subalgebra spanned by elements of the form $D_{m,n}(1)$ is isomorphic to the Lie algebra $\mathcal{H}_2$ of polynomial Hamiltonian vector fields on the plane.

Corollary 7.4. *The Lie algebra $\mathcal{H}_2$ of polynomial Hamiltonian vector fields on the plane acts on*

$$H^*_{\mathrm{taut}}(\mathcal{M}^{\mathrm{ell}}_{r,d,D})[x] = H^*_{\mathrm{taut}}(M^{\mathrm{ell}}_{r,d,D})[x,y]$$

in such a way that $\psi_n(1)$ corresponds to the operator with Hamiltonian y^n.

7.2. **Algebras in the parabolic case.** In order to extend the construction to the parabolic case we consider the parabolic elliptic moduli space $\mathcal{M}^{\mathrm{parell}}_{r,d,D}$ resp. $M^{\mathrm{parell}}_{r,d,D}$, which is the moduli stack resp. coarse moduli space parameterizing D-twisted Higgs bundles of rank r and degree d such that the spectral curve is reduced and irreducible, the residue of the Higgs field over each point of D has distinct eigenvalues, together with a choice of an order of the eigenvalues. There is a map

$$\mathcal{M}^{\mathrm{parell}}_{r,d,D} \to \mathcal{M}^{\mathrm{ell}}_{r,d,D},$$

which is an unramified covering of degree $(r!)^{|D|}$ over its image. Everything in the previous section applies to these moduli spaces, so that we have actions of the respective algebras on $H^*_{\mathrm{taut}}(\mathcal{M}^{\mathrm{parell}}_{r,d,D})$, $H^*_{\mathrm{taut}}(M^{\mathrm{parell}}_{r,d,D})$, which are the subrings generated by the tautological classes of the form $\psi_k(\pi)$, $\psi_k(\pi)_{\mathrm{red}}$ respectively. However, we have new pure classes, as well as new operations.

For each $p \in D$ and each $i = 1, \ldots, r$ we denote by $L_{p,i}$ the line bundle whose fiber is the i-the eigenspace of the residue of the Higgs field θ at p. Let $y_{p,i} = c_1(L_{p,i}) \in H^2(\mathcal{M}^{\mathrm{parell}}_{r,d,D})$. Let $\mathrm{Mod}_{i,p} : \mathcal{M}^{\mathrm{parell}}_{r,d,D} \to \mathcal{M}^{\mathrm{parell}}_{r,d-1,D}$ denote the map sending a Higgs bundle $(\mathcal{E}, \theta)$ to the Higgs bundle

$$\mathrm{Ker}(\mathcal{E} \to \delta_p),$$

where δ_p is the skyscraper sheaf at p and the map $\mathcal{E} \to \delta_p$ is the projection to the i-th eigenspace of the residue of θ. Let $X_{i,p} = \mathrm{Mod}_{i,p}^*$,

$$X_{i,p} : H^*(\mathcal{M}^{\text{parell}}_{r,d-1,D}) \to H^*(\mathcal{M}^{\text{parell}}_{r,d,D}).$$

Some commutation relations between the new generators are given as follows:

Proposition 7.5. *For each $p \in D$, $1 \le i, i' \le r$, $n \ge 0$, $\pi \in H^{>0}$ we have*

$$[X_{i,p}, y_{i',p'}] = 0, \quad [\psi_n(1), X_{i,p}] = n y_{i,p}^{n-1} X_{i,p}, \quad [\psi_n(\pi), X_{i,p}] = 0, \quad [y_{i,p}, T_0(1)] = X_{i,p}.$$

Proof. The first three relations simply measure how the isomorphisms $\mathrm{Mod}_{i,p}$ affect the tautological bundle. The last relation is less trivial. Recall that $T_0(1)$ comes from the Hecke correspondence $\mathcal{Z}$ which parameterizes Higgs subsheaves $\mathcal{E}' \subset \mathcal{E}$ of colength 1. Recall that $\pi_2 : \mathcal{Z} \to \mathcal{M}^{\text{parell}}_{r,d-1,D}$, $\pi_3 : \mathcal{Z} \to \mathcal{M}^{\text{parell}}_{r,d,D}$ denote the projection maps. At p over each point of $\mathcal{Z}$ we have a map

$$\mathcal{E}'(p) \to \mathcal{E}(p).$$

This map is compatible with the action of $\mathrm{res}_p\,\theta$, and therefore we obtain a map of line bundles $\pi_2^* L_{i,p} \to \pi_3^* L_{i,p}$. Hence the difference of the Chern classes is supported at the zero set of this map. It is clear that the zero set is the graph of the map $\mathrm{Mod}_{i,p}$. So we see that the operator

$$y_{i,p} T_0(1) - T_0(1) y_{i,p}$$

is realized by a correspondence where the space is the graph of $\mathrm{Mod}_{i,p}$, but the virtual fundamental class is still unknown. Denote $(y_{i,p} T_0(1) - T_0(1) y_{i,p})(1)$ by c_p. Then we have

$$y_{i,p} T_0(1) - T_0(1) y_{i,p} = c_p X_{i,p}.$$

The constant c_p is the same for all i because of the permutation group action which relabels the eigenvalues of $\mathrm{res}_p\,\theta$. To determine c_p notice that the sum $\sum_{i=1}^r y_{i,p}$ is the first Chern class of the fiber of $\mathcal{E}$ at p, viewed as a bundle of rank r over the moduli space. This is also given by $\psi_1(\omega)$. So we have

$$c_p \sum_{i=1}^{p} X_{i,p} = [\psi_1(\omega), T_0(1)] = T_0(\omega).$$

Applying both sides to the fundamental class we obtain $r c_p = r$, and therefore $c_p = 1$. $\square$

Let $H_{r,D}$ be the ring extension of H obtained by adding generators p_i of degree 2 where $p \in D$, $i = 1, \dots, r$ modulo relations

$$\sum_i p_i = \omega, \qquad p_i H^{>0} = 0, \qquad p_i q_j = 0 \qquad (p, q \in D).$$

Set for each $n \geq 0$

$$\psi_n(p_i) = y_{i,p}^n, \qquad T_n(p_i) = y_{i,p}^n X_{i,p}.$$

This assignment is compatible with the relations $\sum_i p_i = \omega$ because

$$\sum_{i=1}^r y_{i,p}^n = \psi_n(\omega),$$

since $\bigoplus_{i=1}^r L_{i,p}$ is the fiber of $\mathcal{E}$ at p, and

$$\sum_{i=1}^r y_{i,p}^n X_{i,p} = \sum_{i=1}^r \frac{1}{n+1}[y_{i,p}^{n+1}, T_0(1)] = \frac{1}{n+1}[\psi_{n+1}(\omega), T_0(1)] = T_n(\omega).$$

Now we have, for $\pi \in H$,

$$[y_{i,p}^n, T_{n'}(\pi)] = \frac{1}{n'+1}[y_{i,p}^n, [\psi_{n'+1}(\pi), T_0(1)]] = \frac{n}{n'+1} y_{i,p}^{n-1}[\psi_{n'+1}(\pi), X_{i,p}].$$

For $\pi \in H^{>0}$ this vanishes. For $\pi = 1$ we obtain $n y_{i,p}^{n+n'-1} X_{i,p} = n T_{n+n'-1}(p_i)$. Also we have

$$[\psi_n(1), y_{i,p}^{n'} X_{p,i}] = n y_{i,p}^{n+n'-1} X_{p,i} = n T_{n+n'-1}(p_i).$$

So we see that the commutation relations

$$[\psi_n(\pi), T_{n'}(\pi')] = n T_{n+n'-1}(\pi\pi')$$

hold for all $\pi, \pi' \in H_{r,D}$. Finally, we have

$$T_n(p_i)(1) = y_{p,i}^n.$$

In order to interpret this properly, note that the diagonal class, coming from the surface S equals $\omega \otimes \omega$, and therefore annihilates p_i. This implies that for any two symmetric functions f, g of positive degree we have $(fg)(p_i) = 0$. Hence we have

$$h_{n+1}(p_i) = \frac{p_{n+1}(p_i)}{n+1} = \psi_n(p_i) = y_{p,i}^n.$$

The commutation relations between ψ and T, together with the values of T on 1 completely determine the action of T on the polynomial ring generated by $\psi_n(\pi)$ for all n and $\pi \in H_{r,D}$. So the action is as described in Section 4, and in particular the results of that section apply, describing the commutation relations between T's, and Theorems 4.1 and 5.5 also remain valid.

Theorem 7.6. *Corollaries 7.1, 7.2, 7.4 and Proposition 7.3 are valid for the spaces* $\mathcal{M}^{\mathrm{parell}}_{r,d,D}$, $M^{\mathrm{parell}}_{r,d,D}$ *when* H *is replaced by* $H_{r,D}$ *and the tautological ring is enlarged by adding the classes* $y_{p,i}$.

From now on we denote by $H^*_{\mathrm{taut}}(\mathcal{M}^{\mathrm{parell}}_{r,d,D})$, $H^*_{\mathrm{taut}}(M^{\mathrm{parell}}_{r,d,D})$ the rings obtained by enlarging the tautological ring in the old sense with $y_{p,i}$.

7.3. **$\mathfrak{sl}_2$-triples.** In Section 6.6 we have seen that the algebra generated by $\widetilde{D}_{m,n}(\pi)_{\mathrm{red}}$ contains an $\mathfrak{sl}_2$-triple. In the case of Higgs bundles we are in the undeformed situation by Remark 5.6, so the construction simplifies. Recall that by construction the element $\widetilde{D}_{1,0}(\pi)$ vanishes. Hence also $\widetilde{D}_{1,0}(\pi)_{\mathrm{red}} = 0$. We automatically have $y_{\mathrm{red}} = 0$ and $(\partial_y)_{\mathrm{red}} = 0$. The element $D_{0,1}(1) = \psi_1(1) \in H^0(\mathcal{M})$ is the Euler characteristic of the bundle of rank r and degree d. By choosing d appropriately we arrive at the case $D_{0,1}(1) = 0$. So we have

$$\widetilde{D}_{0,1}(1)_{\mathrm{red}} = \widetilde{D}_{1,0}(1)_{\mathrm{red}} = \widetilde{D}_{0,1}(\omega)_{\mathrm{red}} = \widetilde{D}_{1,0}(\omega)_{\mathrm{red}}.$$

This implies that all the extra terms in Proposition 7.3 in the case of the operators $\widetilde{D}_{1,1}(1)_{\mathrm{red}}$, $\widetilde{D}_{0,2}(1)_{\mathrm{red}}$, $\widetilde{D}_{2,0}(1)_{\mathrm{red}}$ vanish. So we have

$$[\widetilde{D}_{2,0}(1)_{\mathrm{red}}, \widetilde{D}_{m,n}(\pi)_{\mathrm{red}}] = -2nD_{m+1,n-1}(\pi)_{\mathrm{red}},$$
$$[\widetilde{D}_{1,1}(1)_{\mathrm{red}}, \widetilde{D}_{m,n}(\pi)_{\mathrm{red}}] = (m-n)D_{m,n}(\pi)_{\mathrm{red}},$$
$$[\widetilde{D}_{0,2}(1)_{\mathrm{red}}, \widetilde{D}_{m,n}(\pi)_{\mathrm{red}}] = 2mD_{m-1,n+1}(\pi)_{\mathrm{red}}.$$

In particular, we obtain

Proposition 7.7. *The operators* $\mathfrak{e} = \frac{1}{2}\widetilde{D}_{0,2}(1)_{\mathrm{red}}$, $\mathfrak{h} = -\widetilde{D}_{1,1}(1)$, $\mathfrak{f} = \frac{1}{2}\widetilde{D}_{2,0}(1)$ *form an* $\mathfrak{sl}_2$*-triple.*

We will need a slightly more general statement. The $\mathfrak{sl}_2$-triple constructed above will be called the *original* $\mathfrak{sl}_2$-triple. Let M be $M^{\mathrm{parell}}_{r,d,D}$ or $M^{\mathrm{ell}}_{r,d,D}$. The space $H^2_{\mathrm{taut}}(M)$ is spanned by $\psi_2(1)$, $\psi_1(\pi)$ for $\pi \in H^2_{r,D}$ and the products $\psi_1(\gamma_i)\psi_1(\gamma_j)$, where we recall that γ_i form a basis of H^1.

Proposition 7.8. *Suppose* $\alpha \in H^2_{\mathrm{taut}}(M)$ *and write*

$$\alpha = A\left(\psi_2(1) + \frac{1}{r}\sum_{i=1}^{g}\psi_1(\gamma_i)\psi_1(\gamma_{i+g})\right) + \sum_{i<j} B_{i,j}\psi_1(\gamma_i)\psi_1(\gamma_j) + \sum_{p,i} C_{p,i}\psi_1(p_i).$$

If $A \neq 0$ and the antisymmetric matrix with off-diagonal entries $B_{i,j}$ is non-degenerate, then there exists an $\mathfrak{sl}_2$-triple with $\mathfrak{e} = \alpha$ for which the $\mathfrak{h}$-degree filtration coincides with the $\mathfrak{h}$-degree filtration of the original $\mathfrak{sl}_2$-triple.

Proof. Consider the two kinds of operators $y(\pi) = \psi_1(\pi)$ and $x(\pi) = \widetilde{D}_{1,0}(\pi)_{\mathrm{red}}$ for $\pi \in H^{>0}_{r,D}$.

By Proposition 7.3 we obtain

$$[y(\pi), x(\pi')] = \psi_0(\pi\pi').$$

If both π, π' are in H^1 we obtain $r\pi \cdot \pi'$, but if one of them is in $H_{r,D}$ we obtain zero. We have

$$[\mathfrak{e}, x(\pi)] = y(\pi).$$

The operators $x(\pi)$, $y(\pi)$ are nilpotent for any π because they have non-zero $\mathfrak{h}$-degree and $H^*(M)$ is finite-dimensional. For $\pi \in H^2_{r,D}$ let

$$X(\pi) = e^{x(\pi)}.$$

We have

$$[\mathfrak{e}, X(\pi)] = y(\pi)X(\pi),$$

which implies

$$X(\pi)^{-1}\mathfrak{e}X(\pi) = \mathfrak{e} + y(\pi).$$

So conjugating by a product of complex powers of $X(p_i)$ (since $x(p_i)$ is nilpotent, complex powers of $X(p_i)$ are well-defined) we can achieve arbitrary complex values for the coefficients $C_{p,i}$. Moreover, since $x(\pi)$ has negative $\mathfrak{h}$-degree, the operators $X(p_i)$ preserve the $\mathfrak{h}$-degree filtration.

For $\gamma, \gamma' \in H^1$ we have

$$[\mathfrak{e}, x(\gamma)y(\gamma')] = y(\gamma)y(\gamma'), \qquad \left[\sum_i y(\gamma_i)y(\gamma_{i+g}), x(\gamma)y(\gamma')\right] = -ry(\gamma)y(\gamma').$$

So the element $\mathfrak{e} + \frac{1}{r}\sum_i y(\gamma_i)y(\gamma_{i+g})$ commutes with any element of the form $x(\gamma)y(\gamma')$. It is easy to check that elements of the form $x(\gamma)y(\gamma')$ generate a Lie algebra isomorphic to $\mathfrak{gl}_{2g}$ which can be integrated to an action of $\mathrm{GL}_{2g}(\mathbb{C})$ acting on elements of the form $y(\gamma)y(\gamma')$ via the exterior square representation. Since any two non-degenerate antisymmetric forms are $\mathrm{GL}_{2g}(\mathbb{C})$-equivalent, we can obtain any matrix from the standard antisymmetric form $B_{i,j} = 1$ if $j = i + g$ and 0 otherwise. Note that the $\mathrm{GL}_{2g}(\mathbb{C})$-action preserves the $\mathfrak{h}$-degree. □

Remark 7.9. It is clear from the proof that the symplectic group $\mathrm{Sp}_{2g}(\mathbb{C}) \subset \mathrm{GL}_{2g}(\mathbb{C})$ acts on $H^*(M)$ respecting the $\mathfrak{sl}_2$-action, preserving the cohomological degree, and acting via the standard representation on the span of $\psi_1(\gamma_i)$.

Corollary 7.10. *For any $\alpha \in H^2_{\mathrm{taut}}(M)$ and for all but finitely many values $\lambda \in \mathbb{C}$ there exists an $\mathfrak{sl}_2$-triple with $\mathfrak{e} = \alpha + \lambda\psi_2(1)$ for which the $\mathfrak{h}$-degree filtration coincides with the $\mathfrak{h}$-degree filtration of the original $\mathfrak{sl}_2$-triple.*

8. $P = W$

8.1. **Relative Lefschetz theory.** Before summarizing the Relative Hard Lefschetz theorem from [dCM05] we develop some algebraic framework.

Definition 8.1. A Lefschetz structure is a finite dimensional vector space V endowed with a linear endomorphism ω and a finite increasing filtration $P_\bullet V$ such that we have $\omega P_i V \subset P_{i+2} V$ for all i, and for all $k \geq 0$ the map

$$\omega^k : P_{-k}V/P_{-k-1}V \to P_kV/P_{k-1}V$$

is an isomorphism. A map of Lefschetz structures is a map of vector spaces compatible with the filtration and commuting with ω.

Denote by $\mathrm{Gr}\, V$ the associated graded

$$\mathrm{Gr}\, V = \bigoplus_i \mathrm{Gr}_i\, V, \qquad \mathrm{Gr}_i\, V = P_iV/P_{i-1}V.$$

This is a graded vector space on which ω induces an operator of degree 2. Any map of Lefschetz structures $\varphi : U \to V$ induces a map $\mathrm{Gr}\,\varphi : \mathrm{Gr}\, U \to \mathrm{Gr}\, V$. The following is well-known.

Proposition 8.2. *The triple of a vector space V, an increasing filtration $P_\bullet V$ and $\omega : V \to V$ satisfying $\omega P_i V \subset P_{i+2}V$ is a Lefschetz structure if and only if there exists an action of $\mathfrak{sl}_2$ on $\mathrm{Gr}\, V$ for which $\mathfrak{e} = \omega$ and $\mathfrak{h}$ acts on $\mathrm{Gr}_i\, V$ as the multiplication by i. For any map of Lefschetz structures $\varphi : U \to V$ the induced map $\mathrm{Gr}\,\varphi$ commutes with $\mathfrak{f}$. In particular, the $\mathfrak{sl}_2$-action above is unique.*

We prove that Lefschetz structures are similar to mixed Hodge structures (see [Del71] in the following sense:

Proposition 8.3. *The category of Lefschetz structures is abelian and the functor* $V \to \operatorname{Gr} V$ *is an exact faithful functor from the category of Lefschetz structures to the category of finite dimensional representations of* $\mathfrak{sl}_2$.

Proof. Let $\varphi : U \to V$ be a map of Lefschetz structures. By snake lemma applied to the short exact sequence

$$0 \to P_i U/P_{i-k}U \to P_{i+1}U/P_{i-k}U \to P_{i+1}U/P_iU \to 0,$$

the corresponding sequence for V and a map between them induced by φ we obtain a long exact sequence

$$0 \to \operatorname{Ker} \varphi_{P_i/P_{i-k}} \to \operatorname{Ker} \varphi_{P_{i+1}/P_{i-k}} \to \operatorname{Ker} \varphi_{P_{i+1}/P_i} \to \operatorname{Coker} \varphi_{P_i/P_{i-k}}$$
$$\to \operatorname{Coker} \varphi_{P_{i+1}/P_{i-k}} \to \operatorname{Coker} \varphi_{P_{i+1}/P_i} \to 0.$$

Here we denote by φ_{P_i/P_j} the induced map $P_iU/P_jU \to P_iV/P_jV$. We claim that the connecting map $\operatorname{Ker} \varphi_{P_{i+1}/P_i} \to \operatorname{Coker} \varphi_{P_i/P_{i-k}}$ is zero. We show this by induction. Notice that $\bigoplus_i \operatorname{Ker} \varphi_{P_{i+1}/P_i} = \operatorname{Ker} \operatorname{Gr} \varphi$ is a representation of $\mathfrak{sl}_2$, and therefore is generated over ω by *primitive* elements, i.e. elements $x \in \operatorname{Ker} \varphi_{P_{-i}/P_{-i-1}}$ for $i \geq 0$ satisfying $\omega^{i+1}x = 0$. The image of such x under the connecting map is some $y \in \operatorname{Coker} \varphi_{P_{-i-1}/P_{-i-1-k}}$ satisfying $\omega^{i+1}y = 0$.

Let us show that ω^i is injective on $\operatorname{Coker} \varphi_{P_{-i}/P_{-i-k}}$ for all $i > 0$. By the induction assumption we have a short exact sequence

$$0 \to \operatorname{Coker} \varphi_{P_{-i-1}/P_{-i-k}} \to \operatorname{Coker} \varphi_{P_{-i}/P_{-i-k}} \to \operatorname{Coker} \varphi_{P_{-i}/P_{-i-1}} \to 0,$$

and the same short exact sequence for i instead of $-i$. By snake lemma applied to the maps induced by ω^i, we have a short exact sequence

$$0 \to \operatorname{Ker}(\operatorname{Coker} \varphi_{P_{-i-1}/P_{-i-k}} \to \operatorname{Coker} \varphi_{P_{i-1}/P_{i-k}}) \to$$
$$\operatorname{Ker}(\operatorname{Coker} \varphi_{P_{-i}/P_{-i-k}} \to \operatorname{Coker} \varphi_{P_i/P_{i-k}}) \to \operatorname{Ker}(\operatorname{Coker} \varphi_{P_{-i}/P_{-i-1}} \to \operatorname{Coker} \varphi_{P_i/P_{i-1}}).$$

Since $\operatorname{Coker} \operatorname{Gr} \varphi$ is a representation of $\mathfrak{sl}_2$, the map $\omega^i : \operatorname{Coker} \varphi_{P_{-i}/P_{-i-1}} \to \operatorname{Coker} \varphi_{P_i/P_{i-1}}$ is an isomorphism, and therefore its kernel vanishes. We know that ω^{i+1} on $\operatorname{Coker} \varphi_{P_{-i-1}/P_{-i-k}}$ is injective, since this was already proved when treating the $k-1$ case. Hence ω^i on it is injective, and therefore $\operatorname{Ker}(\operatorname{Coker} \varphi_{P_{-i-1}/P_{-i-k}} \to \operatorname{Coker} \varphi_{P_{i-1}/P_{i-k}})$ vanishes, and we conclude that the remaining kernel also has to vanish.

Coming back to the element $y \in \operatorname{Coker} \varphi_{P_{-i-1}/P_{-i-1-k}}$, we see that $\omega^{i+1} y = 0$ implies $y = 0$, and therefore the connecting map vanishes on all primitive elements, and therefore on all elements.

Having established vanishing of the connecting map for all k, taking k sufficiently large we see that for all i the map

$$\operatorname{Ker} \varphi \cap P_i U \to \operatorname{Ker} \varphi_{P_i/P_{i-1}},$$

is surjective, which implies that the natural map

$$\operatorname{Ker} \varphi \cap P_i U / \operatorname{Ker} \varphi \cap P_{i-1} U \to \operatorname{Ker} \varphi_{P_i/P_{i-1}}$$

is an isomorphism. Equivalently, this says that $\operatorname{Gr} \operatorname{Ker} \varphi \to \operatorname{Ker} \operatorname{Gr} \varphi$ is an isomorphism. So $\operatorname{Ker} \varphi$ is a Lefschetz structure. Similarly, we obtain that $\operatorname{Coker} \operatorname{Gr} \varphi \to \operatorname{Gr} \operatorname{Coker} \varphi$ is an isomorphism. So $\operatorname{Coker} \varphi$ is a Lefschetz structure. This implies that the category of Lefschetz structures has kernels and cokernels given by the usual kernels and cokernels together with the induced filtrations. To see that it is abelian, we use the commutative square

$$\begin{array}{ccc} \operatorname{Coker} \operatorname{Ker} \operatorname{Gr} \varphi & \xrightarrow{\sim} & \operatorname{Ker} \operatorname{Coker} \operatorname{Gr} \varphi \\ \downarrow \wr & & \uparrow \wr \\ \operatorname{Gr} \operatorname{Coker} \operatorname{Ker} \varphi & \longrightarrow & \operatorname{Gr} \operatorname{Ker} \operatorname{Coker} \varphi \end{array}$$

to see that $\operatorname{Gr} \operatorname{Coker} \operatorname{Ker} \varphi \to \operatorname{Gr} \operatorname{Ker} \operatorname{Coker} \varphi$ is an isomorphism. This implies that the two filtrations on $\operatorname{Im} \varphi$ coincide.

So we have shown that the category of Lefschetz structures is abelian. We also have shown that kernels and cokernels commute with Gr, which means that Gr is exact. It is faithful because $\operatorname{Gr} V = 0$ and finiteness of the filtration implies $V = 0$. $\square$

Definition 8.4. For a nilpotent operator N acting on a vector space the *canonical filtration* or *weight filtration* is the increasing filtration $W_\bullet$ defined by

$$W_k = \sum_{i \geq \max(0,k)} \operatorname{Ker} N^{i+1} \cap \operatorname{Im} N^{i-k}$$

We do not define the perverse filtration here, but we list all the properties we need, proved by de Cataldo and Migliorini:

Theorem 8.5 (See [dCM05], Theorems 2.1.1, 2.1.4, 2.3.3 and Sections 4.5, 4.6)**.** *Let $f : X \to Y$ be a projective map of algebraic varieties. Suppose X is smooth of dimension*

n. Let $\omega \in H^2$ be a relative ample class, i.e. a class whose positive multiple comes from the embedding $X \to Y \times \mathbb{P}^N$. Then the perverse filtration together with the operator of cup product by ω form a Lefschetz structure on $H^*(X)$. If $U \subset Y$ is open, then the restriction $H^*(X) \to H^*(f^{-1}U)$ is a map of Lefschetz structures. If Y is projective and L is the pullback of an ample class on Y, then $P_i H^*(X) = \bigoplus_{b\in\mathbb{Z}} W_{i-b} H^{n+b}(X)$, where $W_\bullet$ is the canonical filtration induced by the nilpotent operator L.

8.2. **Functoriality.** In this section we analyze the effect of pullback and Gysin maps on the perverse filtration. We assume a base A is fixed, which is quasi-projective, $A \subset \mathbb{P}^M$. Let $\overline{A}$ be the closure of A. Any projective map $\pi : X \to A$ induces an embedding $X \subset \mathbb{P}^N \times \mathbb{P}^M$ and resolving the singularities of the closure we obtain a smooth projective compactification $\iota : X \to \overline{X}$ together with a map $\overline{X} \to \mathbb{P}^N \times \mathbb{P}^M$. Let ω be an ample class of $\overline{X}$. By pullback we obtain a map of Lefschetz structures

$$\iota^* : H^*(\overline{X}) \to H^*(X),$$

which induces a Lefschetz structure on $H^*_{\mathrm{pure}}(X)$. In particular, we have

$$P_k H^*_{\mathrm{pure}}(X) = P_k H^*(X) \cap H^*_{\mathrm{pure}}(X) = \iota^* P_k H^*(\overline{X}).$$

Any $\alpha \in H^i(\overline{X})$ induces the cup-product map $H^*(\overline{X}) \to H^{*+i}(\overline{X})$, which commutes with L, which is the pullback of the hyperplane class from $\mathbb{P}^M$. So α preserves the canonical filtration associated with L and therefore we have

$$\alpha P_k H^*(\overline{X}) \subset P_{k+i} H^{*+i}(\overline{X}).$$

Restricting to $H^*(X)$, we obtain the following statement:

Proposition 8.6. *For any $\alpha \in H^i_{\mathrm{pure}}(X)$ we have $\alpha P_k H^*_{\mathrm{pure}}(X) \subset P_{k+i} H^{*+i}_{\mathrm{pure}}(X)$.*

Suppose we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \pi_X} & \swarrow{\scriptstyle \pi_Y} & \\ A & & \end{array} \quad , \tag{8.1}$$

where π_X and π_Y are projective and X, Y are smooth of dimensions d_X, d_Y. Notice that in this case the map f is automatically projective. We pick smooth compactifications

$\iota_X : X \to \overline{X}$, $\iota_Y : Y \to \overline{Y}$ and a commutative diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\ \downarrow{\scriptstyle \pi_{\overline{X}}} & \swarrow{\scriptstyle \pi_{\overline{Y}}} & \\ \overline{A} & & \end{array},$$

which when restricted to A produces (8.1). The pullback $H^*(\overline{Y}) \to H^*(\overline{X})$ and the Gysin map $H^*(\overline{X}) \to H^{*+2d_Y-2d_X}(\overline{Y})$ commute with the multiplication by L and therefore preserve the canonical filtration associated to it. So we obtain

Proposition 8.7. *For a commutative diagram of the form* (8.1) *with π_X, π_Y projective and X, Y smooth of dimensions d_X, d_Y the pullback map and the Gysin map satisfy*

$$f^* P_k H^*_{\mathrm{pure}}(Y) \subset P_{k+d_Y-d_X} H^*_{\mathrm{pure}}(X), \qquad f_* P_k H^*_{\mathrm{pure}}(X) \subset P_{k+d_Y-d_X} H^{*+2d_Y-2d_X}_{\mathrm{pure}}(Y).$$

More generally, suppose we have a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow[\pi_1]{} & Z & \xrightarrow{\pi_2} & Y \\ & {\scriptstyle \pi_X}\searrow & \downarrow{\scriptstyle \pi_Z} & \swarrow{\scriptstyle \pi_Y} & \\ & & A & & \end{array}, \tag{8.2}$$

where X, Y, Z are smooth and π_X, π_Y, π_Z are projective. Let $[Z]^{\mathrm{vir}} \in H^{2k}_{\mathrm{pure}}(Z)$. Then we are interested in the action of the correspondence

$$\alpha \to \pi_{2*}(\pi_1^* \alpha \cup [Z]^{\mathrm{vir}}).$$

Notice that if Z is not smooth or if π_Z is not projective but only proper, we can always replace Z by a birational variety which is both smooth and projective over A using Hironaka's resolution of singularities and Chow's lemma. Suppose the dimensions of X, Y, Z are d_X, d_Y, d_Z respectively. Then we have

Proposition 8.8. *The correspondence sends $P_i H^*(X)$ to $P_{i+d_X+d_Y-2d_Z+2k} H^{*+2d_Y-2d_Z+2k}(Y)$.*

Notice that d_Z and k enter the statement via the difference $d_Z - k$, which is called the *virtual dimension* of Z.

Remark 8.9. In the special case when $2d_Z - 2k = d_X + d_Y$, for instance when $k = 0$, X and Y are symplectic and Z is a Lagrangian in $X \times Y$, the correspondence preserves the perverse filtration.

Remark 8.10. In the case when $d_X = d_Y$ we see that a correspondence whose action has cohomological degree j sends P_i to P_{i+j}.

8.3. **Elliptic case.** Let $M_{r,d} = M^{\text{parell}}_{r,d,D}$. Choosing a generic stability condition we embed $M_{r,d}$ into the moduli space of stable parabolic Higgs bundles, which we denote by $\overline{M}_{r,d}$. More details on $\overline{M}_{r,d}$ will come in Section 8.4, for now we concentrate on $M_{r,d}$. We have the Hitchin map $\overline{\chi} : \overline{M}_{r,d} \to \overline{A}_r$, which is projective, where $\overline{A}_r$ is an affine space independent of d. The map restricts to a projective map $\chi : M_{r,d} \to A_r$, where $A_r \subset \overline{A}_r$ is open. So we have a perverse filtration on $H^*_{\text{pure}}(M_\alpha)$.

By Markman's argument [Mar02], the class of the diagonal in $\overline{M}'_{r,d} \times M_{r,d}$ for certain smooth compactification $\overline{M}'_{r,d}$ of $M_{r,d}$ can be expressed using tautological classes. Since $H^*_{\text{pure}}(M_{r,d})$ is generated by the Künneth components of the diagonal, we obtain that $H^*_{\text{pure}}(M_{r,d})$ is generated by tautological classes:

$$H^*_{\text{taut}}(M_{r,d}) = H^*_{\text{pure}}(M_{r,d}).$$

The Hecke correspondences respect χ, i.e. the projection maps commute with the maps to A_r. Applying Propositions 8.6 and 8.7 we obtain that for $\xi \in H^i(S)$ and $m \geq 0$

- the multiplication by the tautological class $\psi_m(\xi) \in H^{2m-2+i}(M_\alpha)$ satisfies $\psi_m(\xi)P_j \subset P_{j+2m-2+i}$,
- the operation of the Hecke operator $T_m(\xi)$ (of cohomological degree $2m-2+i$) satisfies $T_m(\xi)P_j \subset P_{j+2m-2+i}$,
- more generally, the operator $D_{m,n}(\xi)$ (of cohomological degree $2n-2+i$) satisfies $D_{m,n}(\xi)P_j \subset P_{j+2n-2+i}$.

In particular, the operator $q_1(\omega)$, which was used to identify the cohomologies for different d respects the filtration $q_1(\omega)P_i \subset P_i$, and since the moduli spaces $M_{r,d}$ and $M_{r,d+1}$ are isomorphic we have $q_1(\omega)P_i = P_i$ and therefore $q_1(\omega)^{-1}P_i = P_i$.

This implies

Proposition 8.11. *The operators $\mathfrak{e}, \mathfrak{f}, \mathfrak{h}$ of the original $\mathfrak{sl}_2$-triple, as well as any of the triples constructed in Proposition 7.8 change perversity by $2, -2, 0$ respectively.*

This turns out to completely pin down the perverse filtration:

Proposition 8.12. *The perverse filtration equals the filtration induced by the original $\mathfrak{sl}_2$-triple.*

Proof. Choose a relatively ample class $\alpha \in H^2(M_{r,d})$ and replace it by a linear combination $\alpha + \lambda\psi_2(1)$ for $\lambda \in \mathbb{Q}$ which is ample and is a part of an $\mathfrak{sl}_2$-triple as in Corollary 7.10.

Pass to the associated graded with respect to the perverse filtration. There we have two $\mathfrak{sl}_2$ actions, $\mathfrak{e}, \mathfrak{f}, \mathfrak{h}$ and $\mathfrak{e}', \mathfrak{f}', \mathfrak{h}'$ for which $\mathfrak{e} = \mathfrak{e}'$ and $\mathfrak{h}, \mathfrak{h}'$ commute. Take the operator $\mathfrak{h} - \mathfrak{h}'$

and see that it commutes with both actions. Restrict to a subspace of $\mathfrak{h}-\mathfrak{h}'$ of eigenvalue μ and by taking the trace see that $\mu=0$. This implies $\mathfrak{h}=\mathfrak{h}'$ on the associated graded, which implies equality of the filtrations. □

By $P=W$ we mean the following statement, which clearly follows from $[\mathfrak{h},\psi_m(\xi)] = m\psi_m(\xi)$:

Proposition 8.13. *The subspace $P_m H^*_{\text{pure}}(M^{\text{parell}}_{r,d,D})$ is the span of products $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \leq m+N$, where $-N$ is the $\mathfrak{h}$-weight of $1\in H^0_{\text{pure}}(M^{\text{parell}}_{r,d,D})$.*

8.4. **Twisted parabolic case.** Now we consider the space $\overline{M}_{r,d}$. This space parameterizes stable triples $(\mathcal{E},\theta,(F_p))$ where $\mathcal{E}$ is a vector bundle of rank r and degree d, $\theta:\mathcal{E}\to\mathcal{E}\otimes\Omega(D)$, F_p is a complete flag in the fiber $\mathcal{E}(p)$ for each $p\in D$, $\operatorname{res}_p\theta$ preserves F_p.

The group $\mathbb{C}^*$ acts on $\overline{M}_{r,d}$ by scaling the Higgs field turning it into a semi-projective variety, see [HRV15]. By Corollary 1.3.2 in [HRV15], the cohomology of $\overline{M}_{r,d}$ is pure, and by Markman's argument as in Section 8.3 we obtain

$$H^*_{\text{taut}}(\overline{M}_{r,d}) = H^*_{\text{pure}}(\overline{M}_{r,d}) = H^*(\overline{M}_{r,d}).$$

Let us decompose $\overline{A}_r = A_D\times A'_r$, and correspondingly $\overline{\chi}=\chi_D\times\chi'$, where

$$\chi_D:\overline{M}_{r,d}\to A_D=\mathbb{C}^{r|D|-1}$$

is the map sending $(\mathcal{E},\theta,F)$ to the ordered collection of the eigenvalues of $\operatorname{res}_p\theta$ for all $p\in D$. Note that the sum of all the eigenvalues is always zero.

The map χ_D is smooth, i.e. a surjective submersion. In fact, $\overline{M}_{r,d}$ is known to be Poisson, the fibers of χ_D are the symplectic leaves, and the image of the tangent map at every point is dual to the kernel of the Poisson tensor, so the dimension of the image of the tangent map is constant. Iterating Corollary 1.3.3 in [HRV15] we obtain

Proposition 8.14. *The cohomology of all the fibers of χ_D are all isomorphic and pure.*

In fact, the following easily follows:

Proposition 8.15. *For each $\boldsymbol{\mu}\in A_D$ the restriction map $H^*(\overline{M}_{r,d})\to H^*(\chi_D^{-1}(\boldsymbol{\mu}))$ is an isomorphism.*

As a next step we show

Proposition 8.16. *The restriction map from $H^*(\overline{M}_{r,d})$ to $H^*(M_{r,d})$ is injective, i.e. $H^*(\overline{M}_{r,d}) = H^*_{\text{pure}}(M_{r,d})$.*

Proof. Choosing $\boldsymbol{\mu} \in A_D$ generically, the identity

$$\sum_p \operatorname{res}_p \operatorname{Tr}(\theta) = 0$$

implies that any Higgs bundle in $\chi_D^{-1}(\boldsymbol{\mu})$ is simple, which implies that any spectral curve is reduced and irreducible, and so we have $\chi_D^{-1}(\boldsymbol{\mu}) \subset M_{r,d}$. So the composition

$$H^*(\overline{M}_{r,d}) \to H^*(M_{r,d}) \to H^*(\chi_D^{-1}(\boldsymbol{\mu}))$$

is an isomorphism, and in particular injective. Hence $H^*(\overline{M}_{r,d}) \to H^*(M_{r,d})$ is injective. □

The restriction map $H^*(\overline{M}_{r,d}) \to H^*(M_{r,d})$ preserves perverse filtration (Theorem 8.5) and choosing an ample class becomes a map of Lefschetz structures. So it has to be an isomorphism of Lefschetz structures:

Corollary 8.17. *$P = W$ holds for $H^*(\overline{M}_{r,d})$.*

8.5. **Parabolic case.** Finally, we denote by $M_{r,d}^0 = \chi_D^{-1}(0) \subset \overline{M}_{r,d}$ the moduli space parameterizing stable parabolic Higgs bundles such that $\operatorname{res}_p \theta$ is nilpotent for all $p \in D$. By Proposition 8.15, the restriction map $H^*(\overline{M}_{r,d}) \to H^*(M_{r,d}^0)$ is an isomorphism.

Proposition 8.18. *The restriction isomorphism $H^*(\overline{M}_{r,d}) \to H^*(M_{r,d}^0)$ preserves perverse filtration.*

Proof. Denote $X = \overline{M}_{r,d}$, $X_0 = M_{r,d}^0$. Let $N = \dim X$, $N_0 = \dim X_0$, $n = N - N_0$. Denote the Hitchin map by $\chi : X \to A$. Then $X_0 = \chi^{-1} A_0$ for $A_0 \subset A$ a linear subspace of codimension n.

There is a natural $\mathbb{C}^*$-action on X obtained by scaling the Higgs field. Take the corresponding projectivization (see [Hau98])

$$\overline{X} = \left(X \times \mathbb{C} \setminus \chi^{-1}(0) \times \{0\}\right) / \mathbb{C}^*,$$

and let $\partial X = \overline{X} \setminus X$. The Hitchin map extends to $\overline{\chi} : \overline{X} \to \overline{A}$, where $\overline{A}$ is the corresponding weighted projective space. The spaces $\overline{X}$ and ∂X are projective. They are not necessarily smooth, but the singularities are finite quotient singularities, so they are rationally smooth. Let $L \in H^2(\overline{X})$ be the pullback of the hyperplane class from $\overline{A}$. We claim that we have

$$H^*(X) = H^*(\overline{X}) / L H^*(\overline{X}).$$

To see that, let $U = \overline{X} \setminus \chi^{-1}(0)$. Since U is the total space of a line bundle over ∂X, its cohomology is pure, so the restriction map

$$H^*(\overline{X}) \to H^*(U) = H^*(\partial X)$$

is surjective. The kernel of this map is identified with the homology of $\chi^{-1}(0)$, and therefore $H_*(\chi^{-1}(0))$, and by duality $H^*(\chi^{-1}(0))$ is pure. The restriction map $H^*(X) \to H^*(\chi^{-1}(0))$ is an isomorphism because the $\mathbb{C}^*$-action contracts X to an arbitrary small neighborhood of $\chi^{-1}(0)$. So $H^*(X)$ is pure, which implies surjectivity of the restriction map $H^*(\overline{X}) \to H^*(X)$. Now the long exact sequence in Borel-Moore homology for $X \subset \overline{X}$ splits into short exact sequences

$$0 \to H^{*-2}(\partial X) \to H^*(\overline{X}) \to H^*(X) \to 0.$$

The composition $H^*(\overline{X}) \to H^*(\partial X) \to H^{*+2}(\overline{X})$ is the multiplication by the fundamental class $[\partial X]$, which equals L. So we obtain a short exact sequence

$$H^{*-2}(\overline{X}) \xrightarrow{L} H^*(\overline{X}) \to H^*(X) \to 0,$$

as claimed. In fact, we can dualize this sequence and use the Poincaré duality to obtain a longer exact sequence:

$$0 \to H_{2N-*+2}(X) \to H^{*-2}(\overline{X}) \xrightarrow{L} H^*(\overline{X}) \to H^*(X) \to 0.$$

The same arguments apply for $X_0, \overline{X}_0$. Note that from $H^*(X) \cong H^*(\chi^{-1}(0)) \cong H^*(X_0)$ we obtain another proof that the restriction $H^*(X) \to H^*(X_0)$ is an isomorphism.

Let $\iota : \overline{X}_0 \to \overline{X}$ be the natural closed embedding. The operators ι_*, ι^* both commute with the multiplication by L. The composition $\iota_*\iota^*$ is the operator of multiplication by $[\overline{X}_0] \in H^{2n}(\overline{X})$. On the other hand, since $\overline{X}_0 = \overline{\chi}^{-1}(\overline{A}_0)$ and $\overline{A}_0$ is a projective subspace, we have $L^n = c[\overline{X}_0]$ where $c > 0$ is the multiplicity. So we have

$$\iota_*\iota^* = c^{-1}L^n.$$

We claim the following: $\operatorname{Im}\iota_* \subset \operatorname{Im} L^n$. To see this, let $\alpha = \iota_*\beta$, where $\beta \in H^*(\overline{X}_0)$. Since ι^* induces an isomorphism

$$H^*(\overline{X})/LH^*(\overline{X}) \to H^*(\overline{X}_0)/LH^*(\overline{X}_0),$$

we can "lift" β and write $\beta = \iota^*\beta' + L\beta''$. This implies

$$\alpha = c^{-1}L^n\beta' + L\iota_*\beta''.$$

Let $\alpha' = \iota_* \beta''$. Repeating the argument for α' and continuing in this fashion n times we obtain $\alpha \in L^n H^*(\overline{X})$.

As a next step, we claim that for any $i \geq 0$ we have $(\iota^*)^{-1}(\operatorname{Ker} L^i + \operatorname{Im} L) \subset \operatorname{Ker} L^{i+n} + \operatorname{Im} L$. Indeed, suppose $\alpha \in H^*(\overline{X})$ is such that

$$\iota^* \alpha = \beta + L\beta', \qquad L^i \beta = 0.$$

Applying ι_* and using the previous statement we obtain

$$c^{-1} L^n \alpha = \iota_* \beta + L \iota_* \beta' \quad \Rightarrow \quad c^{-1} L^n \alpha = \iota_* \beta + L^{n+1} \beta''.$$

Hence we have

$$L^{i+n}(\alpha - cL\beta'') = c\iota_* L^i \beta = 0,$$

which shows that $\alpha \in \operatorname{Ker} L^{i+n} + \operatorname{Im} L$.

Now suppose $\alpha \in P_i H^j(X_0)$. This means that α can be represented in $H^*(\overline{X}_0)$ by an element of $W_{i+n-j} \subset \operatorname{Ker} L^{i+N_0-j+1} + \operatorname{Im} L$. By the above, in $H^*(\overline{X})$ can be represented by an element of $\operatorname{Ker} L^{i+N-j+1}$, which implies $\alpha \in P_i H^j(X)$. So the inverse to the pullback map $H^*(X_0) \to H^*(X)$ preserves the perverse filtration, and so it is a map of Lefschetz structures, and so it is an isomorphism of Lefschetz structures. □

Corollary 8.19. *$P = W$ holds for $H^*(M^0_{r,d})$.*

8.6. **Classical case.** Suppose r and d are relatively prime. Let X be the moduli space of stable Higgs bundles of rank r and degree d. Choose any point $p \in C$ and let $D = (p)$. Let X^{par} be the corresponding moduli space of stable parabolic Higgs bundles. This space parameterizes triples $(\mathcal{E}, \theta, F)$ where $\mathcal{E}$ is a vector bundle of rank r and degree d, $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega(p)$, F is a complete flag in the fiber $\mathcal{E}(p)$, $\operatorname{res}_p \theta$ is nilpotent and preserves F, the pair $(\mathcal{E}, \theta)$ is stable. Let $\widetilde{X}$ be the closed subvariety in X^{par} defined by the condition $\operatorname{res}_p \theta = 0$. Then we have the "forget F" map $\pi : \widetilde{X} \to X$, which turns $\widetilde{X}$ into a relative flag variety over X. Denote $\iota : \widetilde{X} \to X^{\mathrm{par}}$ the closed embedding. The codimension of $\widetilde{X}$ in X^{par} equals $\binom{r}{2}$. The relative dimension of $\widetilde{X}$ over X is also $\binom{r}{2}$. So we have maps

$$A = \pi_* \iota^* : H^*(X^{\mathrm{par}}) \to H^{*-2\binom{r}{2}}(X), \qquad B = \iota_* \pi^* : H^*(X) \to H^{*+2\binom{r}{2}}(X^{\mathrm{par}}).$$

Let $\chi : X \to A$, $\chi^{\mathrm{par}} : X^{\mathrm{par}} \to A^{\mathrm{par}}$ be the corresponding Hitchin maps. The space A is a linear subspace of A^{par}, and so we can replace A by A^{par} without changing the perverse filtration on $H^*(X)$. Applying Proposition 8.7 we see that both A and B preserve the perverse filtration. Let

$$\Delta = \prod_{1\le i<j\le r} (y_{p,i} - y_{p,j}) \in H^{2\binom{r}{2}}(X^{\mathrm{par}}).$$

We have $\pi_*(\Delta) = \pm r!$ because the class Δ is, up to a sign, nothing by the Euler class of the relative tangent bundle of $\widetilde{X} \to X$. The composition $\iota^*\iota_*$ is the multiplication by the Euler class of the normal bundle of $\widetilde{X}$ in X^{par}, which is also given by Δ up to a sign. In fact the relative tangent bundle and the normal bundle above are dual to each other, which follows from symplectic geometry. So we obtain

$$AB = \pi_*\iota^*\iota_*\pi^* = \pi_*\Delta\pi^* = \pm r!.$$

Since $\widetilde{X} \subset X^{\mathrm{par}}$ is clearly the zero set of a vector bundle with Euler class $\pm\Delta$, Δ has to be a multiple of the fundamental class $[\widetilde{X}] \in H^{2\binom{r}{2}}(X^{\mathrm{par}})$, and from $\iota^*\iota_* = \pm\Delta$ we obtain $[\widetilde{X}] = \pm\Delta$.

Theorem 8.20. *$P = W$ holds for X, i.e the subspace $P_m H^*(X)$ is the span of products $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \le m + N$, where $-N$ is the perversity of $1 \in H^0(X)$.*

Proof. Let $-N'$ be the perversity of $1 \in H^0(X^{\mathrm{par}})$. Then we have $\pm r! = A(\Delta) \in P_{-N'+\binom{r}{2}} H^0(X)$. So $-N \le -N' + \binom{r}{2}$. Conversely, $\Delta = B(1) \in P_{-N}$. If $-N < -N' + \binom{n}{2}$, since Δ is $\mathfrak{h}$-homogeneous of weight $-N' + \binom{r}{2}$ this implies $\Delta = 0$, a contradiction. Hence we have

$$N' = N + \binom{n}{2}.$$

Consider a product of the form $f = \prod_i \psi_{m_i}(\xi_i) \in H^*(X)$. Then

$$B(f) = \pm\Delta f \in P_{\sum_i m_i - N' + \binom{n}{2}} = P_{\sum_i m_i - N} \quad \Rightarrow \quad f = \pm\frac{1}{r!} ABf \in P_{\sum_i m_i - N}.$$

Conversely, let $f \in P_m H^*(X)$. We assume that f is explicitly written as a polynomial in tautological classes and write

$$B(f) = \Delta f \in P_m H^*(X^{\mathrm{par}}).$$

By $P = W$ for M^{par} we can write

$$\Delta f = \sum_k \lambda_k g_k \qquad (\lambda_k \in \mathbb{C}),$$

where each g_k is a monomial in tautological classes of the form $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \le N' + m$. Using the identification of $H^*(X^{\mathrm{par}})$ with the pure part of the cohomology of the corresponding moduli space $M^{\mathrm{parell}}_{r,d,D}$ we see that the symmetric group S_r acts on $H^*(X^{\mathrm{par}})$ permuting the generators $y_{p,r}$. Let $\mathrm{ASym} : H^*(X^{\mathrm{par}}) \to H^*(X^{\mathrm{par}})$ be the corresponding antisymmetrization operator,

$$\mathrm{ASym} = \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{|\sigma|} \sigma.$$

We obtain

$$\Delta f = \sum_k \lambda_k \, \mathrm{ASym}(g_k).$$

Each $\mathrm{ASym}(g_k)$ is explicitly a product of Δ and a linear combination of monomials of the form $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \le N + m$ not containing the generators $y_{p,i}$. Therefore we can lift each of these and write

$$B(f) = \Delta f = B\left(\sum_k \lambda'_k g'_k\right) \qquad (\lambda'_k \in \mathbb{C}),$$

where each g_k is a monomial in tautological classes of the form $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \le N + m$. Applying A we see that

$$f = \sum_k \lambda'_k g'_k,$$

so the converse claim has also been proved. □

Remark 8.21. The number N, i.e. the negative of the perversity of 1 in the above statements is always the dimension of the fibers of the Hitchin map.

Remark 8.22. It is clear from the above proof that $H^*(X)$ is the antiinvariant part of $H^*(\widetilde{X})$. The corresponding statement on the Betti side was established in [Mel19]. The anti-invariant part is clearly preserved by the operators $\widetilde{D}_{m,n}(\xi)_{\mathrm{red}}$ for $\xi \in H$. In particular, we have an action of $\mathcal{H}_2$ on $H^*(X)[x,y]$ as in Corollary 7.4.

8.7. **The weight filtration.** Let now X be the Betti realization of the moduli space, i.e. the character variety for a compact or non-compact curve with generic local monodromies, see [HLRV11]. There is a decomposition turning $H^*(X)$ into a graded ring

$$H^*(X) = \bigoplus_i W_{2i} H^*(X) \cap F^i H^*(X),$$

where W is the weight filtration and F is the Hodge filtration. Let us call classes $f \in W_{2i} H^*(X) \cap F^i H^*(X)$ classes of *pure weight* i. In the compact curve case, it is explained in [She16] that certain tautological classes are of pure weight. Our classes $\psi_n(\xi)$ are not quite of pure weight because of the Todd contribution in (4.1), but they are modulo products of classes $\prod_i \psi_{n'_i}(\xi'_i)$ with $\sum n'_i < n$. In [Mel19] this claim was extended to the parabolic character varieties. This implies

Proposition 8.23. *The subspace $W_{2m}H^*(X)$ is the span of products $\prod_i \psi_{m_i}(\xi_i)$ satisfying $\sum_i m_i \leq m$.*

So the claims we call $P = W$ in Propositon 8.19 and Theorem 8.20 are equivalent to the claim $P_i = W_{2(i+N)}$.

Acknowledgments

We are thankful to Mark Andrea de Cataldo, Ben Davison, Eugene Gorsky, Lothar Göttsche, Andrei Neguţ, Alexei Oblomkov, Lev Rozansky, Kostya Tolmachov, Eric Vasserot for useful discussions.

Anton Mellit is supported by the consolidator grant No. 101001159 "Refined invariants in combinatorics, low-dimensional topology and geometry of moduli spaces" of the European Research Council. Alexandre Minets is supported by the starter grant "Categorified Donaldson-Thomas Theory" No. 759967 of the European Research Council.

References

[AS13] N. Arbesfeld and O. Schiffmann, *A presentation of the deformed $W_{1+\infty}$ algebra*, Symmetries, integrable systems and representations, Springer Proc. Math. Stat., vol. 40, Springer, Heidelberg, 2013, pp. 1–13. MR 3077678

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR 751966

[dCHM12] Mark Andrea A. de Cataldo, Tamás Hausel, and Luca Migliorini, *Topology of Hitchin systems and Hodge theory of character varieties: the case A_1*, Ann. of Math. (2) **175** (2012), no. 3, 1329–1407. MR 2912707

[dCM05] Mark Andrea A. de Cataldo and Luca Migliorini, *The Hodge theory of algebraic maps*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 5, 693–750. MR 2195257

[dCMS22] Mark Andrea de Cataldo, Davesh Maulik, and Junliang Shen, *Hitchin fibrations, abelian surfaces, and the $P = W$ conjecture*, J. Amer. Math. Soc. **35** (2022), no. 3, 911–953. MR 4433080

[Del71] Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 0498551

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323

[GHM21] Eugene Gorsky, Matthew Hogancamp, and Anton Mellit, *Tautological classes and symmetry in Khovanov-Rozansky homology*, arXiv preprint arXiv:2103.01212 (2021).

[Hau98] Tamás Hausel, *Compactification of moduli of Higgs bundles*, J. Reine Angew. Math. **503** (1998), 169–192. MR 1650276

[Hei10] Jochen Heinloth, *Lectures on the moduli stack of vector bundles on a curve*, Affine flag manifolds and principal bundles, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010, pp. 123–153. MR 3013029

[Hit87] Nigel J. Hitchin, *The self-duality equations on a Riemann surface*, Proceedings of the London Mathematical Society **3** (1987), no. 1, 59–126.

[HLRV11] Tamás Hausel, Emmanuel Letellier, and Fernando Rodriguez-Villegas, *Arithmetic harmonic analysis on character and quiver varieties*, Duke Math. J. **160** (2011), no. 2, 323–400. MR 2852119

[HRV08] Tamás Hausel and Fernando Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties*, Inventiones mathematicae **174** (2008), no. 3, 555–624.

[HRV15] ———, *Cohomology of large semiprojective hyperkähler varieties*, Astérisque (2015), no. 370, 113–156. MR 3364745

[KV19] Mikhail Kapranov and Eric Vasserot, *The cohomological Hall algebra of a surface and factorization cohomology*, arXiv preprint arXiv:1901.07641 (2019), accepted in JEMS.

[Leh99] Manfred Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. **136** (1999), no. 1, 157–207. MR 1681097

[Mar02] Eyal Markman, *Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces*, J. Reine Angew. Math. **544** (2002), 61–82. MR 1887889

[Mel19] Anton Mellit, *Cell decompositions of character varieties*, arXiv preprint arXiv:1905.10685 (2019).

[Min20] Alexandre Minets, *Cohomological Hall algebras for Higgs torsion sheaves, moduli of triples and sheaves on surfaces*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 30, 67. MR 4090584

[MMSV] Anton Mellit, Alexandre Minets, Olivier Schifmann, and Eric Vasserot, *On the cohomological Hall algebra of zero-dimensional sheaves on smooth surfaces*, in preparation.

[MS22] Davesh Maulik and Junliang Shen, *The $P = W$ conjecture for GL_n*, arXiv preprint arXiv:2209.02568 (2022).

[Neg19] Andrei Neguţ, *Shuffle algebras associated to surfaces*, Selecta Math. (N.S.) **25** (2019), no. 3, Paper No. 36, 57. MR 3950703

[OY16] Alexei Oblomkov and Zhiwei Yun, *Geometric representations of graded and rational Cherednik algebras*, Adv. Math. **292** (2016), 601–706. MR 3464031

[She16] Vivek Shende, *The weights of the tautological classes of character varieties*, International Mathematics Research Notices (2016), rnv363.

[Sim90] Carlos T. Simpson, *Harmonic bundles on noncompact curves*, Journal of the American Mathematical Society **3** (1990), no. 3, 713–770.

[Sim91] ———, *Nonabelian Hodge theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 747–756. MR 1159261

[Sim92] ______, *Higgs bundles and local systems*, Publications mathématiques de l'IHÉS **75** (1992), no. 1, 5–95.

[SS20] Francesco Sala and Olivier Schiffmann, *Cohomological Hall algebra of Higgs sheaves on a curve*, Algebr. Geom. **7** (2020), no. 3, 346–376. MR 4087863

[SV13] O. Schiffmann and E. Vasserot, *Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on* $\mathbf{A}^2$, Publ. Math. Inst. Hautes Études Sci. **118** (2013), 213–342. MR 3150250

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