D-MODULES IN REPRESENTATION THEORY

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Contents

Date: February 17, 2021.

We will follow [HTT] and [G2].

Part 1. $\mathcal{D}$-modules

1. Introduction and motivation

1.1. $\mathcal{D}$-modules and linear PDEs. The study of $\mathcal{D}$-modules is motivated by the study of linear differential equations. What is a linear PDE? It is an expression of the form

$$(a_n(t)\partial^n + \ldots + a_1(t)\partial + a_0(t))\, f(t) = 0, \tag{1.1.1}$$

where $f(t)$ is the indeterminate function. If we have multiple variables, partial derivatives can appear. Suppose that all the coefficients $a_i(t)$ are polynomial in t. Then we can consider LHS as a polynomial in two operators t, ∂ acting on f. Note that $\partial(tf) = f + t\partial(f)$, so that we have $[\partial, t] = 1$. Thus, a linear DE is just an element A of the *Weyl algebra*

$$\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle t, \partial\rangle/([\partial, t] = 1).$$

What constitutes a solution of the equation (1.1.1)? First, we need to decide on the space of functions $\mathcal{F}$ among which we are looking for a solution (differentiable, analytic, polynomial etc). Second, we are looking for $f \in \mathcal{F}$ such that $Af = 0$. This means that $\mathcal{F}$ has to be a left $\mathcal{D}_{\mathbb{A}^1}$-module. Then the space of solutions is tautologically given by the following Hom-space:

$$\mathsf{Hom}_{\mathcal{D}_{\mathbb{A}^1}}(\mathcal{D}_{\mathbb{A}^1}/(\mathcal{D}_{\mathbb{A}^1}\cdot A), \mathcal{F}) = \left\{f \in \mathcal{F} : Af = 0\right\}.$$

1.2. $\mathcal{D}$-modules and connections. Let X be a smooth algebraic variety, and $E \longrightarrow X$ a vector bundle. What is a connection? It is a gadget, which allows us to consider PDEs on the spaces of sections of E by providing a rule of how to differentiate sections along a given tangent direction. Algebraically speaking, we have two equivalent definitions of a connection ∇ on E:

- a map $\nabla : \mathsf{Vect}_X \times E \longrightarrow E$, $(\xi, e) \mapsto \nabla_\xi(e)$, which is $\mathcal{O}_X$-linear in ξ, and satisfies the Leibniz rule in e:
$$\nabla_\xi(fe) = f\nabla_\xi(e) + \xi(f)e;$$
- or a map of $\mathcal{O}_X$-modules $\nabla : E \longrightarrow E \otimes_{\mathcal{O}_X} \Omega^1_X$, where Ω^1_X is the sheaf of differentials.

One might also want to require that differentiating in different directions is compatible. One says that a connection ∇ is *flat*, if

- for any $\xi, \eta \in \mathsf{Vect}_X$, $e \in E$ we have
$$\nabla_\xi \circ \nabla_\eta(e) - \nabla_\eta \circ \nabla_\xi(e) = \nabla_{[\xi,\eta]}(e);$$
- or the composition $E \xrightarrow{\nabla} E \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} E \otimes_{\mathcal{O}_X} \Omega^2_X$ is zero.

Comparing with the case of $X = \mathbb{A}^1$ before, we see that to give a flat connection on E amounts to upgrading $\mathcal{O}_X$-module structure on E to a $\mathcal{D}_X$-modules structure. The key insight which leads to the theory of $\mathcal{D}$-modules is that we don't need to require E to be locally free.

1.3. **Geometric representation theory.** Let us give a quick glimpse of where we want to get by the end of the course. Given a simple Lie algebra $\mathfrak{g}$, or equivalently its universal enveloping algebra $U\mathfrak{g}$, we want to study its modules. It is known since the beginning of 20th century that finite-dimensional representations are completely reducible, and all simple ones are classified. The infinite-dimensional case is much richer and more complicated, though. It turns out the category of $U\mathfrak{g}$-modules is equivalent to the category of $\mathcal{D}$-modules on the flag variety G/B (modulo some technical assumptions). This is the content of localization theorem of Beilinson and Bernstein (and its numerous generalizations). Without going into details, one reason to expect such connection is that one can identify $\mathfrak{g}$ with vector fields on G/B. This point of view opens an avenue of attack on completely algebraic/representation theoretic questions through geometric methods.

2. Weyl algebra

2.1. **Definition and examples.** Before going to the definition for general X, let us play around with $X = \mathbb{A}^1$.

Definition 2.1.1. Let $\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle t, \partial\rangle/([\partial, t] = 1)$. A *left $\mathcal{D}$-module* over $\mathbb{A}^1$ is a left module over $\mathcal{D}_{\mathbb{A}^1}$. The category of left $\mathcal{D}$-modules is denoted by $\mathcal{D}\text{-mod}(\mathbb{A}^1)$; it is an abelian category.

Note that in $\mathcal{D}_{\mathbb{A}^1}$ we have $[\partial, f(t)] = \frac{\partial f}{\partial t}$.

Example 2.1.2. (1) $M = \mathbb{C}[t]$ with $\partial(f(t)) = \frac{\partial f}{\partial t}$;
(2) $M = \mathbb{C}[t, t^{-1}]$;
(3) for any $a(t) \in \mathbb{C}[t]$, $M = \mathbb{C}[t]_a = \{p(t)/a(t)^n\}$;
(4) $M = \mathbb{C}(t)$;
(5) $M = \mathcal{D}_{\mathbb{A}^1}/(\partial - \lambda)$. Let us denote the image of 1 by $e^{\lambda t}$. Then we have

$$\partial\left(f(t)e^{\lambda t}\right) = \frac{\partial f}{\partial t} \cdot e^{\lambda t} + f \cdot \partial e^{\lambda t} = \frac{\partial f}{\partial t} \cdot e^{\lambda t} + \lambda f \cdot e^{\lambda t};$$

(6) $M = \mathbb{C}[t, t^{-1}]e^{\lambda t}$;
(7) $M = \mathbb{C}[\partial]$, where t acts by differentiation. We call this module δ_0.

Question 2.1.3. We have an injective map of $\mathcal{D}$-modules $\mathbb{C}[t] \to \mathbb{C}[t, t^{-1}]$. Compute its cokernel.

Question 2.1.4. Does $\mathcal{D}_{\mathbb{A}^1}$ have any finite-dimensional representations? What if we replace $\mathbb{C}$ by $\mathbb{F}_p$?

In general, we have an automorphism F of $\mathcal{D}_{\mathbb{A}^1}$, given by $F(t) = \partial$, $F(\partial) = -t$. It induces an autoequivalence $F : \mathcal{D}\text{-mod}(\mathbb{A}^1) \to \mathcal{D}\text{-mod}(\mathbb{A}^1)$, which we call the *Fourier transform*. For instance, $F(\delta_0) \simeq \mathbb{C}[t]$.

Question 2.1.5. Compute the Fourier transform of $\mathbb{C}[t]e^{\lambda t}$.

2.2. **Filtered algebras.** $\mathcal{D}_{\mathbb{A}^1}$-modules can be seen as a non-commutative version of sheaves on $\mathbb{A}^2$. In order to explain this statement, let us recall some basic facts about filtered algebras.

Definition 2.2.1. A *filtered algebra* is an algebra A together with a chain of subspaces

$$A^{\leq 0} \subset A^{\leq 1} \subset \dots, \quad \bigcup_i A^{\leq i} = A,$$

satisfying

$$A^{\leq i} \cdot A^{\leq j} \subset A^{\leq i+j}.$$

Definition 2.2.2. For a filtered algebra A, define the *associated graded algebra* by

$$\operatorname{gr} A = \bigoplus_{i \geq 0} \operatorname{gr}^i A := \bigoplus_{i \geq 0} A^{\leq i}/A^{\leq i-1},$$

where the product is given by

$$\operatorname{gr}^i A \times \operatorname{gr}^j A \longrightarrow \operatorname{gr}^{i+j} A,$$
$$[a] \cdot [b] = [ab].$$

Question 2.2.3. Let A be a filtered algebra, satisfying $[a, b] \in A^{\leq i+j-1}$ for any $a \in A^{\leq i}$, $b \in A^{\leq j}$. Show that $\operatorname{gr} A$ is commutative.

We call such filtered algebras *almost commutative*.

We have a convenient formal construction, which interpolates between A and its associated graded.

Definition 2.2.4. The *Rees algebra* of A is the $\mathbb{C}[\hbar]$-algebra

$$A_\hbar = \bigoplus_{i \geq 0} A^{\leq i} \hbar^i,$$

where the product is obvious, and $\hbar$ acts by the inclusion $\hbar \left(A^{\leq i} \hbar^i \right) \subset A^{\leq i+1} \hbar^{i+1}$.

Question 2.2.5. Show that $A_\hbar/(\hbar - 1) = A$, and $A_\hbar/(\hbar) = \operatorname{gr} A$.

Example 2.2.6 (Filtration by order). Let us return to the Weyl algebra. Any $f \in \mathcal{D}_{\mathbb{A}^1}$ can be written as $f = \sum_i a_i(t)\partial^i$. We call the maximal i with $a_i \neq 0$ the *degree* of f, and define $\mathcal{D}_{\mathbb{A}^1}^{\leq k} = \{\sum_{i=0}^k a_i(t)\partial^i\}$.

Question 2.2.7. Prove that $\mathcal{D}_{\mathbb{A}^1}$ is almost commutative, and $\operatorname{gr} \mathcal{D}_{\mathbb{A}^1} \simeq \mathbb{C}[t, \partial]$.

Example 2.2.8. For any Lie algebra $\mathfrak{g}$, its universal enveloping algebra $U\mathfrak{g}$ admits a filtration with $\operatorname{gr}(U\mathfrak{g}) = \operatorname{Sym} \mathfrak{g}$ (Poincaré-Birkhoff-Witt theorem).

2.3. **Support of a $\mathcal{D}_{\mathbb{A}^1}$-modules.**

Definition 2.3.1. Let M be a $\mathcal{D}_{\mathbb{A}^1}$-module. A *good filtration* on M is a filtration of $\mathbb{C}[t]$-modules $\ldots \subset M^{i-1} \subset M^i \subset M^{i+1} \subset \ldots$, such that M^{i+1}/M^i is finitely generated for any i, and

$$\mathcal{D}_{\mathbb{A}^1}^{\leq j} M^i \subset M^{i+j}.$$

For any such filtration, $\operatorname{gr} M = \bigoplus_i M^i/M^{i+1}$ is naturally a $\operatorname{gr} \mathcal{D}_{\mathbb{A}^1} = \mathbb{C}[t, \partial]$-module.

Recall that for a commutative ring A and $M \in A\text{-mod}$, we can define

$$\mathsf{Supp}(M) = \mathsf{Spec}(A/\sqrt{\mathsf{Ann}\, M}).$$

Now, for a $\mathcal{D}_{\mathbb{A}^1}$-module M, we define its *singular support* to be $\mathsf{Supp}\, M := \mathsf{Supp}(M)$; we will see later that $\mathsf{Supp}\, M$ does not depend on the choice of a good filtration. Note that $\mathsf{Supp}\, M$ is a subvariety in $\mathsf{Spec}\, \mathbb{C}[t, \partial] = T^*\mathbb{A}^1$.

Example 2.3.2. (1) let $M = \mathbb{C}[t]$. Then $M^i = M$, $i \geq 0$ is a good filtration, and $\mathsf{gr}\, M = \mathbb{C}[t, \partial]/(\partial)$, so that $\mathsf{Supp}\, M = \{\partial = 0\}$;
(2) let $M = \mathcal{D}_{\mathbb{A}^1}$. Then filtration by order is a good filtration, and the singular support is the whole $T^*\mathbb{A}^1$.
(3) let $M = \delta_0$. Again, we filter by order; then $\mathsf{gr}\, M = \mathbb{C}[t, \partial]/(t)$ and $\mathsf{Supp}\, M = \{t = 0\}$.

Question 2.3.3. Compute the singular support of $\mathbb{C}[t, t^{-1}]$.

2.4. Multiple variables.

We can define an analogous ring for $X = \mathbb{A}^n$:

$$\mathcal{D}_{\mathbb{A}^n} = \mathbb{C}\langle t_1, \ldots, t_n, \partial_1, \ldots, \partial_n \rangle / ([\partial_i, t_j] = \delta^{ij}),$$

where δ^{ij} is the Kronecker symbol. Note that we have $\mathcal{D}^{\leq 0}_{\mathbb{A}^n} = \mathbb{C}[t_1, \ldots, t_n]$, and

$$\mathsf{gr}^1\, \mathcal{D}_{\mathbb{A}^n} = \mathcal{D}^{\leq 1}_{\mathbb{A}^n} / \mathcal{D}^{\leq 0}_{\mathbb{A}^n} = \mathbb{C}[t_1, \ldots, t_n]\partial_1 \oplus \ldots \oplus \mathbb{C}[t_1, \ldots, t_n]\partial_n$$

is the space of polynomial vector fields on $\mathbb{A}^n$.

All the examples for $\mathcal{D}_{\mathbb{A}^1}$ can be generalized to our case.

Question 2.4.1. Let $W = \mathbb{A}^m \subset \mathbb{A}^n$, $m \leq n$. Then

$$\delta_W = \mathbb{C}[t_1, \ldots, t_m, \partial_{m+1}, \ldots, \partial_n]$$

has a left $\mathcal{D}$-module structure, where t_i acts by multiplication and ∂_i by derivation for $1 \leq i \leq m$, and vice versa for $m + 1 \leq i \leq n$. Compute the singular support of δ_W.

3. Differential operators on smooth affine varieties

After going through examples, let us give general definitions. In the first half of this course, we are going to concentrate on the case of affine varieties for simplicity. Let X be a smooth affine variety over $\mathbb{C}$, and let $\mathcal{O}_X$ denote the ring of functions on X.

3.1. Definitions.

Recall that we want to generalize the notion of flat connections, which had two equivalent definitions. The first one was in terms of vector bundles on X. Algebraically speaking, we have

$$\mathsf{Vect}_X := \mathsf{Der}(\mathcal{O}_X) = \{A : \mathcal{O}_X \to \mathcal{O}_X \mid A(fg) = A(f)g + fA(g)\}.$$

Differential-geometric Lie bracket of vector fields then corresponds to the commutator $[A, B] = AB - BA$. Let us encapsulate all the properties that we want from Vect_X in one definition:

Definition 3.1.1. A *Lie algebroid* over $\mathcal{O}_X$ is an $\mathcal{O}_X$-module L, equipped with a Lie bracket $[,]$ and a homomorphism of Lie algebras $\rho : L \to \mathsf{Vect}_X$, satisfying

$$[A, fB] = \rho(A)f \cdot B + f[A, B], \qquad A, B \in L, f \in \mathcal{O}_X.$$

Remark 3.1.2. As one might expect, there exists a notion of Lie groupoid, which, when considered infinitesimally, produces a Lie algebroid.

Example 3.1.3. Let $\mathfrak{D}_X = \mathcal{O}_X \oplus \mathsf{Vect}_X$, where $[A, f] = A(f)$ and $[f, g] = 0$, and let $\rho : \mathfrak{D}_X \to \mathsf{Vect}_X$ be the natural projection. This is a Lie algebroid.

Given a Lie algebroid L over $\mathcal{O}_X$, consider its *universal enveloping* algebra $U_{\mathcal{O}_X}L$. Abstractly speaking, one defines $U_{\mathcal{O}_X}(-)$ as a left adjoint of a certain functor. Alternatively, we can define $U_{\mathcal{O}_X}L$ in a hands-on way as follows:

- take the tensor algebra $T_{\mathcal{O}_X}(L) = \bigoplus_n L^{\otimes n}$;
- quotient out the relations

$$[A, fB] = f[A, B] + A(f)B, \quad f \cdot A = fA.$$

The tensor algebra $T_{\mathcal{O}_X}(L)$ is naturally equipped with a filtration

$$(T_{\mathcal{O}_X}(L))^{\leq i} = \bigoplus_{n \geq i} L^{\otimes n},$$

which induces a filtration on $U_{\mathcal{O}_X}L$.

Definition 3.1.4 (First definition)**.** The algebra $\mathcal{D}_X$ is the universal envelope of the Lie algebroid $\mathfrak{D}_X$:

$$\mathcal{D}_X = U_{\mathcal{O}_X}(\mathcal{O}_X \oplus \mathsf{Vect}_X).$$

Remark 3.1.5. For any L locally free over $\mathcal{O}_X$, we have the Poincaré-Birkhoff-Witt isomorphism for algebroids:

$$\mathsf{gr}\, U_{\mathcal{O}_X}(L) = \mathsf{Sym}_{\mathcal{O}_X} L.$$

In particular, $\mathsf{gr}\,\mathcal{D}_X = \mathsf{Sym}_{\mathcal{O}_X} \mathsf{Vect}_X$, and $\mathsf{Spec}(\mathsf{gr}\,\mathcal{D}_X) \simeq T^*X$.

Question 3.1.6. Prove the Poincaré-Birkhoff-Witt isomorphism theorem for Lie algebras (see exercise sheets for details).

Let us define $\mathcal{D}_X$ in a more intrinsic way. Consider the vector space $\mathsf{Hom}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ (attention to the base ring!). We have

$$\mathcal{O}_X \to \mathsf{Hom}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X), \qquad f \mapsto (f \cdot -).$$

Set $\mathcal{D}_X^{\leq 0} = \mathcal{O}_X$, and

$$\mathcal{D}_X^{\leq n} = \left\{ A \mid [A, f] \in \mathcal{D}_X^{\leq n-1} \text{ for all } f \in \mathcal{O}_X \right\}.$$

Question 3.1.7. Show that $\mathcal{D}_X^{\leq 1} = \mathcal{O}_X \oplus \mathsf{Der}(X)$.

Question 3.1.8. Show that $\mathcal{D}_X^{\leq i}\mathcal{D}_X^{\leq j} \subset \mathcal{D}_X^{\leq i+j}$ and $[\mathcal{D}_X^{\leq i}, \mathcal{D}_X^{\leq j}] \subset \mathcal{D}_X^{\leq i+j-1}$.

Definition 3.1.9 (Second definition)**.** We define $\mathcal{D}_X = \bigcup_n \mathcal{D}_X^{\leq n}$.

Proposition 3.1.10. *Let X be smooth. Then the two definitions are equivalent.*

Let us first prove Proposition 3.1.10 for $X = \mathbb{A}^n$, $\mathcal{O}_X = \mathbb{C}[t_1, \dots, t_n]$. The universal enveloping algebra of $\mathcal{O}_X \oplus \mathsf{Vect}_X$ is then just the Weyl algebra in n variables. Let $W^{\leq n}$ be the operators of degree less or equal to n. We need to show the following for all n:

$$W^{\leq n+1} = \{A \in \mathsf{Hom}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X) \mid [A, f] \in W^{\leq n} \text{ for all } f \in \mathcal{O}_X\}.$$

Let A be an element in the right-hand side, and denote $A_i = [A, t_i]$. Since $A_i \in W^{\leq n}$, we can write it as a polynomial in t_i, ∂_i, where all derivatives are on the right. Then one has $[A_i, t_j] = \frac{\partial A_i}{\partial(\partial_j)}$. Moreover, we clearly have

$$\frac{\partial A_i}{\partial(\partial_j)} = [A_i, t_j] = [[A, t_i], t_j] = [[A, t_j], t_i] = \frac{\partial A_j}{\partial(\partial_i)}.$$

The fundamental theorem of multivariate calculus says that then there exists $A' \in W^{\leq n+1}$ such that $[A', t_i] = A_i$ for all i. Consider $B = A - A' \in \mathcal{D}_X^{\leq n+1}$. We have $[B, t_i] = 0$ for all i. We will prove that this implies $B \in \mathcal{O}_X$. Reasoning by induction, we can assume that $B \in \mathcal{D}_X^{\leq 1}$. Then $B = f_0 + \sum_i f_i \partial_i$, $f_i \in \mathcal{O}_X$ by Question 3.1.7. By the assumption, we have $f_i = [B, t_i] = 0$, so that $B \in \mathcal{O}_X$. Thus both $A \in W^{\leq n+1}$, and we may conclude.

Sketch of proof of Proposition 3.1.10. By the universal property and Question 3.1.7, we have a map $U_{\mathcal{O}_X}(\mathcal{O}_X \oplus \mathsf{Vect}_X) \longrightarrow \mathcal{D}_X$. Since it is a map of $\mathcal{O}_X$-modules, it suffices to prove the claim locally. One can check that for any maximal ideal $\mathfrak{m} \subset \mathcal{O}_X$ our map localizes to $U_{\mathcal{O}_{X,\mathfrak{m}}}(\mathcal{O}_{X,\mathfrak{m}} \oplus \mathsf{Vect}_{X,\mathfrak{m}}) \longrightarrow \mathcal{D}_{X,\mathfrak{m}}$. Since X is smooth, this reduces us to the case $X = \mathbb{A}^n$, which was proved above. □

Note that the smoothness condition on X is crucial.

Question 3.1.11. Let $\mathcal{O}_X = \mathbb{C}[t^2, t^3]$.

(1) prove that $\mathsf{Der}(\mathcal{O}_X)$ is generated by $t\partial_t$, $t^2\partial_t$ as an $\mathcal{O}_X$-module;
(2) show that $d := t\partial_t^2 - \partial_t$ belongs to $\mathcal{D}_X^{\leq 2}$;
(3) show that d cannot be expressed as a polynomial of elements in $\mathcal{O}_X$ and $\mathsf{Der}(\mathcal{O}_X)$. Conclude that Proposition 3.1.10 does not hold for X.

Definition 3.1.12. Let X be smooth affine. A *(left) $\mathcal{D}$-module* on X is a (left) $\mathcal{D}_X$-module. Right modules are defined analogously.

Unraveling this definition, the data defining a $\mathcal{D}$-module is:

- an $\mathcal{O}_X$-module M,
- a *covariant derivative* $\mathsf{Vect}_X \times M \longrightarrow M$, $(\xi, m) \mapsto \nabla_\xi(m)$,
- which satisfy the following conditions:

$$\nabla_\xi \circ \nabla_\eta - \nabla_\eta \circ \nabla_\xi = \nabla_{[\xi,\eta]},$$
$$\nabla_{f\xi}(m) = f\nabla_\xi(m), \qquad \nabla_\xi(fm) = \xi(f)m + f\nabla_\xi(m).$$

Given a Lie algebra $\mathfrak{g}$, we have three standard modules: trivial, adjoint, and $U(\mathfrak{g})$. In the global case of $\mathcal{D}$-modules, however, there is no adjoint representation!

Question 3.1.13. Try to construct an adjoint $\mathcal{D}$-module. Which condition breaks?

Example 3.1.14. (1) $M = \mathcal{O}_X$, covariant derivative is the usual action by derivation;
(2) $M = \mathcal{D}_X$, rank 1 free $\mathcal{D}$-module.

Question 3.1.15. Let $M, N \in \mathcal{D}_X$-mod. Equip the module $M \otimes_{\mathcal{O}_X} N$ with the following covariant derivative:

$$\nabla_\xi(m \otimes n) = \nabla_\xi^M(m) \otimes n + m \otimes \nabla_\xi^N(n).$$

Check that this defines a $\mathcal{D}_X$-module structure on $M \otimes_{\mathcal{O}_X} N$. This turns $\mathcal{D}_X$-mod into a monoidal category.

3.2. **Direct and inverse images.**

3.2.1. *The case of $\mathcal{O}_X$-modules.* Consider a map $X \longrightarrow Y$, associated to the map of rings $\mathcal{O}_Y \to \mathcal{O}_X$. When we work with quasi-coherent sheaves, we have a pair of adjoint functors

$$f^* : \mathcal{O}_Y\text{-mod} \rightleftarrows \mathcal{O}_X\text{-mod}.$$

In the affine case $f : X = \operatorname{Spec} A \longrightarrow Y = \operatorname{Spec} B$, this boils down to

$$A \otimes_B - : B\text{-mod} \rightleftarrows A\text{-mod} : \mathrm{res}_{A \to B}$$

3.2.2. *Functors between module categories, bi-modules, and correspondences.* For $\mathcal{D}$-modules, we don't have map in either direction. Roughly speaking, $\mathcal{D}$ is mixture of functions and derivations. Functions pullback, whereas derivation push-forward. However, not all is lost.

Consider an abstract situation where we have rings A, B and we want to constuct a functors

$$F : A\text{-mod} \longrightarrow B\text{-mod}.$$

In general, there are a lot of functors, but let us restrict to functors which respect colimits. Then, all such functors are given by $F(A)$.

First of all note that by functoriality, $F(A)$ has a structure of a right A-module, compatible with the left B-module structure. I.e. $F(A)$ is an object of B-mod-A. Now, for any $M \in A$-mod, we have a resolution

$$A^{\oplus I_1} \longrightarrow A^{\oplus I_0} \longrightarrow M \longrightarrow 0.$$

Applying F, which is right-exact, we get a resolution of $F(M)$:

$$F(A)^{\oplus I_1} \longrightarrow F(A)^{\oplus I_0} \longrightarrow F(M) \longrightarrow 0.$$

The first two terms are precisely

$$F(A) \otimes_A (A^{\oplus I_1} \longrightarrow A^{\oplus I_0}) \tag{3.2.3}$$

and hence, since tensoring is right-exact,

$$F(M) \simeq \operatorname{coker}(3.2.3) \simeq F(A) \otimes_A M.$$

To sum up, *nice* functors A-mod $\longrightarrow$ B-mod are of given by $P \otimes_A -$ where $P \in B$-mod-A. Given such a $P \in B$-mod-A, we can also construct a functor

$$- \otimes_B P : \text{mod-}B \longrightarrow \text{mod-}A.$$

This is what will happen with $\mathcal{D}$-modules. Namely, for $f : X \longrightarrow Y$ between smooth affine varieties, we will define $\mathcal{D}_{X\to Y} \in \mathcal{D}_X$-mod-$\mathcal{D}_Y$, which, as discussed above, will define for us functors

$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} - : \mathcal{D}_Y\text{-mod} \longrightarrow \mathcal{D}_X\text{-mod}$$
$$- \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} : \text{mod-}\mathcal{D}_X \longrightarrow \text{mod-}\mathcal{D}_Y$$

Remark 3.2.4. Assume that A and B are commutative algebras (so that we don't have to worry about left and right modules). Then, $A\text{-mod} \simeq \mathrm{QCoh}(\operatorname{Spec} A)$ and $A\text{-mod-}B \simeq (A \otimes B)\text{-mod} \simeq \mathrm{QCoh}(\operatorname{Spec}(A \otimes B)) \simeq \mathrm{QCoh}(\operatorname{Spec} A \times \operatorname{Spec} B)$. Given a module $P \in \mathrm{QCoh}(\operatorname{Spec} A \times \operatorname{Spec} B)$, we can cook up a functor $\mathrm{QCoh}(\operatorname{Spec} A) \to \mathrm{QCoh}(\operatorname{Spec} B)$ by pulling, tensoring with P, and pushing along the following correspondence

$$\begin{array}{ccc} & \operatorname{Spec} A \times \operatorname{Spec} B & \\ \swarrow^{f} & & \searrow^{g} \\ \operatorname{Spec} A & & \operatorname{Spec} B \end{array}$$

I.e. we are looking at the following functor $g_*(f^*(-) \otimes P)$.

More generally, given a correspondence of affine schemes $\operatorname{Spec} A \leftarrow \operatorname{Spec} C \to \operatorname{Spec} B$ and $M \in \mathrm{QCoh}(\operatorname{Spec} C)$ we also obtain a functor. Moreover, the module on $\mathrm{QCoh}(\operatorname{Spec} A \times \operatorname{Spec} A)$ could be obtained by pushing forward along $\operatorname{Spec} C \to \operatorname{Spec} A \times \operatorname{Spec} A$. Note that this works more generally: for a correspondence $X \leftarrow C \to Y$, we only need $C \to X \times Y$ affine.

Remark 3.2.5. $\mathcal{D}$-modules are supposed to be a deformed version of $\mathrm{QCoh}(T^*X)$. So let's now think about T^*X a bit. Consider a map $X \to Y$ of schemes, then, we have the following

$$\begin{array}{ccccc} TX & & \longrightarrow & & \\ & \searrow^{\cdots} & & & \searrow \\ & & X \times_Y TY & \longrightarrow & TY \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

where the map $TX \to X \times_Y TY$ is a map vector bundles. Taking the dual, we obtain the following correspondence

$$\begin{array}{ccc} & X \times_Y T^*Y & \\ \swarrow & & \searrow \\ T^*X & & T^*Y \end{array}$$

It is easy to check that $X \times_Y T^*Y \to T^*X \times T^*Y$ is affine. The above discussion allows us to form functors between $\mathrm{QCoh}(T^*X)$ and $\mathrm{QCoh}(T^*Y)$. Our goal now is to actual do this for $\mathcal{D}$-modules.

3.2.6. *The case of $\mathcal{D}$-modules.* Since $\mathcal{D}$-modules are defined using sheaf of differential operators and vector fields, it's more convenient to say what we've discussed above in terms of sheaves.

Consider the natural map, analogous to the left arrow in the correspondence,

$$\mathrm{Vect}_X \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathrm{Vect}_Y,$$
$$\xi \mapsto f_{(1)} \otimes \xi_{(2)}.$$

Here, we have used Sweedler's notation.

Definition 3.2.7. Let $\mathcal{D}_{X\to Y}$ be the following bimodule:

- as an $\mathcal{O}_X$-module, $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$,
- right $\mathcal{D}_Y$-module structure is given by multiplication on the right,

- left $\mathcal{D}_X$-module structure is given by the covariant derivative
$$\xi(f \otimes m) = \xi(f) \otimes m + f f_{(1)} \otimes \xi_{(2)}(m).$$

This allows us to define the following functors:
$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} - : \mathcal{D}_Y\text{-mod} \to \mathcal{D}_X\text{-mod}$$
$$- \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} : \text{mod-}\mathcal{D}_X \to \text{mod-}\mathcal{D}_Y$$

Remark 3.2.8. The underlying $\mathcal{O}_X$-module of a $\mathcal{D}$-module pullback is particularly simple. Indeed,
$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{D_Y} M \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} M,$$
which is the same as the $\mathcal{O}$-module pullback. Thus, the $\mathcal{D}$-module pullback construction above could be viewed as defining a $\mathcal{D}_X$-module struture on the $\mathcal{O}$-module pullback. In fact, $\mathcal{D}_{X\to Y}$ is this construction applied to the $\mathcal{D}_Y$-module $\mathcal{D}_Y$.

Question 3.2.9. Let $y_1, \ldots, y_n, \partial_1, \ldots, \partial_n$ be a local coordinate system on Y. Show that the map $\mathsf{Vect}_X \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathsf{Vect}_Y$ associated to $f : X \to Y$ sends ξ to $\sum_i \xi(y_i \circ f) \otimes \partial_i$. Thus, in terms of local coordinates, we have
$$\xi(f \otimes m) = \xi(f) \otimes m + f \sum_i \xi(y_i \circ f) \otimes \partial_i m.$$

Remark 3.2.10. Let's phrase everything in terms of differentials. First, for any affine scheme $X = \mathsf{Spec}\, A$, $\mathsf{Vect}_X = \Omega_A^\vee = \mathsf{Hom}_A(\Omega_A, A) = \mathsf{Der}(A, A)$, where Ω_A is the module of Kähler differentials. For any $f : \mathsf{Spec}\, A \to \mathsf{Spec}\, B$, by universal property, we have $A \otimes_B \Omega_B \to \Omega_A$. Dualizing, we obtain
$$\mathsf{Der}(A, A) = \mathsf{Hom}_A(\Omega_A, A) \to \mathsf{Hom}_A(A \otimes_B \Omega_B, A) \simeq \mathsf{Hom}_B(\Omega_B, A) \simeq A \otimes_B \Omega_B^\vee \simeq \mathsf{Der}(B, A).$$

Example 3.2.11 (Open affine embedding). Let $j : X \hookrightarrow Y$ be an open embedding of affine schemes defined by $X = \{f \neq 0\}$. In other words, $\mathcal{O}_X = (\mathcal{O}_Y)_f$ (localized at f). Then, $\mathcal{D}_X = (\mathcal{D}_Y)_f \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \simeq \mathcal{D}_{X\to Y}$. Thus, in this case, we have $\mathcal{D}_Y \to \mathcal{D}_X$ and $\mathcal{D}_{X\to Y} = \mathcal{D}_X$.

Thus, the pullback of a $\mathcal{D}_Y$-module M is just M_f with the obvious $\mathcal{D}_X$-module structure. Similarly, if N is a right $\mathcal{D}_X$-module, then its pushforward is $N \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \simeq N \otimes_{\mathcal{D}_X} \mathcal{D}_X \simeq N$.

Let's consider an example of this example: $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$. Writing $\mathbb{A}^1 = \mathsf{Spec}\, \mathbb{C}[t]$ and $\mathbb{A}^1 \setminus \{0\} = \mathsf{Spec}\, \mathbb{C}[t, t^{-1}]$, then $\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle t, \partial \rangle$. Pulling back is just localizing by inverting t. For example, $\mathbb{C}[t]$ pulls-back to $\mathbb{C}[t, t^{-1}]$.

For push-forward, since we haven't thought much about examples of right $\mathcal{D}$-modules. In the case of $\mathbb{C}\langle t, \partial \rangle$, note that there is an isomorphism of rings
$$\mathbb{C}\langle t, \partial \rangle \to \mathbb{C}\langle t, \partial \rangle^{\text{op}}$$
$$t \mapsto t$$
$$\partial \mapsto -\partial$$

Under this isomorphism, we have an equivalence of categories $\mathbb{C}\langle t, \partial \rangle\text{-mod} \simeq \text{mod-}\mathbb{C}\langle t, \partial \rangle$. Thus, we can view $\mathbb{C}[t, t^{-1}]$ as a right $\mathcal{D}_{\mathbb{A}^1 \setminus \{0\}}$-module. The push-forward is still just $\mathbb{C}[t, t^{-1}]$.

Note that the right $\mathcal{D}$-module of $\mathbb{C}[t, t^{-1}]$ is somewhat unusual. So we need to be careful when computing. For example, $t \cdot \partial = -\partial(t) = -1$.

Note also that in this case, both pullback and pushforward of $\mathcal{D}$-modules agree with the corresponding functors for $\mathcal{O}$-modules.

Example 3.2.12 (Projection to a point). Let $f : X \to \mathrm{pt} = \operatorname{Spec} \mathbb{C}$. In this case, $\mathcal{D}_{X\to\mathrm{pt}} = \mathcal{O}_X$. Thus, the pullback of $\mathbb{C}$ is $\mathcal{O}_X$ with the usual action of $\mathcal{D}_X$. Let N be a right $\mathcal{D}_X$-module. Then, the pushforward is $N \otimes_{\mathcal{D}_X} \mathcal{O}_X$.

Let's compute the pushforward for $\mathbb{A}^1$. Namely, we need to compute $\mathbb{C}[t] \otimes_{\mathbb{C}\langle t,\partial\rangle} \mathbb{C}[t]$. This looks a bit confusing, so let us resolve the right copy of $\mathbb{C}[t]$

$$0 \to \mathbb{C}\langle t, \partial\rangle \xrightarrow{\cdot\partial} \mathbb{C}\langle t, \partial\rangle \to \mathbb{C}[t] \to 0.$$

Applying $\mathbb{C}[t] \otimes_{\mathbb{C}\langle t,\partial\rangle} -$ to the first two terms, we have the following chain complex

$$0 \to \mathbb{C}[t] \xrightarrow{\cdot\partial} \mathbb{C}[t] \to 0$$

whose zeroth homology is $\mathbb{C}[t] \otimes_{\mathbb{C}\langle t,\partial\rangle} \mathbb{C}[t] \simeq 0$ and first homology is $\mathbb{C}$.

Example 3.2.13 (Closed immersions). We will consider the closed embedding $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$ of the LHS to the first n coordinates in the RHS. We have

$$\mathcal{D}_{\mathbb{A}^n\to\mathbb{A}^{n+m}} \simeq \mathbb{C}\langle t_1, \dots, t_n, \partial_1, \dots, \partial_{n+m}\rangle$$

Pulling back a left $\mathcal{D}_{\mathbb{A}^{n+m}}$-module M gives

$$\mathbb{C}[t_1, \dots, t_n] \otimes_{\mathbb{C}[t_1,\dots,t_{n+m}]} M \simeq M/(t_{n+1}, \dots, t_{n+m})M$$

Similary, pushing forward a $\mathcal{D}_{\mathbb{A}^n}$-right module N gives

$$N \otimes_{\mathbb{C}\langle t_1,\dots,t_n,\partial_1,\dots,\partial_n\rangle} \mathbb{C}\langle t_1, \dots, t_n, \partial_1, \dots, \partial_{n+m}\rangle \simeq N[\partial_{n+1}, \dots, \partial_{n+m}].$$

In particular, pushing forward $\mathbb{C}$ along $\{0\} \hookrightarrow \mathbb{A}^n$ gives $\mathbb{C}[\partial_1, \dots, \partial_n]$.

3.3. **Right vs. left $\mathcal{D}$-modules.** For a morphism of affine schemes $f : X \to Y$, using $\mathcal{D}_{X\to Y}$, we can pullback left $\mathcal{D}_Y$-modules and pushforward right $\mathcal{D}_X$-modules. But how do we pushforward left $\mathcal{D}_X$-modules? As it turns out, the category of left and right $\mathcal{D}$-modules are equivalent.

First, let us unravel the definition of a right $\mathcal{D}$-module.

Lemma 3.3.1. *Let M be an $\mathcal{O}_X$-module. Giving a right $\mathcal{D}_X$-module on M extending the $\mathcal{O}_X$-module structure is equivalent to giving a $\mathbb{C}$-linear morphism*

$$\nabla' : \mathsf{Vect}_X \to \mathsf{End}_{\mathbb{C}}(M)$$

satisfying the following conditions for all $s \in M$, $\xi, \xi_1, \xi_2 \in \mathsf{Vect}_X$, and $f \in \mathcal{O}_X$

(i) $\nabla'_{(f\xi)}(s) = \nabla'_\xi(fs)$

(ii) $\nabla'_\xi(fs) = \xi(f)s + f\nabla'_\xi(s)$

(iii) $\nabla'_{[\xi_1,\xi_2]}s = [\nabla'_{\xi_1}, \nabla'_{\xi_2}]s$

In terms of ∇', the right $\mathcal{D}_X$-module is given by

$$s\xi = -\nabla'_\xi(s). \tag{3.3.2}$$

Remark 3.3.3. The confusing part about this lemma is that we are trying to define a "right structure" using "left notation." If our ring were commutative, then left modules are the same as right modules on the nose. In the case of $\mathcal{D}$-modules, the x's and ∂'s commute among themselves but not with each other. The minus sign appearing in (3.3.2) is designed to make (ii) works (the minus sign essentially comes from $[\xi, f]$ vs $[f, \xi]$).

Philosophically speaking, one should think of left $\mathcal{D}$-modules as functions and right $\mathcal{D}$-modules as distributions. If we view distributions as the dual of functions, then the left action of differential operators on functions should induce a right action of distributions. We will now define a right module structure on $\Omega_X^{\text{top}} = \wedge^{\text{top}}\Omega_X$.

There is a natural action of Vect_X on Ω_X^{top} called *Lie* derivative. For $\xi \in \mathsf{Vect}_X$, we define

$$((\mathsf{Lie}\,\xi)\omega)(\xi_1, \dots, \xi_n) := \xi(\omega(\xi_1, \dots, \xi_n)) - \sum_{i=1}^{n} \omega(\xi_1, \dots, [\xi, \xi_i], \dots, \xi_n).$$

In general, for a d-th form ω,

$$(\mathsf{Lie}\,\xi)\omega = i_\xi d\omega + d(i_\xi \omega),$$

where,

$$i_\xi : \Omega_X^d \to \Omega_X^{d-1}$$
$$(i_\xi \omega)(\xi_1, \dots, \xi_{d-1}) = \omega(\xi, \xi_1, \dots, \xi_{d-1}).$$

When ω is a top form, however, we get

$$(\mathsf{Lie}\,\xi)\omega = d(i_\xi \omega).$$

Remark 3.3.4. To show that the two formula are the same, we need to use the following identity

$$(d\omega)(\xi_1, \dots, \xi_{n+1}) = \sum_i (-1)^i \xi_i \omega(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{n+1}) + \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{n+1}).$$

Lemma 3.3.5. *We have*

(*i*) $(\mathsf{Lie}(f\xi))\omega = (\mathsf{Lie}\,\xi)(f\omega)$
(*ii*) $(\mathsf{Lie}\,\xi)(fw) = \xi(f)w + f\,\mathsf{Lie}(\xi)w$
(*iii*) $\mathsf{Lie}([\xi_1, \xi_2])w = [\mathsf{Lie}\,\xi_1, \mathsf{Lie}\,\xi_2]w.$

Thus, Lemma 3.3.1 above equips Ω_X^{top} with a right $\mathcal{D}_X$-module.

Proof. Exercise. □

Lemma 3.3.6. *In terms of a local coordinate system $\{x_i, \xi_i\}$, we have*

$$(f dx_1 \wedge \cdots \wedge dx_n) \cdot P(x, \partial) = ({}^t P(x, \partial) \cdot f) dx_1 \wedge \cdots \wedge dx_n,$$

where for $P(x, \partial) = \sum_\alpha x^\alpha \partial^\alpha$, ${}^t P(x, \partial) = \sum_\alpha (-\partial)^\alpha x^\alpha$ is the transpose *of $P(x, \partial)$.*

Proof. I will only prove a special case of one variable

$$f\,dx \cdot \partial_x = -\,\mathsf{Lie}(\partial_x)(f\,dx) = -d(f\,i_{\partial_x} dx) = -df = -\partial_x f\,dx.$$

□

Note that for any ring R, being a right R-module is the same as being a left R^{op}-module. We will thus use $\mathcal{D}_X^{\mathrm{op}}$-mod to denote the category of right $\mathcal{D}$-modules. The lemma above implies that once we've chosen a local coordinate system, we can use the adjoint operator to identify $\mathcal{D}_X$ and $\mathcal{D}_X^{\mathrm{op}}$ and hence, left and right modules. This was exactly what we did in the example above.

Proposition 3.3.7. *Let $M, N \in \mathcal{D}_X$-mod and $M', N' \in \mathcal{D}_X^{\mathrm{op}}$-mod. Then*

(*i*) $M \otimes_{\mathcal{O}_X} N \in \mathcal{D}_X\text{-mod}, \qquad \xi(s \otimes t) = (\xi s) \otimes t + s \otimes (\xi t).$
(*ii*) $M' \otimes_{\mathcal{O}_X} N \in \mathcal{D}_X^{\mathrm{op}}\text{-mod}, \qquad (s \otimes t)\xi = (s\xi) \otimes t - s \otimes (\xi t).$
(*iii*) $\mathsf{Hom}_{\mathcal{O}_X}(M, N) \in \mathcal{D}_X\text{-mod}, \qquad (\xi p)(s) = \xi(p(s)) - p(\xi(s)).$
(*iv*) $\mathsf{Hom}_{\mathcal{O}_X}(M', N') \in \mathcal{D}_X\text{-mod}, \qquad (p\xi)(s) = -p(s)\xi + p(s\xi).$
(*v*) $\mathsf{Hom}_{\mathcal{O}_X}(M, N') \in \mathcal{D}_X^{\mathrm{op}}\text{-mod}, \qquad (p\xi)(s) = p(s)\xi + p(\xi s).$

Corollary 3.3.8. *We have mutually inverse functors*

$$\Omega_X^{\mathrm{top}} \otimes_{\mathcal{O}_X} - : \mathcal{D}_X\text{-mod} \rightleftarrows \mathcal{D}_X^{\mathrm{op}}\text{-mod} : - \otimes_{\mathcal{O}_X} (\Omega_X^{\mathrm{top}})^{-1}.$$

Proof. Use (ii) and (iv) in the proposition above, where $M' = \Omega_X^{\mathrm{top}}$. □

Now, we can define push-forward of left $\mathcal{D}_X$-modules as follows. Let $f : X \to Y$ be a morphism between smooth affine schemes and $M \in \mathcal{D}_X$-mod. Then, the push forward of M is defined to be

$$((\Omega_X^{\mathrm{top}} \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{O}_Y} (\Omega_Y^{\mathrm{top}})^{-1}.$$

3.4. **Twisted differential operators.**

Let us recap what we did last week. For any map of smooth algebraic varieties $f : X \to Y$ we defined a natural $(\mathcal{D}_X, \mathcal{D}_Y)$-bimodule $\mathcal{D}_{X\to Y}$. Further, we defined a right $\mathcal{D}_X$-module $\omega_X := \Omega_X^{\mathrm{top}}$ of top degree differential forms on X. Tensoring with ω_X over $\mathcal{O}_X$ gives an equivalence of categories $\mathcal{D}_X\text{-mod} \to \mathcal{D}_X^{\mathrm{op}}\text{-mod}$. Using this equivalence, we can define

$$\mathcal{D}_{Y\leftarrow X} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \in \mathcal{D}_Y\text{-mod-}\mathcal{D}_X$$

This produces us the functors

$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} - : \mathcal{D}_Y\text{-mod} \to \mathcal{D}_X\text{-mod},$$
$$\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} - : \mathcal{D}_X\text{-mod} \to \mathcal{D}_Y\text{-mod},$$

as well as their right modules counterparts.

Recall that in order to define the equivalence $\omega_X \otimes_{\mathcal{O}_X} -$, we used Proposition 3.3.7. However, the tensor product between two right $\mathcal{D}$-modules, as well as one of the possible Hom-spaces were notably absent from the statement. Let us explain this omission in a conceptual fashion.

Definition 3.4.1. Let $E_1, E_2 \in \mathcal{O}_X\text{-mod}$. Similarly to Definition 3.1.9, we define

$$\mathcal{D}_X^{\leq n}(E_1, E_2) := \left\{ A \in \mathrm{Hom}_{\mathbb{C}}(E_1, E_2) \mid [A, f] \in \mathcal{D}_X^{\leq n-1}(E_1, E_2) \text{ for all } f \in \mathcal{O}_X \right\},$$

and set $\mathcal{D}_X(E_1, E_2) = \bigcup_n \mathcal{D}_X^{\leq n}(E_1, E_2) \subset \mathrm{Hom}_{\mathbb{C}}(E_1, E_2)$.

An analog of Question 3.1.8 shows that we have a composition

$$\mathcal{D}_X(E_1, E_2) \times \mathcal{D}_X(E_2, E_3) \longrightarrow \mathcal{D}_X(E_1, E_3).$$

In particular, we call $\mathcal{D}_X(E) := \mathcal{D}_X(E, E)$ the ring of differential operators *twisted by* E.

Question 3.4.2. Let E be a locally free $\mathcal{O}_X$-module. Show that $\mathcal{D}_X^{\leq 1}(E)$ is a Lie algebroid over $\mathcal{O}_X$.

One can show like before that for a locally free $\mathcal{O}_X$-module E, we have an isomorphism $\mathcal{D}_X(E) \simeq U_{\mathcal{O}_X}\left(\mathcal{D}_X^{\leq 1}(E)\right)$.

Question 3.4.3. Show that $\left(\mathcal{D}_X^{\leq 1}\right)^{\mathrm{op}} \simeq \mathcal{D}_X^{\leq 1}(\omega_X)$. Deduce that $\mathcal{D}_X^{\mathrm{op}} \simeq \mathcal{D}_X(\omega_X)$.

As in Question 3.1.15, we can define a tensor product functor

$$\otimes : \mathcal{D}_X(E_1)\text{-mod} \times \mathcal{D}_X(E_2)\text{-mod} \longrightarrow \mathcal{D}_X(E_1 \otimes_{\mathcal{O}_X} E_2)\text{-mod}.$$

In particular, we have maps

$$\mathcal{D}_X\text{-mod} \times \text{mod-}\mathcal{D}_X = \mathcal{D}_X\text{-mod} \times \mathcal{D}_X(\omega_X)\text{-mod} \longrightarrow \mathcal{D}_X(\omega_X)\text{-mod},$$
$$\text{mod-}\mathcal{D}_X \times \text{mod-}\mathcal{D}_X = \mathcal{D}_X(\omega_X)\text{-mod} \times \mathcal{D}_X(\omega_X)\text{-mod} \longrightarrow \mathcal{D}_X(\omega_X^2)\text{-mod}.$$

This explains why the tensor product of two right $\mathcal{D}$-modules is neither right nor left $\mathcal{D}$-module in general. Analogous reasoning also works for Hom-functors.

3.5. Derived category of $\mathcal{D}_X$-modules.

Definition 3.5.1. The *bounded derived category* $D^b(\mathcal{D}_X\text{-mod})$ is the bounded homotopy category of complexes of projective $\mathcal{D}_X$-modules.

Given a map $f : X \to Y$, we define derived functors in the usual way. Let $R_X(\mathcal{D}_{X\to Y})$, $R_Y(\mathcal{D}_{X\to Y})$ be finite resolutions of $\mathcal{D}_{X\to Y}$ as a $\mathcal{D}_X$- and $\mathcal{D}_Y$-module respectively. Then[1]

$$f_*(M) = R_X(\mathcal{D}_{Y\leftarrow X}) \otimes_{\mathcal{D}_X} M,$$
$$f^!(N) = R_Y(\mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_Y} N[\dim X - \dim Y].$$

We are working in the affine setting, so we always have free resolutions. However, in order to justify that these functors land in the *bounded* derived category, we need finite resolutions. Let us construct them.

Lemma 3.5.2. *For any composition* $X \to Y \to Z$, *we have*

$$\mathcal{D}_{X\to Z} = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z} = \mathcal{D}_{X\to Y} \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}.$$

[1] We will explain the homological shift later.

Proof. The proof is a formal manupulation of tensor products:

$$\begin{aligned} \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z} &= (\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \otimes_{\mathcal{D}_Y} (\mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{D}_Z) \\ &= (\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) \otimes_{\mathcal{O}_Z} \mathcal{D}_Z \\ &= \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{D}_Z = \mathcal{D}_{X\to Z}. \end{aligned}$$

The second equality is deduced analogously. We simply need to use the fact that $\mathcal{D}_X$ is a locally free $\mathcal{O}_X$-module, and replace some tensor porducts by their derived counterparts. □

Note that f always factors as a composition of a regular embedding and a projection:

$$X \xrightarrow{\mathrm{id}, f} X \times Y \xrightarrow{pr_Y} Y.$$

3.5.3. *Pushforward.* Let $X \to Y$ be a closed embedding. Then analogously to Example 3.2.13 we have

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathsf{Sym}_{\mathcal{O}_X} N_X Y.$$

In particular, $\mathcal{D}_{X\to Y}$ is free over $\mathcal{D}_X$, and the functor $- \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ is exact. Thus it suffices to construct a resolution for projection maps $X \times Y \to Y$. In order to keep notations simple, we will only consider projection to a point $X \to \mathrm{pt}$. In this case $\mathcal{D}_{X\to Y} = \mathcal{O}_X$ and $\mathcal{D}_{Y\leftarrow X} = \omega_X$. Let us denote

$$DR^k(\mathcal{O}_X) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge_{\mathcal{O}_X}^k \mathsf{Vect}_X,$$

and consider the following differential:

$$\begin{aligned} \mathrm{d} : DR^k(\mathcal{O}_X) &\to DR^{k-1}(\mathcal{O}_X), \\ d(u \otimes \xi_1 \wedge \ldots \wedge \xi_k) &= \sum_i (-1)^{i+1} u\xi_i \otimes \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k \\ &\quad + \sum_{i<j} (-1)^{i+j} u \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k. \end{aligned}$$

Question 3.5.4. Show that $\mathrm{d}^2 = 0$, $H^0 = \mathcal{O}_X$, and other cohomology groups vanish.

Thus we have obtained a resolution $\mathcal{O}_X \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^\bullet \mathsf{Vect}_X$.

Dually, we have $\omega_X \simeq \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X[\dim X]$, with the differential given by

$$\mathrm{d}(\omega \otimes u) = \mathrm{d}\omega \otimes u + \sum_i dx_i \wedge \omega \otimes \partial_i u,$$

where $\{x_i, \partial_i\}$ are local coordinates on X.

Thus the pushforward of $M \in \mathcal{D}_X\text{-mod}$ to a point is given by

$$\mathrm{pr}_*(M) = \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{D}_X} M = \Omega_X^\bullet \otimes_{\mathcal{O}_X} M.$$

For instance, when $M = \mathcal{O}_X$ we get the algebraic de Rham cohomology:

$$\mathrm{pr}_*(\mathcal{O}_X) = H^*(\Omega_X^\bullet).$$

3.5.5. *Pullback.* Let $f : X \times Y \to Y$ be a projection. Then $\mathcal{D}_{X\times Y\to Y} = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{D}_Y$ by Example 3.2.12. This is a free right $\mathcal{D}_Y$-module, so that the functor $\mathcal{D}_{X\times Y\to Y} \otimes_{\mathcal{D}_Y} -$ is exact. Thus it suffices to construct a resolution for regular embeddings.

Suppose that $i : X \hookrightarrow Y$ is a closed embedding, given by a regular sequence $(f_1, \dots, f_n) = I_X \subset \mathcal{O}_Y$. Then we have the *Koszul resolution* of $\mathcal{O}_X$:

$$\mathcal{O}_X \simeq K(I_X) := \bigotimes_i [\mathcal{O}_Y \xrightarrow{f_i} \mathcal{O}_Y].$$

More explicitly, we can write

$$K(I_X) = \left(K^n \to K^{n-1} \to \dots \to K^0\right),$$

where $K^k = \bigwedge^k \left(\bigoplus_{i=1}^n \mathcal{O}_Y dy_i\right)$, with dy_i formal symbols, and the differentials are given by

$$\mathrm{d}\left(g dy_{i_1} \wedge \dots \wedge dy_{i_k}\right) = \sum_{j=1}^k (-1)^{j+1} f_{i_j} g dy_{i_1} \wedge \dots \wedge \widehat{dy_{i_j}} \wedge \dots \wedge dy_{i_k}.$$

Question 3.5.6. Check that $K(I_X)$ is indeed a resolution of $\mathcal{O}_X$.

In particular, we have $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \simeq K(I_X) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$.

Corollary 3.5.7. *Let $X \to Y$ be a closed embedding of codimension n, where X and Y are smooth schemes. Then, $i^! M$ is concentrated in cohomological degrees $[0, n]$ for $M \in \mathcal{D}_X$-mod.*

Example 3.5.8. Consider the diagonal embedding $\Delta : X \hookrightarrow X \times X$. Then one can check that $\Delta^!(M \boxtimes N)[\dim X] = M \otimes_{\mathcal{O}_X} N$ as defined before.

3.6. **Kashiwara's theorem.** Once again, let $i : X \hookrightarrow Y$ is a closed embedding of smooth varieties, given by an ideal $I_X \subset \mathcal{O}_Y$.

Definition 3.6.1. A $\mathcal{D}_Y$-module M is *topologically supported* on X, if the action of I_X on M is *locally nilpotent*, that is every $m \in M$ is annihilated by some power of I_X:

$$\exists n > 0 : I_X^n m = 0.$$

Theorem 3.6.2 (Kashiwara's theorem)**.** *Let $(\mathcal{D}_Y\text{-mod})_X$ denote the full subcategory of $\mathcal{D}_Y$-mod, consisting of modules topologically supported on X. Then the pushforward functor is an equivalence of categories*:

$$i_* : \mathcal{D}_X\text{-mod} \xrightarrow{\sim} (\mathcal{D}_Y\text{-mod})_X.$$

Moreover, the inverse is given by $i^! \simeq \mathrm{H}^0(i^!)$.

There is also a derived version. We start with the following

Definition 3.6.3. Let $i : X \to Y$ be a closed embedding of smooth affine schemes. Let $D^b(\mathcal{D}_Y\text{-mod})_X$ be the full subcategory of $D^b(\mathcal{D}_Y\text{-mod})$ consisting of chain complexes in $\mathcal{D}_Y$-mod whose cohomologies are in $\mathcal{D}_Y\text{-mod}_X$.

Corollary 3.6.4 (Kashiwara's theorem, derived version)**.** *Let $i : X \to Y$ be a closed embedding of smooth affine schemes. Then the pushforward $i_* : D^b(\mathcal{D}_X\text{-mod}) \to D^b(\mathcal{D}_Y\text{-mod})$ induces an equivalence of categories*

$$D^b(\mathcal{D}_X\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{D}_Y\text{-mod})_X.$$

Before proving this theorem, let us have a quick discussion about adjoint functors. Let A, B be two rings, and take an (A, B)-bimodule P. We have a natural adjunction

$$- \otimes_A P : \text{mod-}A \rightleftarrows \text{mod-}B : \mathsf{Hom}_B(P, -).$$

Question 3.6.5. Prove this adjunction, as well as its derived version.

In particular, let $A = \mathcal{D}_X$, $B = \mathcal{D}_Y$, and $P = \mathcal{D}_{X\to Y}$. We obtain an adjunction

$$i_* : \text{mod-}\mathcal{D}_X \rightleftarrows \text{mod-}\mathcal{D}_Y : \mathsf{Hom}_{\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, -)$$

Lemma 3.6.6. *For any right $\mathcal{D}_Y$-module M we have*

$$i^! M = \mathsf{RHom}_{\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, M).$$

Before proving this lemma, let us give the main idea behind the proof. Forgetting the $\mathcal{D}$-module structure (and remembering just the $\mathcal{O}$-module structure), up to a shift, pulling back is just pulling back of $\mathcal{O}$-modules. For $\mathcal{O}$-modules, there are two ways to pullback: the usual way $- \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathcal{O}_X$, and the "exceptional" way $\mathsf{RHom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)$. The content of this result is that for a closed embedding of $\mathcal{D}$-modules, exceptional pullback of a free $\mathcal{O}_Y$-module differs from the usual one by a shift and a twist by the determinant of the normal bundle.

Proof. We have isomorphisms of right $\mathcal{D}_X$-modules

$$\begin{aligned} \mathsf{RHom}_{\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, M) &= \mathsf{RHom}_{\mathcal{D}_Y}(\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, M) \\ &\simeq \mathsf{RHom}_{\mathcal{O}_Y}(\mathcal{O}_X, M) \\ &\simeq M \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathsf{RHom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y), \end{aligned}$$

where the last equivalence can be seen by picking a perfect resolution of $\mathcal{O}_X$ as an $\mathcal{O}_Y$-module. This exists in a very big generality, but in our case we can just use the Koszul resolution.

Let us consider the right $\mathcal{D}_X$-module[2] $R\,\mathsf{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)$. Since the statement is local in Y, we can shrink it and assume that X is a complete intersection. In this case, $\mathcal{O}_X$ has a Koszul resolution

$$K(I_X) = (K_d \to K_{d-1} \to \cdots \to K_0),$$

as an $\mathcal{O}_Y$-module, where $d = \dim Y - \dim X$. Each K_i is locally free, and we have a perfect pairing $K_i \otimes_{\mathcal{O}_Y} K_{d-i} \to K_d$. In particular, we have

$$\begin{aligned} \mathsf{RHom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) &= K(I_X)^\vee_\bullet \simeq K(I_X)_\bullet[-d] \otimes_{\mathcal{O}_Y} (K^d)^\vee \\ &\simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} (K^d)^\vee[-d] \simeq \wedge^d (I_X/I_X^2)^\vee[-d] \simeq \omega_X \otimes_{\mathcal{O}_Y} \omega_Y^\vee[-d]. \end{aligned}$$

In the last line, we have used the short exact sequence

$$0 \to I_X/I_X^2 \to i^*\Omega_Y \to \Omega_X \to 0$$

and the fact that I_X/I_X^2 is a free $\mathcal{O}_X$-module. For more details, see the proof of [H, Theorem III.7.11].

Question 3.6.7. Prove that I_X/I_X^2 is a locally free $\mathcal{O}_X$-module.

[2]or rather a complex thereof.

Finally, we have

$$\begin{aligned}\mathsf{RHom}_{\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, M) &\simeq M \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathsf{RHom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)\\ &\simeq M \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \left(\omega_X \otimes_{\mathcal{O}_Y} \omega_Y^\vee\right)[-d]\\ &\simeq M \otimes^{\mathsf{L}}_{\mathcal{D}_Y} \left(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}\right)[-d]\\ &\simeq M \otimes^{\mathsf{L}}_{\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X}[-d],\end{aligned}$$

and we conclude by definition of $i^! M$. □

Remark 3.6.8. Technically, we define $f^!$ for left (and not right) $\mathcal{D}$-modules. In order to fix this, we should do some side switching in the proof above. The actual statement for left $\mathcal{D}$-module is as follows: we have an equivalence of functors $i^! \simeq \mathsf{RHom}_{\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, -)$.

We obtain a pair of adjoint functors $i_* \dashv i^!$. We will prove that i_* is fully-faithful, with the essential image being given by $\mathcal{D}_Y\text{-mod}_X$. Note that fully-faithfullness of i_* is equivalent to $\mathrm{id} \to i^! i_*$ being an equivalence.

Proof of Theorem 3.6.2. Since we formulated adjunctions for right modules, we will proceed with the proof in this context. Moreover, since we only prove the non-derived statement, for brevity's sake, we will use $i^!_0$ to denote $\mathsf{H}^0(i^!)$ in this proof.

Consider the right action of $I_X \subset \mathcal{O}_Y$ on $\mathcal{D}_{X\to Y}$. Let $f \otimes A$ be a differential operator in $\mathcal{D}_{X\to Y}$, where $f \in \mathcal{O}_Y/I_X$ and $A \in \mathcal{D}_Y^{\leq n}$. For any $a_0, \ldots, a_n \in I_X$, we have the following:

$$\begin{aligned}(f\otimes A)(a_n \ldots a_0) &= \left((f a_n)\otimes A + f \otimes [A, a_n]\right)(a_{n-1}\ldots a_0)\\ &= (f \otimes [A, a_n])(a_{n-1}\ldots a_0).\end{aligned}$$

Note that $[A, a_n] \in \mathcal{D}_Y^{\leq n-1}$. By induction on n, we obtain that $(f \otimes A)(a_n \ldots a_0) = 0$. Thus I_X acts locally nilpotently on $\mathcal{D}_{X\to Y}$, and so on $i_*(M) = M \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}$ as well. This proves that the image of i_* lies in $(\mathcal{D}_Y\text{-mod})_X$.

Let us now prove that i_* is an equivalence. Choose a regular sequence $(f_1, \ldots, f_n)$ of generators of I_X. Write $X_k = \mathsf{Spec}\, \mathcal{O}_Y/(f_1, \ldots, f_k)$. Then $X_{k+1} \subset X_k$ is a smooth embedding of codimension 1 for any k. Reasoning by induction, we therefore assume that $X \subset Y$ is of codimension 1 from now on.

Suppose that $X = (f = 0) \subset Y$, so that $\mathcal{O}_X = \mathcal{O}_Y/(f)$. For any right $\mathcal{D}_Y$-module M, we write

$$M^f := \{m \in M \mid mf = 0\}.$$

This coincides with $i^!_0(M)$ as an $\mathcal{O}_X$-module. The rest of the proof is basically an exercise in linear algebra.

As before, in local coordinates we can always find a dual vector $\partial \in \mathsf{Vect}(Y)$ such that $[\partial, f] = 1$. In this case, $\mathcal{D}_{X\to Y}$ is isomorphic to $\mathcal{D}_X[\partial]$ as a left $\mathcal{D}_X$-module, and we have

$$i_*(N) = N[\partial] = \bigoplus_{n\geq 0} N\partial^n$$

for any right $\mathcal{D}_X$-module N.

Consider the grading element $s = f\partial$. It is clear that $[s, f] = 1$ and $[s, \partial] = -1$.

Question 3.6.9. Check that $N\partial^n$ is precisely the $(-n)$-th eigenspace of s acting on N. Moreover, f induces an isomorphism of vector spaces $N\partial^n \to N\partial^{n-1}$ for any $n > 0$.

In particular, we see that

$$i_0^! i_*(N) = (i_*(N))^f = N \otimes 1 \subset N[\partial],$$

and thus $N \to i^! i_*(N)$ is an isomorphism, which means that i_* is fully-faithful.

To characterize the image, note that for any $N \in (\mathcal{D}_Y\text{-mod})_X$, we can show that

$$N \simeq \bigoplus_{n \geq 0} N^i,$$

where N^i is the $(-i)$-eigenspace for $s = f\partial$. Moreover, f and ∂ are as above, i.e. ∂ increases i, f decreases i, $f : N^i \to N^{i-1}$ iso for $i > 0$ and surjective otherwise. This is proved by using local nilpotence of N.

By Corollary 3.5.7, we know that $i^!$ has cohomological amplitudes $[0, 1]$, where

$$\mathrm{H}^0(i^! N) = N^f = \ker(N \xrightarrow{f} N) = N^0,$$

$$\mathrm{H}^1(i^! N) = \operatorname{coker}(N \xrightarrow{f} N) = 0.$$

The second line shows that $i^! N \simeq \mathrm{H}^0(i^! N)$. It is easy to check that

$$i_* i^! = N^0[\partial] \simeq N,$$

which means that N is in the essential image of i_*. Hence, we are done. □

Example 3.6.10. Consider the easiest example of $0 \hookrightarrow \mathbb{A}^1$. Kashiwara's theorem says that the $(\mathcal{D}_{\mathbb{A}^1}\text{-mod})_0 \simeq \mathsf{Vect}$. In particular, $\mathcal{D}_{\mathbb{A}^1}$-modules supported at 0 do not have non-trivial extensions. This is decidedly not true for coherent sheaves on $\mathbb{A}^1$!

Let us now demonstrate a couple of applications of Kashiwara's theorem.

Proof of Corollary 3.6.4. We will prove fully-faithfulness of i_* and characterize its essential image by induction on cohomological length.

For any $M \in D^b(\mathcal{D}_X\text{-mod})$, we will show that $i^! i_* M \simeq M$. If M has length 1, we are done. Suppose we already know up to length n, for M of length $n + 1$, we have

$$\begin{array}{ccccccc}
\tau^{\leq m} M & \longrightarrow & M & \longrightarrow & \mathrm{H}^{m+1}(M)[-m-1] & \longrightarrow & \cdots \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq & & \\
i_* i^! \tau^{\leq m} M & \longrightarrow & i_* i^! M & \longrightarrow & i_* i^! \mathrm{H}^{m+1}(M)[-m-1] & \longrightarrow & \cdots
\end{array}$$

and we conclude by 2-out-of-3.

Essential image is done similarly. □

3.7. **Base change.**

Let $f : X \to Y$ be a map of smooth (affine) schemes, $y \in Y$, and $M \in \mathcal{D}_X\text{-mod}$. Let $X_y = f^{-1}(y)$ be the fiber of f at y and $f_y : X_y \to Y$. We want to understand how the "fiber" of $f_* M$ at y is related to $(f_y)_* M$. As it turns out, they are the same. In fact, we have a very general statement.

Theorem 3.7.1 (Base change)**.** *Let $f : Y \to X$, $g : Z \to X$ be two morphisms of smooth affine varieties, and consider the cartesian square*

$$\begin{array}{ccc} Y_Z & \xrightarrow{\tilde{g}} & Y \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & X \end{array}$$

Assume that $Y_Z = Y \times_X Z$ is smooth. Then we have an isomorphism of functors $\tilde{f}_* \tilde{g}^! \xrightarrow{\sim} g^! f_*$.

Remark 3.7.2. Let $f : 0 \hookrightarrow \mathbb{A}^1$. Then we have $f^! f_* = \mathrm{id}$. If we worked with coherent sheaves (say, in K-theory), the Euler class of the conormal bundle would pop up on the right-hand side. This happens because $\mathbb{C}$ is not a $\mathcal{D}$-module.

Before proving Theorem 3.7.1, we need a preliminary result.

Proposition 3.7.3 (Dévissage)**.** *Let X be a smooth variety, Z a smooth closed subvariety and $U = X \setminus Z$. Let i and j be the corresponding closed and open embeddings.*

(*i*) *We have $j^! i_* \simeq 0$ and $i^! j_* \simeq 0$;*

(*ii*) *For $M \in D^b(\mathcal{D}_X\text{-mod})$ we have a distinguished triangle*

$$i_* i^! M \to M \to j_* j^! M \xrightarrow{+1}$$

Proof. First claim is an exercise. For the second claim, recall that $i_* \dashv i^!$. It is also easy to see that $j^! \dashv j_*$ and moreover, $\mathrm{id} \simeq j^! j_*$, i.e. j_* is fully-faithful. Consider the unit map $M \to j_* j^! M$. Let K be the co-cone of this, i.e. we have the following triangle

$$K \to M \to j_* j^! M \xrightarrow{+1}$$

We want to show that $K \simeq i_* i^! M$. Applying $j^!$ to this triangle, we see that $j^! K \simeq 0$. Since $j^!$ is exact, we know that $j^! \mathrm{H}^*(K) = \mathrm{H}^*(j^! K) \simeq 0$. Thus, K is topologically supported along Z; in other words, $K \in D^b(X)_Z$.[3] Now, applying $i^!$ to the triangle above, we get $i^! K \simeq i^! M$ since $i^! j_* \simeq 0$. By Kashiwara, we get that $K \simeq i_* i^! M$. □

We are now ready to prove the base change theorem.

Proof of Theorem 3.7.1. As before, we can decompose g in a composition of a closed embedding with aa projection. Since pullbacks are functorial, it suffices to prove the claim for projections and closed embeddings.

First, assume $g = \mathrm{pr}_X : T \times X \to X$ is a projection. Then $\tilde{g} = \mathrm{pr}_Y$, $\tilde{f} = \mathrm{id}_T \times F$, and for any $M \in D^b(\mathcal{D}_Y\text{-mod})$ we have

$$\tilde{f}_* \tilde{g}^!(M) \simeq \mathrm{id}_T \times f_*(\mathcal{O}_T \boxtimes M)[\dim T] \simeq \mathcal{O}_T \boxtimes f_*(M)[\dim T] \simeq g^! f_*(M).$$

Now, suppose $g : Z \hookrightarrow X$ is a closed embedding then so is $\tilde{g}$. Using Kashiwara's theorem, we have (we ignore all the tildes here as they should be clear from the context)

$$f_* g^! \simeq g^! g_* f_* g^! \simeq g^! f_* g_* g^!.$$

[3]Here, we use the fact that for a module M over a ring R and $f \in R$. If $M_f = 0$, then M consists of only locally f-nilpotent elements.

It remains to show that $g^! f_* g_* g^! \simeq g^! f_*$. Consider the following diagram, where all squares are cartesian, and j's are open complement of g's

$$\begin{array}{ccccc} Y_Z & \xrightarrow{g} & Y & \xleftarrow{j} & U_Y \\ \downarrow f & & \downarrow f & & \downarrow f \\ Z & \xrightarrow{g} & X & \xleftarrow{j} & U \end{array}$$

Let $M \in D^b(X)$. Then we have the following exact triangle

$$g_* g^! M \to M \to j_* j^! M \xrightarrow{+1}$$

Applying $g^! f_*$, we get

$$g^! f_* g_* g^! M \to g^! f_* M \to g^! f_* j_* j^! M \xrightarrow{+1}$$

But now

$$g^! f_* j_* j^! M \simeq g^! j_* f_* j^! M \simeq 0,$$

since $g^! j_* \simeq 0$ by Proposition 3.7.3. Thus $g^! f_* g_* g^! M \simeq g^! f_* M$, and we are done. □

Corollary 3.7.4 (Projection formula)**.** *Let $f : X \to Y$ be a morphism of smooth affine varieties. Then for any $M \in D^b(\mathcal{D}_X\text{-mod})$, $N \in D^b(\mathcal{D}_Y\text{-mod})$, we have*

$$f_*(M \otimes^L_{\mathcal{O}_X} f^!(N)) \simeq f_*(M) \otimes^L_{\mathcal{O}_Y} N.$$

Proof. Exercise. □

3.7.5. *$\mathcal{D}$-modules on singular varieties.* Let X be a singular Noetherian affine scheme. Then embed it in some affine space $X \hookrightarrow \mathbb{A}^N$, and *define* the category of $\mathcal{D}$-modules on X to be

$$\mathcal{D}\text{-mod}(X) := (\mathcal{D}_{\mathbb{A}^N}\text{-mod})_X .$$

Kashiwara's theorem then guarantees that this category does not depend on the embedding, since we can always embed into a bigger affine space:

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{A}^M \\ \downarrow & & \downarrow \\ \mathbb{A}^M & \hookrightarrow & \mathbb{A}^{M+N}. \end{array}$$

Another application of Kashiwara's theorem is the proof that the projective space $\mathbb{P}^n$ is $\mathcal{D}$-affine; we will do this later.

4. Holonomic $\mathcal{D}$-modules

4.1. **Coherent $\mathcal{D}$-modules.** In order to continue our study of $\mathcal{D}_X$-modules in a more geometric direction, we need to impose some finiteness conditions. Recall that for $\mathcal{O}_X$-modules such condition is provided by the notion of coherent sheaves. Our first, very naive guess is thus to consider $\mathcal{D}_X$-modules with the underlying $\mathcal{O}_X$-modules being coherent. Unfortunately, this essentially suffocates our field of study, as the following proposition shows.

Proposition 4.1.1. *A $\mathcal{D}_X$-module M is coherent over $\mathcal{O}_X$ if and only if it is locally free.*

In particular, this definition only captures vector bundles with flat connections.

Proof of Proposition 4.1.1. The "if" part is obvious. For the "only if" part, it suffices to show it locally, that is that M_x is free for all $x \in X$. Let us pick local coordinates $(x_i, \partial_i)_{i=1}^d$ at x. Let $\bar{s}_1, \ldots, \bar{s}_n$ be a basis of $M_x/\mathfrak{m}_x M_x$ over $\mathbb{C}$, and choose lifts $s_1, \ldots, s_n \in M_x$. By Nakayama's Lemma, s_i's span M_x as an $\mathcal{O}_{X,x}$-module. This induces a surjective map $\mathcal{O}_{X,x}^{\oplus n} \to M_x$. Our goal is to show that it is injective as well.

Suppose that we have a non-trivial relation in M_x

$$\sum_{i=1}^n f_i s_i = 0, \quad f_i \in \mathcal{O}_{X,x}. \tag{4.1.1}$$

Passing to the completion $\widehat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[\![x_1, \ldots, x_d]\!]$, the same non-trivial relation holds in $\widehat{M}_x$.

For any $f \in \mathbb{C}[\![x_1, \ldots, x_d]\!]$, define $\mathsf{ord}(f)$ to be the degree of the lowest term. The natural action of ∂_i's on $\mathbb{C}[\![x_1, \ldots, x_d]\!]$ obviously decreases the degree of series, where the lowest term contains x_j. Note that $\partial_j s_i$ is a linear combination of s_i's in $\widehat{\mathcal{O}}_{X,x}$. Applying ∂_j to the relation (4.1.1), we get

$$0 = \sum_{i=1}^n ((\partial_j f_i) s_i + f_i(\partial_j s_i)) = \sum_{i=1}^n g_i s_i, \quad g_i \in \widehat{\mathcal{O}}_{X,x}.$$

Let i_0 be such that f_{i_0} has the smallest degree among f_i's. Pick j such that some lowest degree term in f_i contains x_j. Then $\mathsf{ord}(g_i) \leq \mathsf{ord}(\partial_j f_i) < \mathsf{ord}(f_i)$, and so

$$\min_i(\mathsf{ord}(f_i)) > \min_i(\mathsf{ord}(g_i)).$$

We can continue doing this until the minimum degree reaches 0. At this point, we arrive at a non-trivial relation in $M_x/\mathfrak{m}_x M_x$

$$\sum_{i=1}^n \bar{h}_i \bar{s}_i = 0, \quad \bar{h}_i \in \mathbb{C}.$$

But this contradicts the fact that $\bar{s}_i$'s form a basis for $M_x/\mathfrak{m}_x M_x$. Thus we are done. □

Recall that $\mathcal{D}_X$ is supposed to be a deformation of $\mathcal{O}_{T^*X}$, so in a sense it is not surprising that we have failed. Let us consider the following definition instead.

Definition 4.1.2. We say that a $\mathcal{D}_X$-module M is *coherent* if it is finitely generated as a $\mathcal{D}_X$-module. We denote the category of coherent $\mathcal{D}_X$-modules by $\mathsf{Coh}(\mathcal{D}_X)$.

Note that $\mathcal{D}_X$ is Noetherian, so "finitely generated" is the same as "finitely presented."

Example 4.1.3. The $\mathcal{D}_{\mathbb{A}^1}$-module $\delta_0 = \mathbb{C}[\partial]$ is infinitely generated over $\mathbb{C}[t]$, but is clearly generated over $\mathbb{C}[t, \partial]$ by 1.

We see that this definition is less restrictive. However, we soon face another serious problem: the category $\mathsf{Coh}(\mathcal{D}_X)$ is not preserved under the natural functors.

Example 4.1.4. Let $\pi : \mathbb{A}^1 \to \mathrm{pt}$ be the projection, and $M = \mathcal{D}_{\mathbb{A}^1}$. We have

$$\pi_*(M) = \mathbb{C}[t, \partial] \otimes_{\mathbb{C}[t,\partial]} \mathbb{C}[t] = \mathbb{C}[t].$$

This is an infinite-dimensional $\mathbb{C}$-vector space.

Example 4.1.5. Let $i : \{0\} \hookrightarrow \mathbb{A}^1$, and $M = \mathcal{D}_{\mathbb{A}^1}$. We have

$$i^*(M) = \mathbb{C}[t,\partial] \otimes_{\mathbb{C}[t,\partial]} \mathbb{C}[\partial] = \mathbb{C}[\partial].$$

Remark 4.1.6. Later we will prove *Bernstein's b-function lemma*, which says that push-forward along an open embedding preserves coherency.

In order to resolve this issue, let us make a quick detour.

4.2. **Singular supports.** Recall that we had the following definition.

Definition 4.2.1. Let $M \in \mathsf{Coh}(\mathcal{D}_X)$. A *good filtration* on M is a filtration by $\mathcal{O}_X$-modules

$$\ldots \subset F^{i-1} \subset F^i \subset F^{i+1} \subset \ldots,$$

exhaustive and bounded from below, such that

$$\mathcal{D}_X^{\leq j} F^i \subset F^{i+j},$$

and $\mathrm{gr}^F M = \bigoplus_i F^i/F^{i+1}$ is a coherent $\mathrm{gr}\,\mathcal{D}_X = \mathcal{O}_{T^*X}$-module.

Proposition 4.2.2. *For any $M \in \mathsf{Coh}(\mathcal{D}_X)$, there exists a good filtration. If $\{F_1^i\}$ and $\{F_2^i\}$ are two good filtrations on M, then for some N we have*

$$F_1^i \subset F_2^{i+N} \subset F_1^{i+2N}$$

for all i.

Proof. Since M is coherent, it is generated by a finite collection of elements $m_1, \ldots, m_k \in M$. Consider the associated map $\mathcal{D}_X^k \twoheadrightarrow M$. It is easy to check that for any surjection of $\mathcal{D}_X$-modules $N_1 \twoheadrightarrow N_2$ the image of a good filtration on N_1 is a good filtration on N_2. Since $\mathcal{D}_X^k$ has a good filtration by degree, we obtain a good filtration on M.

For the second claim, let p_i be such that $F_1^p = \sum_{p \geq p_i} \mathcal{D}_X^{\leq p-p_i} m_i$ for any p. Such a choice exists, because we can assume that images of m_i's generate $\mathrm{gr}^{F_1} M$. Further, let q_i be such that $m_i \in F_2^{q_i}$. Denote by N the maximal value of $q_i - p_i$. Then we have

$$F_1^p = \sum_{p \geq p_i} \mathcal{D}_X^{\leq p-p_i} m_i \subset \sum_{p \geq p_i} \mathcal{D}_X^{\leq p-p_i} F_2^{q_i} \subset \sum_{p \geq p_i} F_2^{p+(q_k-p_k)} \subset F_2^{p+N},$$

and we may conclude. □

Corollary 4.2.3. *The closed subvariety*

$$\mathsf{Supp}(\mathrm{gr}^F M) = V\left(\sqrt{\mathsf{Ann}(\mathrm{gr}^F M)}\right) \subset T^*X \tag{4.2.1}$$

does not depend on the choice of a good filtration F.

Proof. We use Proposition 4.2.2. Shifting the filtrations, we can assure that for some N we have

$$F_1^i \subset F_2^i \subset F_1^{i+N}.$$

Let us first assume that $N = 1$. Consider the natural map $\phi_i : F_1^i/F_1^{i-1} \to F_2^i/F_2^{i-1}$. Then we have

$$\ker \phi_i = F_2^{i-1}/F_1^{i-1} = \mathrm{coker}\,\phi_{i-1}.$$

Consider the exact sequence

$$0 \to \ker \phi \to \operatorname{gr}^{F_1} M \to \operatorname{gr}^{F_2} M \to \operatorname{coker} \phi \to 0,$$

where ϕ is the natural map. We have

$$\mathsf{Supp}(\operatorname{gr}^{F_1} M) = \mathsf{Supp}(\ker \phi) \cup \mathsf{Supp}(\operatorname{im} \phi) = \mathsf{Supp}(\operatorname{coker} \phi) \cup \mathsf{Supp}(\operatorname{im} \phi) = \mathsf{Supp}(\operatorname{gr}^{F_2} M),$$

and so we can conclude.

The general case is obtained by considering the intermediate filtrations $F^i_{1,k} = F^i_1 + F^{i+k}_2$, $1 \le k \le n$. □

Remark 4.2.4. The claims above remain true for any Noetherian almost commutative filtered ring.

Definition 4.2.5. We call the subvariety (4.2.1) the *singular support* (or *characteristic variety*) of $M \in \mathsf{Coh}(\mathcal{D}_X)$, and denote it by $\mathsf{Ch}(M)$.

Lemma 4.2.6. *Let* $0 \to M_1 \to M \to M_2 \to 0$ *be a short exact sequence of coherent* $\mathcal{D}_X$*-modules. Then* $\mathsf{Ch}\, M = \mathsf{Ch}\, M_1 \cup \mathsf{Ch}\, M_2$.

Proof. See exercise sheet 1. □

Recall that giving a grading on a commutative algebra R is equivalent to defining a $\mathbb{C}^*$-action on $\mathsf{Spec}\, R$. Graded ideals then correspond to $\mathbb{C}^*$-invariant subvarieties. In our case, the grading on $\operatorname{gr} \mathcal{D}_X \simeq \mathcal{O}_{T^*X}$ coincides with the one obtained from the action $\mathbb{C}^* \curvearrowright T^*X$ by homotheties along the cotangent direction. In particular, $\mathsf{Ch}(M)$ is preserved by $\mathbb{C}^*$, or in other words a *conical* subvariety in T^*X. Furthermore, let $p : T^*X \to X$ be the natural projection. Then $p(\mathsf{Ch}\, M) = X \cap \mathsf{Ch}\, M = \mathsf{Supp}(M)$ by conicality.

Proposition 4.2.7 (Bernstein's inequality)**.** *For any irreducible component* $\mathsf{Ch}(M)_i \subset \mathsf{Ch}(M)$, *we have* $\operatorname{codim} \mathsf{Ch}(M)_i \le \dim X$.

In order to prove this inequality, we will have to do some preparatory work. However, note that for any vector bundle with flat connection M, we have $\mathsf{Ch}(M) = X$, so that Bernstein's inequality becomes an equality.

4.3. **Holonomic modules.**

Definition 4.3.1. A coherent $\mathcal{D}_X$-module is called *holonomic*, if $\dim \mathsf{Ch}(M) = \dim X$. The category of holonomic $\mathcal{D}_X$-modules will be denoted by $\mathsf{Hol}\, X$.

Let us think of M in terms of differential equations, see Section 1. If the singular support is as "small" as possible, this means that the corresponding system of PDEs is as "large" as possible, e.g. we cannot add any boundary conditions. In effect, one can show that the solution spaces of holonomic $\mathcal{D}$-modules are finite-dimensional.

It turns out that this is precisely the finiteness condition that we need.

Theorem 4.3.2 (Main theorem A)**.** *Let* $f : X \to Y$ *be a map of smooth affine varieties. Then the functors* f_*, $f^!$ *preserve holonomicity.*

After a series of reductions, this theorem reduces to Bernstein's b-function lemma, but we will have to work quite a bit to get there. Let us begin with a couple of statements about $\mathsf{Hol}\, X$. The first one tells us that we did not go *that* far from vector bundles with flat connections.

Proposition 4.3.3. *Let $M \in \mathsf{Hol}(X)$. There exists an open $U \subset X$, such that $M|_U$ is locally free over $\mathcal{O}_X$.*

Proof. Let $p : T^*X \to X$ be the natural projection. If $p(\mathsf{Ch}\, M) \neq X$, then the proposition is trivially true. We therefore assume that $p(\mathsf{Ch}\, M) = X$. There exists an irreducible component $(\mathsf{Ch}\, M)_i$ which surjects to X. Since $\mathsf{Ch}\, M$ is conical, this implies that $(\mathsf{Ch}\, M)_i = X$. The projection of any other irreducible component has to be closed in X, so that there exists an open $U \subset X$ with $\mathsf{Ch}(M|_U) = U$. We conclude by invoking Proposition 4.1.1. □

The next statement shows that $\mathsf{Hol}\, X$ is extremely rigid.[4] Let us start with a definition.

Definition 4.3.4. Let $\mathsf{Ch}(M) = \bigcup_i C_i$ be the decomposition into irreducible components. We define the *characteristic cycle* of M to be the formal sum

$$\mathsf{CC}(M) := \sum_i m_{C_i}(M) \cdot C_i,$$

where $m_{C_i}(M)$ is the multiplicity of $\mathsf{gr}\, M$ along C_i.

Proposition 4.3.5. $\mathsf{CC}(M)$ *does not depend on the choice of good filtration. If* $0 \to M_1 \to M \to M_2 \to 0$ *is a short exact sequence, we have* $\mathsf{CC}(M) = \mathsf{CC}(M_1) + \mathsf{CC}(M_2)$.

Proof. Exercise. □

Corollary 4.3.6. *Every $M \in \mathsf{Hol}(X)$ has finite length.*

Proof. By additivity of characteristic cycle, the length of M is bounded above by $\sum_i m_{C_i}(M)$. □

Remark 4.3.7. This stands in stark contract with $\mathsf{Coh}\, X$, where finite length implies zero-dimensional support.

4.4. **Proof of Bernstein's inequality.** Let us first recall that for a possibly singular variety Z, its *dimension* is defined as the maximum of dimensions of irreducible components of the smooth locus Z^{sm}.

Definition 4.4.1. Let $M \in \mathsf{Coh}(\mathcal{D}_X)$. We define G^iM to be the largest submodule $N \subset M$ with $\dim \mathsf{Ch}(N) \leq i$. The increasing filtration $\{G^iM\}_i$ of M is called the *Gabber filtration*.

Proposition 4.4.2. *We have* $\dim \mathsf{Ch}(G^iM/G^{i-1}M) = i$. *Moreover, this variety is equidimensional.*

Unfortunately, we will need more technology in order to prove Proposition 4.4.2. Let us first show how this implies Bernstein's inequality.

[4]Not a mathematical term.

Lemma 4.4.3. *Let $i : X \hookrightarrow Y$ be a closed embedding of a smooth subvariety, $M \in \mathcal{D}_X$-mod, and $N = i_* M$. Consider the correspondence*

$$\begin{array}{ccccc} & & T^*Y|_X & & \\ & \overset{g}{\swarrow} & & \overset{h}{\searrow} & \\ T^*X & & & & T^*Y \end{array}$$

Then we have

$$\mathsf{Ch}(N) = hg^{-1}\,\mathsf{Ch}(M).$$

Proof. The question is local in X. Moreover, by induction on codimension we can assume that $X = \mathsf{Spec}\,\mathcal{O}_Y/(f)$ for some $f \in \mathcal{O}_Y$. Pick local coordinates $\{x_i, \partial_i\}$ on Y such that $f = x_1$, and set $\partial = \partial_1$. Then $N \simeq i_* M[\partial]$ as in the proof of Theorem 3.6.2. Take a good filtration $F^i M$, such that $F^{-1}M = 0$. Then

$$F^i N = \sum_{k \le j \le i} \mathbb{C}\partial^k \otimes i_* F^{i-j} M.$$

is a good filtration of N, which satisfies

$$F^i N / F^{i-1} N = \bigoplus_{j=0}^{i} \mathbb{C}\partial^j \otimes i_*(F^{i-j}M / F^{i-j-1}M).$$

Therefore we have

$$\mathsf{Ch}(N) = \mathsf{Supp}(\mathsf{gr}^F N) = \mathsf{Supp}(\mathsf{gr}^F M[\partial]) = hg^{-1}\,\mathsf{Supp}(\mathsf{gr}^F M) = hg^{-1}\,\mathsf{Ch}(M).$$

□

Proof of Proposition 4.2.7. By Proposition 4.4.2, it suffices to show that $\dim \mathsf{Ch}(M) \ge \dim X$ for any coherent $\mathcal{D}_X$-module M. We do it by induction on $\dim X$. If $\mathsf{Supp}\,M = X$, then $X \subset \mathsf{Ch}(M)$, and thus $\dim \mathsf{Ch}(M) \ge \dim X$. We can therefore assume that $S = \mathsf{Supp}\,M$ is a closed subscheme of X. Passing to an open subset of X, we can further assume that S is smooth. Let $i : S \hookrightarrow X$ be the embedding. By Kashiwara's theorem we have $M = i_* L$ for some $L \in \mathcal{D}_S$-mod. By Lemma 4.4.3 we have

$$\dim \mathsf{Ch}(M) = \dim(hg^{-1}\,\mathsf{Ch}(L)) = \dim \mathsf{Ch}(L) + \mathrm{codim}_X S.$$

On the other hand, we have $\dim \mathsf{Ch}(L) \ge \dim S$ by induction, and so we may conclude. □

Remark 4.4.4. One can prove that $\mathsf{Ch}(M)$ is *coisotropic*; however, this is a much deeper theorem.

In order to prove Proposition 4.4.2, we need another characterization of Gabber filtration. Its definition uses Verdier duality.

4.5. **Verdier duality.** Recall that for $\mathcal{O}_X$-modules, we have a duality functor $\mathsf{R}\mathcal{H}om(-, \mathcal{O}_X) : D^b(\mathsf{Coh}\, X) \to D^b(\mathsf{Coh}\, X)^{\mathrm{op}}$. The reason we take the derived Hom-functor is that the non-derived one obviously kills all torsion sheaves. We have a similar definition in the world of $\mathcal{D}_X$-modules.

Definition 4.5.1. The functor

$$\mathbf{D} : D^b(\mathsf{Coh}(\mathcal{D}_X)) \to D^b(\mathsf{Coh}(\mathcal{D}_X))^{\mathrm{op}},$$
$$\mathbf{D}(M_\bullet) = \mathsf{RHom}_{\mathcal{D}_X}(M_\bullet, \mathcal{D}_X)[\dim X] \otimes_{\mathcal{O}_X} \omega_X^{-1}$$

is called the *Verdier duality*.

Proposition 4.5.2. *The Verdier duality* $\mathbf{D}$ *is an auto-equivalence of categories. We have* $\mathbf{D} \circ \mathbf{D} \simeq \mathrm{id}$.

Proof. This is obvious for free, and therefore projective $\mathcal{D}_X$-modules. The general case follows by induction on the length of projective resolution. □

Recall that for any complex $M_\bullet = \quad \cdots \to M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \to \cdots$ we can define its i-th truncation by

$$\tau_{\geq i} M_\bullet = \quad \cdots \to 0 \to \mathrm{im}\, d_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \to \cdots .$$

We have a natural map $M_\bullet \to \tau_{\geq i} M_\bullet$.

Definition 4.5.3. Let $M \in \mathsf{Coh}(\mathcal{D}_X)$. We define $S^i M$ to be the image of the natural map

$$H^0(\mathbf{D}\, \tau_{\geq i}\, \mathbf{D}\, M) \to H^0(\mathbf{D}\,\mathbf{D}\, M) = H^0(M) = M.$$

The increasing filtration $\{S^i M\}_i$ of M is called the *Sato-Kashiwara filtration*.

It is obvious that the Sato-Kashiwara filtration is compatible with morphisms of $\mathcal{D}_X$-modules.

Lemma 4.5.4. *We have* $\dim \mathsf{Ch}(S^i M) \leq i$, *and* $S^{\dim \mathsf{Ch}(M)} M = M$.

Proof. Let us reformulate the statement as follows:

$$\dim \mathsf{Ch}(S^i M / S^{i-1} M) \leq i, \qquad S^i M / S^{i-1} M \text{ for } i > \dim \mathsf{Ch}(M).$$

The natural equivalence $\mathbf{D}^2 \simeq \mathrm{id}$ yields a spectral sequence

$$E_2^{ij} = \mathsf{Ext}^j_{\mathcal{D}_X}(\mathsf{Ext}^{\dim \mathsf{Ch}(M) - i}_{\mathcal{D}_X}(M, \mathcal{D}_X), \mathcal{D}_X) \Rightarrow \begin{cases} S^i M / S^{i-1} M & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The claim then follows from the lemma below; see [G2] for more details and references. □

Lemma 4.5.5. *Let* $M \in \mathsf{Coh}(\mathcal{D}_X)$. *Then*

(1) $\operatorname{codim} \mathsf{Ch}(\mathsf{Ext}^j_{\mathcal{D}_X}(M, \mathcal{D}_X)) \geq j$,
(2) $\mathsf{Ext}^j_{\mathcal{D}_X}(M, \mathcal{D}_X) = 0$ *unless* $\operatorname{codim} \mathsf{Ch}(M) \leq j \leq 2 \dim X$.

Proof. In the exercise sheet 3. □

Theorem 4.5.6. *The Gabber filtration is equal to the Sato-Kashiwara filtration.*

Proof. By Lemma 4.5.4 we have $S^iM \subset G^iM$. In the opposite direction, consider the inclusion $G^iM \hookrightarrow M$. By functoriality of S^i we have $S^i(G^iM) \subset S^iM$. Since $\dim \mathsf{Ch}(G^iM) = i$, we have $S^iG^iM = G^iM$, and so we may conclude. □

In particular, this theorem buys us functoriality of Gabber filtration, which is not at all obvious from the original definition. It is also easy to check that Sato-Kashiwara filtration commutes with pullbacks along open embeddings.

Proof of Proposition 4.4.2. Suppose that $\mathsf{Ch}(G^iM/G^{i-1}M)$ has a component C of dimension strictly less than i. Choose an open affine $j : U \hookrightarrow X$ such that

$$U \cap \mathsf{Ch}(G^iM) \subset C^{\mathrm{sm}}.$$

We have $j^!(M/G^{i-1}M) \neq 0$, because its characteristic variety is non-empty. On the other hand, we clearly have $\dim \mathsf{Ch}(j^!M) \leq i-1$, so that $j^!M = G^{i-1}j^!M = j^!G^{i-1}M$. In particular,

$$j^!(M/G^{i-1}M) = j^!M/j^!(G^{i-1}M) = 0,$$

and so we have arrived at a contradiction. □

4.6. ***b*-function lemma.** One important property of holonomicity is that it is preserved under pulling back and pushforward. We have seen before that coherence, by itself, is not preserved. Let us start with a simple but instructive example.

Example 4.6.1. Let $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$ and $j : U = \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$ be the open immersion. Let $M = \mathbb{C}[t, t^{-1}]$ be a D-module over U. Then, $j_*M = \mathbb{C}[t, t^{-1}]$ is a $\mathcal{D}_X$-module. We want to see that it is holonomic.

First, we check that it's coherent. Observe that

$$\partial t^{-n} = -nt^{-n-1}, n \geq 1$$

and hence, as a $\mathcal{D}_X$-module, j_*M is generated by t and t^{-1}. Thus, it is coherent.

To see that it's holonomic, we use the following exact sequence of $\mathcal{D}_X$-modules

$$0 \to \mathbb{C}[t] \to \mathbb{C}[t, t^{-1}] \to t^{-1}\mathbb{C}[t^{-1}] \to 0.$$

Note that here, $t^{-1}\mathbb{C}[t^{-1}] = i_*i^!\mathbb{C}[t][1]$. The singular support of $i_*i^!\mathbb{C}[t][1]$ is 1-dimensional. Thus, it is holonomic. $\mathbb{C}[t]$ is obviously holonomic. We thus conclude.

The main upshot of this example is that even though $\{t^{-n}\}_{n\geq 1}$ show up, there's nothing to fear since we can reach negative powers of t by differentiating t^{-1}. This is in stark contrast to the case of $\mathcal{O}_X$-modules.

The proof of the fact that holonomicity is compatible with pushforward and pulling back will follow a series of reduction. One important case is that of an affine open embedding, a generalization of the example above.

Until the end of the subsection, we use the following notation. Let X be smooth affine variety, $f \in \mathcal{O}_X$, and $U = X_f$, the complement of $\{f = 0\} \subseteq X$. Let $j : X_f \to X$ denote the open immersion. Moreover, let $M \in \mathcal{D}_U$-mod.

Theorem 4.6.2. *Let $j : U \to X$ as above. Suppose that M is holonomic, then $j_*M \in \mathcal{D}_X\text{-mod}_{\mathsf{hol}}$.*

The proof of this theorem needs a technical lemma called the b-function lemma. It is a generalization of the observation above where we see that negative powers of t don't cause any trouble. We need some preparation to state the result.

Let $O = \mathbb{C}[s]$ and $K = \mathbb{C}(s)$. We can base change and obtained U_K and X_K as well as U_O and X_O. Consider $\mathcal{D}_{U_K/K} = \mathcal{D}_U \otimes K$, $\mathcal{D}_{X_K/K} = \mathcal{D}_X \otimes K$ as well as subrings $\mathcal{D}_{U_O/O} = \mathcal{D}_U \otimes O$ and $\mathcal{D}_{X_O/O} = \mathcal{D}_X \otimes O$. Let $Mf^s = \{a(s)mf^s : a(s) \in K, m \in M\}$. Here, f^s is just a formal symbol and some places write $a(s)m$"f^s".

Mf^s has an $\mathcal{O}_X$-module structure given by acting on m. It also has a $\mathcal{D}_X$-module structure given by

$$\xi(mf^s) = \xi(m)f^s + s\frac{\xi(f)}{f}mf^s, \qquad \xi \in \mathsf{Vect}_X.$$

Note that this also gives Mf^s the structure of a $\mathcal{D}_{U_K/K}$-module.

Remark 4.6.3. If f^s looks a bit strange, here's another way to think about it. There is an automorphism τ of $\mathcal{D}_{U_O/O}$ given by

$$\tau(\xi) = \xi + s\frac{\xi f}{f}.$$

Then, we can twist the natural $\mathcal{D}_{U_O/O}$-module structure of M to get a new module M^τ. It is clear that this is the same as the one above.

Remark 4.6.4. In general, if N is a $\mathcal{D}_X$-module that is holonomic, then so is Nf^s, as a $\mathcal{D}_{X_K/K}$-module, by extending the filtration by scalars. Moreover, if κ is a field automorphism of K, then it induces an automorphism of $\mathcal{D}_{X_K/K}$-module, which doesn't affect holonomicity.

Remark 4.6.5. There's also a geometric way to think about f^s, via Kummer sheaf. First, fix $s \in \mathbb{C}$ (rather than a variable). Consider the following D-module on $\mathbb{A}^1 \setminus \{0\}$

$$\text{“}x^s\text{”} = \mathbb{C}[x,\partial]/(\mathbb{C}[x,\partial](\partial - \frac{s}{x})).$$

By abuse of notation, we also denote by "x^s" by its pushforward to $\mathbb{A}^1$. Given a function $f : X \to \mathbb{A}^1$, one obtain "f^s" by pulling back "x^s" along f.

Now, we have to perform this construction "in family" to get the variable s. Consider $\operatorname{Spec}\mathbb{C}[x,s] = \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 = \operatorname{Spec}[s]$ and relative D-modules "x^s" given by

$$\text{“}x^s\text{”} = \mathbb{C}[x,s,\partial_x]/(\mathbb{C}[x,s,\partial_x](\partial_x - \frac{s}{x})).$$

For any map $f : X \to \mathbb{A}^1$, we obtain $f_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ via $(x,s) \mapsto (f(x),s) \mapsto s$. Pulling back "$x^s$" along this map, we get "f^s", where s is now a variable. We can also take the generic fiber and obtain "f^s" as a $\mathcal{D}_{X_K/K}$-module. For any $\mathcal{D}_X$-module M, we can consider M_K, which is a $\mathcal{D}_{X_K/K}$-module and hence, also $M_K \otimes_{\mathcal{O}_{X_K}}$ "f^s" which is the same as Mf^s above.

We are now ready to state the lemma on b-function.

Lemma 4.6.6. *Suppose M is a holonomic $\mathcal{D}_U$-module. Then, for any section $m \in M$, there exists $u(s) \in \mathcal{D}_{X_O/O}$ and a polynomial $b(s) \in O = \mathbb{C}[s]$ such that*

$$u(s)(mf^s) = b(s)\frac{m}{f}f^s.$$

Remark 4.6.7. Suppose that s were a number and not a variable. Then since $b(s)$ has only finitely many roots, when $s = n \gg 0$, $b(-n) \neq 0$ and we have

$$\frac{u(-n)}{b(-n)}(mf^{-n}) = mf^{-n-1},$$

which says that we get get lower powers of f by differentiating.

Proof of Lemma 4.6.6. We will start by proving a slightly more general statement. Let $M_0 \subset M$ be a $\mathcal{O}_X$-coherent module. We will show that $N = \mathcal{D}_{X_K/K}M_0f^s$ is a holonomic $D_{X_K/K}$-module.

Let $N' \subset N$ be the lowest term of the Gabber–Sato–Kashiwara filtration. By definition N' is holonomic. From Sato–Kashiwara formulation of the filtration, we see that it's well-behaved with respect to localization. Hence, $N'|_{U_K} \simeq N|_{U_K}$, since N is holonomic over U_K, being a submodule of M, which is holonomic over U_K. In particular, $N' \neq 0$.

Consider N/N', which is now a $\mathcal{D}_{X_K}$-module supported on $X \setminus U$. Consider the image of M_0f^s in N/N'. Since M_0 is finitely generated, we know that its image is annihilated by a large enough power of f. In particular, there exists some k such that $f^kM_0f^s \subset N'$, and hence, $\mathcal{D}_{X_K/K}M_0f^{s+k} \simeq \mathcal{D}_{X_K/K}f^kM_0f^s \subset N'$. Thus, $\mathcal{D}_{X_K/K}M_0f^{s+k}$ is holonomic, being a sub-module of a holonomic D-module. Now, note that $\mathcal{D}_{X_K/K}M_0f^{s+k} \simeq \mathcal{D}_{X_K/K}M_0f^s$, by a an automorphism of K given by $s \mapsto s+k$. Thus, the latter, which is N by definition, is also holonomic.

Consider the following decreasing chain of $\mathcal{D}_{X_K/K}$-modules

$$\mathcal{D}_{X_K/K}M_0f^s \supseteq \mathcal{D}_{X_K/K}fM_0f^s \supseteq \mathcal{D}_{X_K/K}f^2M_0f^s \supseteq \cdots,$$

which necessarily stabilizes, since the category of holonomic $\mathcal{D}$-modules is Artinian. Thus, there exists $k \gg 0$ such that $\mathcal{D}_{X_K/K}f^kM_0f^s = \mathcal{D}_{X_K/K}f^{k+1}M_0f^s$. Using the change of variable trick on K again, we get $\mathcal{D}_{X_K/K}M_0f^s = \mathcal{D}_{X_K/K}fM_0f^s$.

Now, suppose that M_0 is generated by 1 element m, then, we get $\mathcal{D}_{X_K/K}mf^s = \mathcal{D}_{X_K/K}fmf^s$. In particular, there exists $u'(s) \in \mathcal{D}_{X_K/K}$such that $u'(s)(fmf^s) = mf^s$. Write $u'(s) = \frac{u(s)}{b(s)}$ where $u(s) \in \mathcal{D}_{X_O/O}$ and $b(s) \in O$, we get

$$u(s)mf^s = b(s)\frac{m}{f}f^s,$$

and the proof is complete. □

We will now return to the proof of Theorem 4.6.2.

Proof of Theorem 4.6.2. Since M is holonomic, it's generated, over $\mathcal{D}_U$, by a coherent $\mathcal{O}_X$-module $M_0 = \mathcal{O}_X\langle m_1, \dots, m_k\rangle$. By Lemma 4.6.6 and Remark 4.6.7, we see that j_*M is generated over $\mathcal{D}_X$ by $m_1f^{-l}, m_2f^{-l}, \dots, m_kf^{-l}$ for some $l \gg 0$. Thus, j_*M is coherent. It remains to show holonomicity.

We have

$$\mathcal{D}_{X_K/K} M_0 f^s = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_{X_K/K} M_0 f^{s-k} = \mathcal{D}_{U_K/K} M_0 f^s = j_* M f^s.$$

Here, the first equality is due to Lemma 4.6.6. Moreover, from the first part of the proof of Lemma 4.6.6, we know that $\mathcal{D}_{X_K/K} M_0 f^s$, and hence $j_* M f^s$, is a holonomic $\mathcal{D}_{X_K/K}$-module. Assume that M_0 is generated by one element m (otherwise, do an induction on the number of generators), then $\mathcal{D}_{X_O/O} m f^s = \mathcal{D}_{X_O/O}/I$. By what we saw above, over K, $\mathcal{O}_{T^*X_K}/\mathrm{gr}(I_K)$ defines a $\dim_K X_K$ dimensional subvariety of T^*X_K. By upper semi-continuity of fiber dimension [G3, Theorem 13.1.3], we know that for all but finitely many values of s, $\mathcal{O}_{T^*X}/\mathrm{gr}(I_s)$ defines also a $\dim X$-dimensional variety of T^*X over $k(s) \simeq \mathbb{C}$. Specializing $s = -l$, $l \gg 0$, we get $\mathcal{D}_X M_0 f^{-l} = j_* M$ (see first part of the proof) is holonomic and we are done. □

4.7. **Functoriality of holonomicity.** The main goal of this subsection is to show that pulling and pushing preserve holonomicity.

The lemma on b-function allows us to show that pushing forward along an open affine embedding preserves holonomicity. This is true in more general (though admittedly, we haven't really talked about D-modules on a non-affine variety).

Lemma 4.7.1. *Let U be an open subset of X. If $M \in D^b_{\mathrm{hol}}(U)$, then $j_* M \in D^b_{\mathrm{hol}}(X)$.*

Proof. By induction on cohomological length, it suffices to prove it for the case when $M \in (\mathcal{D}_U\text{-mod})_{\mathrm{hol}}$. Since holonomicity is a local property, without loss of generality, we can assume that X is affine.

When $j : U \to X$ is an open affine embedding, then it is the content of Theorem 4.6.2. Now suppose $U = \cup_i U_i$ is a (finite) affine covering of U, then

$$j_* M \simeq \bigoplus_i (j_0)_* M|_{U_i} \to \bigoplus_{i_0<i_1} (j_{i_0 i_1})_* M|_{U_{i_0}\cap U_{i_1}} \to \bigoplus_{i_0<i_1<i_2} (j_{i_0 i_1 i_2})_* M|_{U_{i_0}\cap U_{i_1}\cap U_{i_2}} \to \cdots$$

This is a chain complex of holonomic $\mathcal{D}_X$-modules. Thus, its cohomology is holonomic and we are done. □

Corollary 4.7.2. *For any closed embedding $i : Z \to X$, $i^!$ preserves holonomicity.*

Proof. Let $M \in D^b_{\mathrm{hol}}(X)$, then we have the following excision sequence

$$i_* i^! M \to M \to j_* j^! M \to .$$

Clearly $j^! M$ is holonomic. By Lemma 4.7.1, $j_* j^! M$ is holonomic. Thus, $i_* i^! M$ is holonomic. Since we know precisely what the singular support of $i_* N$ looks like for any N. It's easy to see that $i^! M$ is also holonomic. □

This allows us to conclude that pulling back always preserves holonomicity.

Theorem 4.7.3. *Let $f : X \to Y$ be a morphism between smooth varieties. Then $f^!$ preserves holonomicity.*

Proof. We factors f into a composition of a closed embedding and a smooth projection

$$X \to X \times Y \to Y$$

Pulling back along a closed immersion preserves holonomicity by Corollary 4.7.2. For a smooth projection, it's easy to see that coherence is preserved. Moreover, the singular support of pulling back along a smooth projection is also easy to compute by going from right to left in the following correspondence

$$T^*(X \times Y) \hookleftarrow (X \times Y) \times_Y T^*Y \to T^*Y.$$

Thus, we are done. □

It remains to show that pushforward preserves holonomicity. We start with the following easy lemma.

Lemma 4.7.4. *Let $i : Y \to X$ be a locally closed immersion of smooth variety. Let M be a $\mathcal{D}_Y$-module. Then, M is holonomic if and only if i_*M is.*

Proof. We factor i as $Y \xrightarrow{i'} U \xrightarrow{j} X$ where i' is a closed embedding and j an open embedding. By what we have seen above, j preserves holonomicity. It's also easy to see that it reflects holonomicity. Since i' is a closed embedding of smooth schemes, a simple dimension count implies that it both preserves and reflects holonomicity. □

We will now prove an important criterion for holonomicity.

Theorem 4.7.5. *Let $M \in D^b_{\mathrm{coh}}(X)$. Then, the following are equivalent*

(i) *M is holonomic,*

(ii) *for any point $x \in X$, $i^!_x M$ is finite dimensional.*

(i) implies (ii) is a direct consequence of Corollary 4.7.2. For the other direction, we will need the following general.

Lemma 4.7.6. *Let M be a coherent $\mathcal{D}_X$-module. Then, there exists an open dense subset U of X such that $M|_U$ is flat. Furthermore, if for any $x \in U$, $M \otimes_A k(x)$ is finite dimensional, then $M|_U$ is coherent over $\mathcal{O}_U$.*

Proof. Choose a good filtration F on M, then $\mathrm{gr}^F M$ is a coherent sheaf on T^*X. By generic flatness, [Stacks, Tag 052A], we know that there exists an open dense subset U of X such that $\mathrm{gr}^F M|_U$ is flat. In particular, for each i, $\mathrm{gr}^F_i M|_U$ is flat. This implies that $F_i M|_U$ is also flat for each i, by induction on i. Since $M|_U = \mathrm{colim}_i F^i M|_U$, we get that $M|_U$ is also flat.

For the second part, we assume that $X = U$, and we will prove that M is a coherent $\mathcal{O}_X$-module. Something stronger is true: each $\mathrm{gr}^F_i M_X$ is a finitely generated, i.e. coherent, $\mathcal{O}_X$-module. Thus, the fact that it's flat implies that it's locally free and hence projective, see [Stacks, Tag 00NX]. By the splitting property of projective modules, we get $M \simeq \bigoplus_i \mathrm{gr}^F_i M$ as an $\mathcal{O}_X$-module.

Let $x \in X$ be an arbitrary closed point. Then, $M \otimes_{\mathcal{O}_X} k(x)$ is finite dimensional, which means that $\mathrm{gr}^F_i M \otimes_{\mathcal{O}_X} k(x) \simeq 0$ when $i \gg 0$. By Nakayama's lemma, we see that $(\mathrm{gr}^F_i M)_x \simeq 0$ when $i \gg 0$, and hence, assuming that X is connected, $\mathrm{gr}^F_i M \simeq 0$ when $i \gg 0$. Thus, $M \simeq \bigoplus_i^N \mathrm{gr}^F_i M$ for some large, but finite, N. In particular, M is a finitely generated locally free sheaf. □

Proof of Theorem 4.7.5. (i) implies (ii) is a direct consequence of Corollary 4.7.2. It remains to prove the other direction. And we will do this by downward induction on $\dim \operatorname{Supp} M$.

Let $S = \operatorname{Supp} M$, let U be an open dense subset of S^{reg} such that $\dim(S \setminus U) < \dim S$. Let V be an open subset of X such that $V \cap S = U$. Let $j : V \to X$ denote the open embedding, and consider

$$M \to j_* j^! M \to C \to \cdots$$

Note that C has lower dimensional support, so we are covered by induction. Thus, it suffices to show that $j_* j^! M$ is holonomic. Since j_* preserves holonomic, it suffices to show that $j^! M$ is holonomic. But, $j^! M \simeq i_* N$ where $i : U \hookrightarrow V$ is a closed embedding and N a coherent $\mathcal{D}_V$-module. Thus, $j^! M$ is holonomic if and only if N is. Shrinking U and V if necessary, Lemma 4.7.6 implies that we can take N to be flat over $\mathcal{O}_U$. For any point $x \in U$, its fiber at x is, up to a shift, just tensoring with $k(x)$ (there's no higher cohomology, due to flatness). Lemma 4.7.6 then allows us to conclude that N is coherent, and in fact, locally free. Thus, N is an vector bundle with integrable connection. In particular, N is holonomic, and we are done. □

Theorem 4.7.7. *Let $f : X \to Y$ be a morphism between smooth affine varieties. Then f_* preserves holonomicity.*

Proof. Factoring f into a closed immersion followed by a projection, we reduce to the case $p : X \times Y \to Y$ since we already know that pushing forward along a closed immersion preserves holonomicity.

Now, embed X and Y as closed sub-varieties of $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively and consider the following commutative diagram

$$\begin{array}{ccc} X \times Y & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ \mathbb{A}^m \times \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \end{array}$$

By Lemma 4.7.4, which says that pushing forward along a closed embedding both preserves and reflects holonomicity, it suffices to show that pushing forward along the projection map $p : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^n$ preserves holonomicity.

Without loss of generality, we can assume that $m = 1$. Write $V = \mathbb{A}^{n+1}$ and $W = \mathbb{A}^n$, then the projection $p : V \to W$ induces a closed embedding $i = p^\vee : W^\vee \to V^\vee$. Consider the Fourier transform $F_V : \mathcal{D}_V \simeq \mathcal{D}_{V^\vee}$ given by

$$x_i \mapsto -\partial_i^\vee \qquad \text{and} \qquad \partial_i \mapsto x_i^\vee .$$

It is known that F preserves and reflects holonomicity, see Remark 4.7.9. Observe that

$$F_W p_* \simeq i^! F_V . \tag{4.7.8}$$

Indeed, up to a shift, $i^!$ is given by

$$(\mathbb{C}[x_1^\vee, \ldots, x_{n+1}^\vee] \xrightarrow{x_{n+1}^\vee} \mathbb{C}[x_1^\vee, \ldots, x_{n+1}^\vee]) \otimes_{\mathbb{C}[x_1^\vee, \ldots, x_{n+1}^\vee]} M \simeq \left(M \xrightarrow{x_{n+1}^\vee} M \right),$$

whereas p_*M is given by

$$M \xrightarrow{\partial_{n+1}} M,$$

see also Example 3.2.12. These two are clearly related by a Fourier transform as written in (4.7.8). Now, in order to show that p_* preserves holonomicity, it suffices to show that $i^!$ does. But this is already done in Corollary 4.7.2. □

Remark 4.7.9. Besides filtration by order, the Weyl algebra $\mathcal{D}_{\mathbb{A}^n}$ has *Bernstein filtration* B, defined as follows:

$$B^k\mathcal{D}_{\mathbb{A}^n} = \sum_{\deg p + \deg q \leq k} p(x_1, \dots, x_n)q(\partial_1, \dots, \partial_n),$$

where p, q are polynomials. Since x_i's and ∂_i's play symmetric role here, the Bernstein filtration is compatible with Fourier transform. Furthermore, $B^k\mathcal{D}_{\mathbb{A}^n}$ is finite-dimensional for any k. Thus for any good (with respect to B) filtration F on a coherent $\mathcal{D}_{\mathbb{A}^n}$-module M each filtered piece F^iM is finite-dimensional. This allows us to import the theory of Hilbert polynomials from the commutative setting. We have

$$\dim F^iM = \chi(M, F; i)$$

for $i \gg 0$, $\chi(M, F; t) \in \mathbb{Q}[t]$. Moreover, the degree of χ is independent of F, and equal to the dimension of the support of $\operatorname{gr}^F M$. One can easily check that $\dim \operatorname{gr}^F M = \dim \mathsf{Ch}(M)$. This shows that the dimension of singular support is preserved under Fourier transform. In particular, Fourier transform preserves and reflects holonomic $\mathcal{D}$-modules.

4.8. **Compatibility with Verdier duality.** We finish this section by proving that holonomic modules behaves nicely with respect to Verdier duality. For any $M_\bullet \in D^b(\mathsf{Coh}(\mathcal{D}_X))$, we define $\mathsf{Ch}(M_\bullet) := \bigcup_i \mathsf{Ch}\, H^i(M_\bullet)$.

Proposition 4.8.1. *Let $M_\bullet \in D^b(\mathsf{Coh}(\mathcal{D}_X))$. We have $\mathsf{Ch}(M_\bullet) = \mathsf{Ch}(\mathbf{D}\, M_\bullet)$.*

Proof. By symmetry, we only need to show that $\mathsf{Ch}(\mathbf{D}\, M_\bullet) \subset \mathsf{Ch}(M_\bullet)$. Let us first consider the case when $M \in \mathsf{Hol}\, X$. Given a good filtration F on M, there exists a good filtration F on $\mathsf{Ext}^i(M, \mathcal{D}_X)$ such that $\operatorname{gr}^F \mathsf{Ext}^i(M, \mathcal{D}_X)$ is isomorphic to a subquotient of $\mathsf{Ext}^i(\operatorname{gr}^F M, \mathcal{O}_{T^*X})$, see [HTT, Lemma D.2.5]. As a consequence, we have

$$\begin{aligned} \mathsf{Ch}(\mathbf{D}\, M) &= \bigcup_i \mathsf{Ch}(\mathsf{Ext}^i(M, \mathcal{D}_X)) = \bigcup_i \mathsf{Supp}(\operatorname{gr}^F \mathsf{Ext}^i(M, \mathcal{D}_X)) \\ &\subset \bigcup_i \mathsf{Supp}(\mathsf{Ext}^i(\operatorname{gr}^F M, \mathcal{O}_{T^*X})) \subset \mathsf{Supp}(\operatorname{gr}^F M) = \mathsf{Ch}(M). \end{aligned}$$

For the general case, we use induction on the cohomological length of $M_\bullet$. Suppose $M_\bullet \simeq \tau_{\geq k} M_\bullet$. We have a distinguished triangle

$$H^k(M_\bullet)[-k] \to M_\bullet \to \tau_{\geq k} M_\bullet \xrightarrow{+1}$$

We have $\mathsf{Ch}(M_\bullet) = \mathsf{Ch}(\tau_{\geq k}M_\bullet) \cup \mathsf{Ch}(H^k(M_\bullet))$. Applying $\mathbf{D}$ to the distinguished triangle above, we have

$$\begin{aligned}\mathsf{Ch}(\mathbf{D}\,M_\bullet) &\subset \mathsf{Ch}(\mathbf{D}\,\tau_{\geq k}M_\bullet) \cup \mathsf{Ch}(\mathbf{D}\,H^k(M_\bullet)) \\ &\subset \mathsf{Ch}(\tau_{\geq k}M_\bullet) \cup \mathsf{Ch}(H^k(M_\bullet)) = \mathsf{Ch}(M_\bullet)\end{aligned}$$

by the induction hypothesis. □

Corollary 4.8.2. *Verdier duality* $\mathbf{D}$ *induces an autoequivalence of* $D^b(\mathsf{Hol}\,X)$.

One can ask if Verdier duality can be restricted to the abelian category $\mathsf{Hol}\,X$, i.e. if Verdier dual of a holonomic module is of cohomological length 1. The answer is yes; in fact, the inverse statement also holds true.

Lemma 4.8.3. *M is holonomic if and only if $H^i(\mathbf{D}\,M) = 0$ for $i \neq 0$.*

Proof. We use Lemma 4.5.5. Note that $H^i(\mathbf{D}\,M) = \mathsf{Ext}^{i+\dim X}(M, \mathcal{D}_X)$. By (1), we have $\operatorname{codim} \mathsf{Ch}(H^i(\mathbf{D}\,M)) > \dim X$ for $i > 0$, and so $H^i(\mathbf{D}\,M) = 0$ by Bernstein inequality. By (2), we have $H^i(\mathbf{D}\,M) = 0$ for $i < 0$. This proves the "only if" part.

For the "if" part, assume that $\mathbf{D}\,M \simeq H^0(\mathbf{D}\,M) =: M'$. We have $\mathbf{D}\,M' \simeq M$, and $H^0(\mathbf{D}\,M') \simeq M$. On the other hand, (1) tells us that $\operatorname{codim} \mathsf{Ch}(H^0(\mathbf{D}\,M')) \geq \dim X$, so that M is holonomic. □

5. $\mathcal{D}$-modules over general smooth varieties

Let us generalize our theory of $\mathcal{D}$-modules to arbitrary smooth varieties.

5.1. Definitions.

Let X be a smooth variety over $\mathbb{C}$, $\mathcal{O}_X$ is the sheaf of functions on X, Θ_X the tangent sheaf, $\Omega^1_X = \Theta_X^\vee$ the sheaf of 1-forms, and $\omega_X = \bigwedge^{\mathrm{top}} \Omega^1_X$ the dualizing sheaf.

Definition 5.1.1. The sheaf of differential operators $\mathcal{D}_X$ is defined as the union $\bigcup \mathcal{D}_X^{\leq n} \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$, where $\mathcal{D}_X^{\leq 0} = \mathcal{O}_X$, and

$$\mathcal{D}_X^{\leq n} = \left\{ A \mid [A, f] \in \mathcal{D}_X^{\leq n-1} \text{ for all } f \in \mathcal{O}_X \right\}.$$

On an open affine $U \subset X$, this definition recovers Definition 3.1.9. Let us underline that $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ is a quasi-coherent *sheaf* of $\mathcal{O}_X$-modules and a sheaf of $\mathbb{C}$-algebras, but not a sheaf of $\mathcal{O}_X$-algebras.

Definition 5.1.2. A *$\mathcal{D}$-module* on X is a quasi-coherent sheaf on X together with a structure of a module over the sheaf of algebras $\mathcal{D}_X$. A $\mathcal{D}$-module is *coherent* if it is locally finitely generated over $\mathcal{D}_X$.

As before, we have $\operatorname{gr} \mathcal{D}_X = \bigoplus_k \mathsf{Sym}^k_{\mathcal{O}_X}(\Theta_X)$, so that

$$\mathsf{Spec}_X(\operatorname{gr} \mathcal{D}_X) = T^*X.$$

The whole theory of characteristic varieties trivially generalizes to the non-affine situation, since all the constructions and dimension estimates are local. In particular, for any $M \in \mathsf{Coh}(\mathcal{D}_X)$ we can define $\mathsf{Ch}(M) \subset T^*X$, and the notion of holonomicity makes sense.

5.2. **Pullbacks, pushforwards and duality.** Recall that in the affine case, we defined pullback and pushforward as tensor product by a certain bimodule $\mathcal{D}_{X\to Y}$. Back then, we worked within the category of $\mathbb{C}$-vector spaces. Now that X, Y are not necessarily affine, we need to keep track of the space over which we consider sheaves. A careful consideration leads to the following definition.

Definition 5.2.1. Let $f : X \to Y$ be a morphism of smooth varieties. We define $\mathcal{D}_{X\to Y}$ to be the following sheaf on X:

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

This sheaf is naturally equipped with a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule structure as in Definition 3.2.7.

The pullback is defined almost verbatim as in the affine case.

Definition 5.2.2. The $\mathcal{D}$*-module pullback* $f^!$ is defined by

$$f^! : D^b(\mathcal{D}_Y\text{-mod}) \to D^b(\mathcal{D}_X\text{-mod}),$$
$$M \mapsto \left(\mathcal{D}_{X\to Y} \otimes^{\mathsf{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}(M)\right)[\dim X - \dim Y],$$

where $f^{-1}(M)$ is the sheaf-theoretic pullback of M.

Note that we have implicitly used the fact that f^{-1} is an exact functor.

In order to define pushforward, we need to be a little bit more careful. First of all, let's talk about right $\mathcal{D}$-modules in order to not deal with additional canonical sheaves. The formula that we had in the affine case is $f_*(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$. In the current setting, this would be a sheaf over X, so we need to push it to Y sheaf-theoretically. However, contrary to pullback the sheaf-theoretic pushforward $f_*^{\mathcal{O}}$ is only left exact. Since tensor product above is right exact, this means that the composition cannot be written as a derived functor of a functor between abelian categories. Still, we have the following definition.

Definition 5.2.3. The $\mathcal{D}$*-module pushforward* f_* is defined by

$$f_* : D^b(\text{mod-}\mathcal{D}_X) \to D^b(\text{mod-}\mathcal{D}_Y),$$
$$M \mapsto Rf_*^{\mathcal{O}}\left(M \otimes^{\mathsf{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right),$$

where $f_*^{\mathcal{O}}$ is the sheaf-theoretic pushforward.

Remark 5.2.4. If $f : X \hookrightarrow Y$ is a closed embedding, then $f_*^{\mathcal{O}}$ *is* exact, as well as $- \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$, so f_* is an exact functor, and the derived decorations above can be omitted.

Finally, the Verdier duality $\mathbf{D}$ is defined in the same way as before, see Definition 4.5.1.

It is clear that $\mathbf{D}$ preserves holonomic modules, since this is a statement of local nature. Furthermore, the proof of Theorem 4.7.3 works for non-affine varieties without any changes, so that pullbacks preserve holonomic modules. Let us check that the same hold for pushforwards. First of all, using factorization through graph of f and Lemma 4.7.4 it suffices to consider the case when $f : X \times Y \to Y$ is a projection. Without loss of generality, we may assume that Y is affine, and $M \in \mathsf{Hol}X$ is a holonomic module (and not complex). Cover $X = \bigcup_{i=0}^{r} X_i$ by affine open charts, assuming that and let $j_i : X_i \times Y \hookrightarrow X \times Y$ be the inclusions. Then M is quasi-isomorphic to the Čech complex

$$M \simeq \left(0 \to \bigoplus_i j_{i*} M|_{X_i \times Y} \to \bigoplus_{i_0 < i_1} (j_{i_0 i_1})_* M|_{(X_{i_0} \cap X_{i_1}) \times Y} \to \cdots \right).$$

Now $f_*(M)$ can be computed via a spectral sequence associated to this complex. However, we know that $f \circ j_i$, $f \circ j_{i_0 i_1}$ (and so on) are maps between affine varieties, so that Theorem 4.7.7 is applicable. Thus all terms in the spectral sequence are holonomic, so $f_*(M)$ must be holonomic as well.

Remark 5.2.5. In the next subsection, we will see another proof that pushforward preserves holonomicity, which will use Theorem 4.7.5.

5.3. $\mathcal{D}$-affine varieties.

Definition 5.3.1. A smooth variety X is called *$\mathcal{D}$-affine* if the following conditions are satisfied:

1. the global section functor $\Gamma(X, -) : \mathcal{D}_X\text{-mod} \to \Gamma(X, \mathcal{D}_X)\text{-mod}$ is exact,
2. if $\Gamma(X, M) = 0$, then $M = 0$.

Proposition 5.3.2. *Let X be a $\mathcal{D}$-affine variety. Then*

1. *any $M \in \mathcal{D}_X\text{-mod}$ is generated by its global sections,*
2. *the functor $\Gamma(X, -)$ gives an equivalence of categories.*

Proof. Let $M \in \mathcal{D}_X\text{-mod}$, and let M_0 be the image of the natural map $\mathcal{D}_X \otimes_{\mathbb{C}} \Gamma(X, M) \to M$. By the first condition, we get an exact sequence

$$0 \to \Gamma(X, M_0) \to \Gamma(X, M) \to \Gamma(X, M/M_0) \to 0.$$

The first map is an isomorphism by definition of M_0, therefore $\Gamma(X, M/M_0) = 0$. By the second condition, we have $M/M_0 = 0$. Thus $M = M_0$, which proves (1).

For (2), let us write $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$, and consider the left adjoint $\mathcal{D}_X \otimes_{\mathcal{D}(X)} -$ to $\Gamma(X, -)$. It is sufficient to show that the unit and counit maps

$$\alpha_M : \mathcal{D}_X \otimes_{\mathcal{D}(X)} \Gamma(X, M) \to M, \qquad \beta_V : V \to \Gamma(X, \mathcal{D}_X \otimes_{\mathcal{D}(X)} V)$$

are isomorphisms.

Pick a partial free resolution of V

$$\mathcal{D}(X)^{\oplus I} \to \mathcal{D}(X)^{\oplus J} \to V \to 0.$$

By the first condition, the functor $\Gamma(X, \mathcal{D}_X \otimes_{\mathcal{D}(X)} -)$ is right exact. Hence we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{D}(X)^{\oplus I} & \longrightarrow & \mathcal{D}(X)^{\oplus J} & \longrightarrow & V & \longrightarrow & 0 \\
\| & & \| & & \downarrow{\scriptstyle \beta_V} & & \\
\mathcal{D}(X)^{\oplus I} & \longrightarrow & \mathcal{D}(X)^{\oplus J} & \longrightarrow & \Gamma(X, \mathcal{D}_X \otimes_{\mathcal{D}(X)} V) & \longrightarrow & 0
\end{array}$$

and so β_V is an isomorphism.

The map α_M is surjective by (1). Hence we have an exact sequence

$$0 \to K \to \mathcal{D}_X \otimes_{\mathcal{D}(X)} \Gamma(X, M) \to M \to 0.$$

Applying $\Gamma(X, -)$ and using that β is an isomorphism, we obtain $\Gamma(X, K) = 0$. This implies that $K = 0$ by the second condition. Thus α_M is an isomorphism, which concludes the proof. □

It is obvious that any affine variety is $\mathcal{D}$-affine. However, the latter class of varieties is richer, as the following theorem shows.

Theorem 5.3.3. *The projective space $\mathbb{P}^n$ is $\mathcal{D}$-affine for any $n > 0$.*

Proof. Let us write $V = \mathbb{C}^{n+1}$, $V^\circ = V \setminus \{0\}$, and $\mathbb{P} = V^\circ/\mathbb{G}_m = \mathbb{P}^n$. We have the following maps:

$$\begin{array}{ccccc} 0 & \xhookrightarrow{i} & V & \xhookleftarrow{j} & V^\circ \\ & & & & \big\downarrow \pi \\ & & & & \mathbb{P} \end{array}$$

Note that j_* coincides with the sheaf-theoretic pushforward, and $\pi^!$ with sheaf-theoretic pullback π^* (up to a shift).

Let $M \in \mathcal{D}_{\mathbb{P}}\text{-mod}$. Recall that $\pi^* M = \mathcal{O}_{V^\circ} \otimes_{\pi^{-1}\mathcal{O}_{\mathbb{P}}} \pi^{-1} M$. In particular, the space of global sections $\Gamma(V^\circ, \pi^* M)$ acquires a $\mathbb{G}_m$-action. Note that

$$\Gamma(V^\circ, \pi^* M) = \mathsf{Hom}(j^* \mathcal{O}, \pi^* M) = \mathsf{Hom}(\mathcal{O}, j_* \pi^* M) = j_* \pi^* M.$$

We can write down the weight space decomposition

$$\Gamma(V^\circ, \pi^* M) = j_* \pi^* M = \bigoplus_i (j_* \pi^* M)_i.$$

It is easy to check that $(j_*\pi^*M)_i = \Gamma(\mathbb{P}, M(n))$; in particular, $(j_*\pi^*M)_0 = \Gamma(\mathbb{P}, M)$. Thus, we need to prove that the functor $M \mapsto (j_*\pi^*M)_0$ is exact, and sends non-zero objects to non-zero objects.

Let us express the weight decomposition above in a different way. Namely, akin to the proof of Kashiwara's theorem consider the Euler operator

$$\theta = \sum_{i=0}^{n} x_i \partial_i \in \mathcal{D}_V.$$

Its k-th eigenspace on $j_*\pi^*M$ coincides with the weight space $(j_*\pi^*M)_k$. Using this definition, it's easy to check that

$$x_i (j_*\pi^*M)_k \subset (j_*\pi^*M)_{k+1}, \qquad \partial_i (j_*\pi^*M)_k \subset (j_*\pi^*M)_{k-1}. \tag{5.3.1}$$

Let us begin with exactness of $j_*\pi^*(-)_0$. Consider an exact sequence of $\mathcal{D}_{\mathbb{P}}$-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Since π is faithfully flat, the map π^* is exact. Applying j_*, we obtain a long exact sequence

$$0 \to j_*\pi^*M_1 \to j_*\pi^*M_2 \to j_*\pi^*M_3 \to R^1 j_*\pi^*M_1 \to \cdots$$

Since j is an open embedding, the sheaf $R^1 j_* \pi^* M_1$ is supported at 0. Applying Kashiwara's theorem, we have

$$R^1 j_* \pi^* M_1 \simeq i_* N = N[\partial_0, \dots, \partial_n],$$

where N is a vector space. As we have seen before, $x_i(\partial_j^k \otimes u) = -k\delta_{ij}(\partial_j^{k-1} \otimes u)$ for $u \in N$. Let $\partial^\alpha = \partial_0^{\alpha_0} \dots \partial_n^{\alpha_n}$. Then

$$\theta(\partial^\alpha \otimes u) = -\sum_i (\alpha_i + 1)\partial^\alpha \otimes u = -(|\alpha| + n + 1)\,\partial^\alpha \otimes u,$$

so that θ acts on $R^1 j_* \pi^* M_1$ with negative eigenvalues. Taking 0-th eigenspaces in the long exact sequence above, we therefore obtain

$$0 \longrightarrow (j_* \pi^* M_1)_0 \longrightarrow (j_* \pi^* M_2)_0 \longrightarrow (j_* \pi^* M_3)_0 \longrightarrow 0,$$

which proves exactness of $j_* \pi^* (-)_0$.

Next, assume that $M \neq 0$ and $(j_* \pi^* M)_0 = 0$. Note that $j_* \pi^* M \neq 0$; in particular, there exists $l \in \mathbb{Z}$ such that $(j_* \pi^* M)_l \neq 0$.

Suppose $l > 0$, and $u \in (j_* \pi^* M)_l$. If $\partial_i u = 0$ for all i, then $\theta u = 0$ and thus $l = 0$. Pick i such that $\partial_i u \neq 0$. Then the formulas (5.3.1) imply that $\partial_i u \in (j_* \pi^* M)_{l-1}$. Repeating this procedure, we obtain a non-zero element of $(j_* \pi^* M)_0$, and thus arrive at a contradiction.

Suppose $l < 0$, and $u \in (j_* \pi^* M)_l$. If $x_i u = 0$ for all i, then u is supported at 0, which cannot happen by definition. Pick i such that $\partial_i u \neq 0$. Then the formulas (5.3.1) imply that $x_i u \in (j_* \pi^* M)_{l+1}$. Repeating this procedure, we obtain a non-zero element of $(j_* \pi^* M)_0$, and thus arrive at a contradiction. This concludes the proof. □

Remark 5.3.4. The same proof shows that for any $n > 0$, X affine variety the product $\mathbb{P}^n \times X$ is $\mathcal{D}$-affine.

Remark 5.3.5. Note that $\mathcal{D}$-affineness is not equivalent to $\mathcal{D}^{\mathrm{op}}$-affineness. In effect, $\mathbb{P}^1$ is $\mathcal{D}$-affine, while $\Gamma(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = 0$.

Corollary 5.3.6. *Let $f : X \longrightarrow Y$ be a morphism between smooth quasi-projective varieties. Then f_* preserves holonomicity.*

Proof. We already know the claim for open and closed embeddings, see Lemmas 4.7.1 and 4.7.4. Thus it remains to prove it for projections $\pi : Y = \mathbb{P}^n \times X \longrightarrow X$.

Let M be a coherent $\mathcal{D}_Y$-module. Let us prove that $\pi_* M$ is a coherent $\mathcal{D}_X$-module. In view of Theorem 5.3.3, it is enough to assume that $M = \mathcal{D}_{\mathbb{P}^n \times X}$. We have

$$\begin{aligned} \pi_*(\mathcal{D}_Y) = R\pi_*^{\mathcal{O}}(\mathcal{D}_{X \leftarrow Y}) = R\pi_*^{\mathcal{O}}(\pi^* \mathcal{D}_X \otimes \omega_{\mathbb{P}^n}) = \mathcal{D}_X \otimes R\pi_*^{\mathcal{O}}(\omega_{\mathbb{P}^n}) \\ = \mathcal{D}_X[-n], \end{aligned}$$

and the latter $\mathcal{D}$-module is clearly coherent.

Let M be a holonomic $\mathcal{D}_{\mathbb{P}^n \times X}$-module. For any $x \in X$, we have the following pull-back square

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{k_x} & \mathbb{P}^n \times X \\ \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle \pi} \\ \{x\} & \xrightarrow{i_x} & X \end{array}$$

By base change, $i_x^! \pi_* M \simeq \pi_{x*} k_x^! M$. Since k_x is a closed embedding, $k_x^!$ preserves holonomicity by Corollary 4.7.2. But then, what we said above implies that $(p_x)_* k_x^! M$ is finite-dimensional. Thus, we conclude by invoking the criterion for holonomicity (Theorem 4.7.5). □

From now on, we will keep the assumptions of Corollary 5.3.6 and only consider quasi-projective varieties.[5]

Let us describe what the algebra $\Gamma(\mathcal{D}_X)$ looks like for $X = \mathbb{P}^1$.

Question 5.3.7. Let t be a local coordinate on $\mathbb{P}^1$ around 0, and ∂ the corresponding derivation. Recall that vector fields on $\mathbb{P}^1$, that is global sections of $\Theta_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$, are spanned by

$$\partial, \quad t\partial, \quad t^2\partial. \tag{5.3.2}$$

(i) Show that, up to rescaling, vector fields (5.3.2) satisfy the $\mathfrak{sl}_2$-relations
$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$
(ii) Show that $\mathcal{D}(\mathbb{P}^1)$ is generated by the vector fields (5.3.2). Thus, we obtain a surjective homomorphism of algebras $\alpha : U(\mathfrak{sl}_2) \to \mathcal{D}(\mathbb{P}^1)$.
(iii) Show that the Casimir element $c := ef + fe + h^2/2$ is mapped to 0 by α.

One can prove that the map $U(\mathfrak{sl}_2)/(c = 0) \to \mathcal{D}(\mathbb{P}^1)$ obtained above is an isomorphism. This example is the simplest manifestation of Beilinson-Bernstein localization theorem.

[5]We have already implicitly done this, by assuming that there exist enough projective objects to construct derived functors.

Part 2. Representation theory of semi-simple Lie algebras

As hinted at the end of the first part, the main goal of the course is to establish a result which relates representation theory of $U(\mathfrak{g})$ with $\mathcal{D}$-modules on the flag variety G/B, where $\mathfrak{g} = \mathsf{Lie}(G)$ and G is a semi-simple group. In this part, we will study representation theory of semi-simple Lie algebras. We will mostly follow [G1].

6. Basics of the category $\mathcal{O}$

6.1. **Preliminaries on semi-simple Lie algebras.** Fix $\mathfrak{g}$ a semi-simple Lie algebra. We fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Sometimes, we will use $\mathfrak{b}^+$ to denote $\mathfrak{b}$.

Let $\mathfrak{n}^+ = \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilpotent radical of $\mathfrak{b}$ and similarly for $\mathfrak{n}^-$. We have $\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}$.

We have the following decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha = \bigoplus_{\alpha\in\Delta^+} \mathfrak{n}_\alpha,$$

where $\Delta \subset \mathfrak{h}^*$ is the set of *roots*, and $\Delta^+ \subset \Delta$ is the set of *positive* roots. All vector spaces $\mathfrak{g}_\alpha$ are one-dimensional. The complement $\Delta^- = \Delta \setminus \Delta^+$ is the set of *negative* roots. We write

$$\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha = \bigoplus_{\alpha\in\Delta^-} \mathfrak{n}^-_\alpha.$$

Then $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$ is a Borel subalgebra of $\mathfrak{g}$ which is opposite to $\mathfrak{b}^+$, that is it satisfies $\mathfrak{b}^+ \cap \mathfrak{b}^- = \mathfrak{h}$. By abuse of notation, for $\alpha \in \Delta^+$ we write $\mathfrak{n}^-_\alpha = \mathfrak{n}^-_{-\alpha}$.

Let Q^+ be the sub-semigroup of $\mathfrak{h}^*$ given by the non-negative span of Δ^+. For $\lambda, \mu \in \mathfrak{h}^*$, we say that $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. Each root $\alpha \in \Delta$ admits a dual *coroot* $\check{\alpha} \in \mathfrak{h}$, and these coroots form a dual root system $\check{\Delta}$. We denote by $P^+ \subset \mathfrak{h}^*$ the sub-semigroup of dominant integral weights, i.e. those λ such that $\langle \lambda, \check{\alpha}\rangle \in \mathbb{Z}^+$ for all $\alpha \in \Delta^+$. We also denote by P, Q the sublattices of $\mathfrak{h}^*$, generated by P^+, Q^+ respectively.

Example 6.1.1. Let $\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f$ with the usual relations. Then, $\Delta = \{2h^*, -2h^*\}$, $\check{\Delta} = \{h, -h\}$, $Q^+ = 2\mathbb{Z}_+ h^*$ and $P^+ = \mathbb{Z}_+ h^*$. We will use $\alpha = 2h^*$ and $\check{\alpha} = h$ to denote the unique positive root/coroot of $\mathfrak{sl}_2$.

In a certain sense, all semi-simple Lie algebras are built from the example above. Namely, for each $\alpha \in \Delta^+$, let $e_\alpha \in \mathfrak{n}_\alpha$ and $f_\alpha \in \mathfrak{n}^-_\alpha$ be non-zero vectors. Then[6]

$$[e_\alpha, f_\alpha] = k h_{\check{\alpha}}, \quad k \in \mathbb{C}^\times.$$

In particular, the span of $e_\alpha, f_\alpha, h_\alpha$ inside $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2$.

We write

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha \in \mathfrak{h}^* \qquad \text{and} \qquad \check{\rho} = \frac{1}{2}\sum_{\alpha\in\check{\Delta}^+} \check{\alpha} \in \mathfrak{h}.$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then, in general (i.e. no extra assumption on $\mathfrak{g}$ needed)

$$\mathfrak{g}\text{-mod} \simeq U(\mathfrak{g})\text{-mod}.$$

[6] h_α is just $\check{\alpha}$.

When $\mathfrak{g}$ is semi-simple, we have an isomorphism

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}) \tag{6.1.1}$$

as $(U(\mathfrak{n}^-), U(\mathfrak{n}))$-bimodules given by left and right multiplications respectively.

6.2. **Borel–Weil theorem.** Let G be a semi-simple simply connected algebraic group and $\mathfrak{g} = \mathsf{Lie}(G)$. Let $\mathsf{Rep}(G)$ denote the category of representations of G. Taking differentials at the identity, we obtain a functor

$$\mathsf{Rep}(G) \to \mathfrak{g}\text{-}\mathsf{mod}\,.$$

Unlike in the case of Lie algebras, all representations of G are unions of finite dimensional representations. Thus it suffices to consider only finite dimensional representations. Moreover, it can be shown that the category of G-representations is semi-simple with irreducible representations given by integral dominant weights. These coincide precisely with finite dimensional irreducible representations of $\mathfrak{g}$. In particular, the essential image of the functor above is precisely the full subcategory spanned by locally finite $\mathfrak{g}$-modules.

The theorem of Borel–Weil gives a way to realize these irreducible representations geometrically. Consider the quotient map $G \to G/B$, which is naturally a B-principal bundle. Let λ be a character of T, i.e. a group homomorphism $T \to \mathbb{G}_{\mathrm{m}}$. Then in particular, we get group homomorphism $B \twoheadrightarrow T \to \mathbb{G}_{\mathrm{m}}$. The B-principal bundle $G \to G/B$ is compatible with G-action on the left. Thus, it can can be induced to obtain a principal $\mathbb{G}_{\mathrm{m}}$-bundle, and hence a line bundle, to be denoted by $\mathcal{L}_\lambda$, which is G-equivariant. The cohomology group $\mathsf{H}^0(G/B, \mathcal{L}_\lambda)$ thus has an action of G.

Remark 6.2.1. The discussion above could be made cleaner using the language of stacks. Indeed, consider the following correspondence

$$\begin{array}{ccc} BB & \longrightarrow & BT \\ \big\downarrow & & \\ BG & & \end{array}$$

For any algebraic group H, $\mathsf{QCoh}(BH) \simeq \mathsf{Rep}(H)$. In particular, a character λ of T induces a line bundle $\mathcal{L}_\lambda$ on BT. Pulling and pushing along this correspondence gives an element in $\mathsf{QCoh}(BG) = \mathsf{Rep}\, G$. The relation to the discussion above is realized via the following pullback diagram

$$\begin{array}{ccc} G/B & \longrightarrow & BB \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & BG \end{array}$$

In this subsection, we will always assume that λ is integral.

Theorem 6.2.2 (Borel–Weil)**.** *If $-\lambda$ is not a dominant weight, then $\mathsf{H}^0(G/B, \mathcal{L}_\lambda) \simeq 0$. If $-\lambda$ is dominant, then $\mathsf{H}^0(G/B, \mathcal{L}_\lambda)$ is an irreducible representation with lowest weight λ (and hence, is the dual of the irreducible representation of highest weight $-\lambda$).*

Proof. Since G/B is proper, $V = \mathsf{H}^0(G/B, \mathcal{L}_\lambda)$ is finite dimensional. Thus, we obtain a finite dimensional representation of G. Let U^- denote the negative unipotent subgroup of G and $V_0 \subseteq V$ denote the subspace of lowest weight vectors, i.e. $V_0 = V^{U^-}$.

The space of global sections $\mathsf{H}^0(G/B, \mathcal{L}_\lambda)$ could be identified with the space of functions v on G such that $v(xb) = \lambda(b)v(x)$ for each $b \in B$. The action of G on V can be realized explicitly as $(gv)(x) = v(xg)$. The space V_0 thus consists of those v such that $v(ux) = v(x)$ for all $u \in U^-$. In particular, $v(ub) = \lambda(b)v(1)$ for all $u \in U^-, b \in B$. In other words, over U^-B, v is completely determined by the value $v(1)$. Since U^-B is dense in G we see V_0 has dimension at most 1. If it's non-zero, it's generated by v with $v(1) = 1$. Let $t \in T$, we have

$$(tv)(1) = v(t) = \lambda(t)v(1),$$

so that $tv = \lambda(t)v$ and hence, V is an irreducible representation of lowest weight λ. But this is not possible unless $-\lambda$ is dominant.

It remains to show that V_0 is non-trivial if $-\lambda$ is dominant. Exercise, at least for SL_2. □

6.3. **Verma modules.** The map of Lie algebras $\mathfrak{b} \to \mathfrak{g}$ induces a map of algebras $U(\mathfrak{b}) \to U(\mathfrak{g})$ from which we obtain a pair of adjoint functors

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} - \simeq \mathsf{Ind}_{\mathfrak{b}}^{\mathfrak{g}} : \mathfrak{g}\text{-mod} \rightleftarrows \mathfrak{b}\text{-mod} : \mathsf{res}_{\mathfrak{b}}^{\mathfrak{g}}$$

For any $\lambda \in \mathfrak{h}$, let $\mathbb{C}^\lambda$ denote the corresponding $\mathfrak{h}$-module. By abuse of notation, we will also use $\mathbb{C}^\lambda$ to denote the $\mathfrak{b}$-module given by $\mathfrak{b} \to \mathfrak{h}$. We let

$$M_\lambda = \mathsf{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}^\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}^\lambda.$$

By adjunction, for any $N \in \mathfrak{g}\text{-mod}$,

$$\mathsf{Hom}_{\mathfrak{g}}(M_\lambda, N) \simeq \mathsf{Hom}_{\mathfrak{b}}(\mathbb{C}^\lambda, N).$$

By construction, M_λ is generated over $U(\mathfrak{g})$ by a vector v_λ.

Lemma 6.3.1. *We have an equivalence of $U(\mathfrak{n}^-)$-modules $M_\lambda = U(\mathfrak{n}^-) \otimes \mathbb{C}^\lambda$. In other words, M_λ is freely generated over $U(\mathfrak{n}^-)$ by v_λ.*

Proof. Indeed,

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}^\lambda \simeq U(\mathfrak{n}^-) \otimes U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}^\lambda \simeq U(\mathfrak{n}^-) \otimes \mathbb{C}^\lambda.$$

□

Proposition 6.3.2. *The action of $\mathfrak{h}$ on M_λ is locally finite and semi-simple. The weights (i.e. eigen-values of $\mathfrak{h}$) on M_λ are of the form*

$$\lambda - \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \quad n_\alpha \in \mathbb{Z}^+.$$

Proof. We have

$$M_\lambda = \bigcup_{k \geq 0} U(\mathfrak{n}^-)_k v_\lambda.$$

Each element in $U(\mathfrak{n}^-)_k$ is given by $\sqcap_i f_{\alpha_i}$ for $f_{\alpha_i} \in \mathfrak{n}^-_{\alpha_i}$. Thus, for each $h \in \mathfrak{h}$, we have

$$h(\prod_i f_{\alpha_i})v_\lambda = \sum_j f_{\alpha_1} \cdots f_{\alpha_{j-1}}[h, f_{\alpha_j}]f_{\alpha_{j+1}} \cdots f_{\alpha_n} v_\lambda + \prod_i f_{\alpha_i} h v_\lambda$$
$$= (\lambda - \sum_i \alpha_i) \prod_i f_{\alpha_i} v_\lambda.$$

This shows that $U(\mathfrak{n}^-)_k v_\lambda$ is $\mathfrak{h}$-stable. Moreover, since it's finite dimensional, the action of $\mathfrak{h}$ on M_λ is locally finite. The computation above also show that the action is semi-simple with the correct weights. □

Corollary 6.3.3. *The multiplicities of weights of $\mathfrak{h}$ on M_λ are finite.*

Proof. Let $\mu < \lambda$ be a weight of M_λ. Write

$$\lambda - \mu = \sum_{\alpha \in \Delta^-} n_\alpha \alpha, \qquad n_\alpha \in \mathbb{Z}^+.$$

From the computation in the Proposition above, we see that each choice of n_α's contributes one to the multiplicity. However,

$$\langle \check{\rho}, \lambda - \mu \rangle = \sum_{\alpha \in \Delta^-} \langle \check{\rho}, \alpha \rangle \geq \sum_{\alpha \in \Delta^-} n_\alpha,$$

since $\langle \check{\rho}, \alpha \rangle \geq 1$. Thus, the n_α's are bounded and hence, there are only finitely many possibilities. □

For each $\nu \in Q^+$, we will refer to the integer $\langle \check{\rho}, \nu \rangle$ as its length.

6.4. **The case of $\mathfrak{sl}_2$.** Let us now consider Verma modules for $\mathfrak{sl}_2$. For each each number λ is uniquely determined by a complex number $l = \langle \check{\alpha}, \lambda \rangle$, where $\check{\alpha}$ is the unique positive coroot of $\mathfrak{sl}_2$. The weights of $\mathfrak{h}$ on M_λ are of the form $\lambda - n\alpha$, i.e. $l - 2n$. For $l' = l - 2n$, we will use $v_{l'}$ to denote $f^n v_l$. Moreover, we will write M_l instead of M_λ.

Proposition 6.4.1. *The $\mathfrak{sl}_2$-module M_l is irreducible unless $l \in \mathbb{Z}^+$. In the latter case, we have a short exact sequence*

$$0 \to M_{-l-2} \to M_l \to V_l \to 0$$

where V_l is the irreducible finite dimensional $\mathfrak{sl}_2$-module of highest weight l.

Proof. Suppose M_l contains a proper sub-module N. Then in particular, N is $\mathfrak{h}$-stable and hence, $\mathfrak{h}$-diagonalizable (this can be seen by, for example, using invertibility of Vandermonde matrix). Let $l' = l - 2n$ be the maximal weight of N. Clearly, $l' \neq l$ since since otherwise, $v_l \in N$ and hence, $M_l = N$, since M_l is generated by v_l. Equivalently, we have $n > 0$.

Since $ev_{l'}$ can either have weight $l' + 2$ or 0, maximality of l' implies that $ev_{l'} = 0$. Thus,

$$ef^n v_l = \sum_{1 \leq i \leq n} f^{n-i}[e, f]f^{i-1}v_l + f^n e v_l = \sum_{1 \leq i \leq n} (l - 2(i-1)) f^{n-1} v_l = n(l - (n-1)) f^{n-1} v_l.$$

This can only be 0 if $l = n - 1$, which implies, in particular, that $l \geq 0$, since $n > 0$. In this case, $l' = l - 2n = l - 2(l+1) = -l - 2$, and it's easy to see that the vector v_{-l-2} generates a module isomorphic to M_{-l-2}. □

6.5. Irreducible quotients of Verma modules.

Proposition 6.5.1. *M_λ admits a unique irreducible quotient.*

Proof. Let $N \subset M_\lambda$ be a proper sub-module. Then,

$$N \simeq \bigoplus_{\mu < \lambda} N(\mu),$$

where $N(\mu)$ is the weight space associated to weight μ; in particular, $N(\mu) \subset M_\lambda(\mu)$. Let M_λ^0 be the union of all of those N above, then we still have $M_\lambda^0(\lambda) = 0$. Hence, M_λ^0 is a proper sub-module of M_λ.

Take $L_\lambda = M_\lambda / M_\lambda^0$. It is clear from the construction that this is the unique irreducible quotient. □

Corollary 6.5.2. *If $L_\lambda \simeq L_{\lambda'}$, then $\lambda = \lambda'$.*

Proof. Clear. □

6.6. **Category $\mathcal{O}$.** Proposition 6.5.1 shows that it is beneficial to study infinite-dimensional representations of $\mathfrak{g}$. On the other hand, we would want *some* finiteness conditions to be able to deduce reasonable structural results. The following definition is due to Bernstein-Gelfand-Gelfand.

Definition 6.6.1. The *category* $\mathcal{O}$ of a Lie algebra $\mathfrak{g}$ to be the full subcategory of $\mathfrak{g}\text{-mod}$ consisting of representations M, which satisfy the following properties

(i) The action of $\mathfrak{n}$ on M is locally finite, i.e. for any $v \in M$, $U(\mathfrak{n})v$ is finite dimensional.
(ii) The action of $\mathfrak{h}$ on M is locally finite and semi-simple.
(iii) M is finitely generated as a $U(\mathfrak{g})$-module.

Lemma 6.6.2. *Verma modules belong to the category $\mathcal{O}$.*

Proof. The only thing that needs to be proved is local-finiteness of the $\mathfrak{n}$-action. For any $\lambda \in \mathfrak{h}^*$, we have

$$M_\lambda = \bigcup_k U(\mathfrak{g})_k v_\lambda$$

where $U(\mathfrak{g})_k$ is the k-th step in the PBW filtration. It suffices to show that $U(\mathfrak{g})_k$ is stable under $\mathfrak{n}$. For any $x \in U(\mathfrak{g})_k$ and $u \in \mathfrak{n}$, we have

$$uxv_\lambda = [u, x]v_\lambda + xuv_\lambda = [u, x]v_\lambda.$$

But it is clear that $[u, x] \in U(\mathfrak{g})_k$, and hence, we are done. □

Remark 6.6.3. In a precise sense, $\mathcal{O}$ is generated by Verma modules, see Proposition 6.6.10.

Lemma 6.6.4. *$\mathcal{O}$ is stable under taking submodules and quotient modules. Moreover $\mathcal{O}$ is Noetherian.*

Proof. This follows from the fact that $U(\mathfrak{g})$ is Noetherian. □

We will shortly prove that $\mathcal{O}$ is Artinian, i.e. every object has finite length; see Theorem 8.1.4. Note that we have already encountered a category of geometric origin with this property: the category of holonomic $\mathcal{D}$-modules.

Let us recall Lie's and Engel's theorems.

Theorem 6.6.5 (Lie's theorem). *Let $\mathfrak{g}$ be a solvable Lie algebra and $V \in \mathfrak{g}\text{-mod}$ that is finite dimensional. We have the following equivalent statements*:

(*i*) *V has a non-zero invariant $\mathfrak{g}$-vector.*
(*ii*) *There exists a basis of V such that all elements of $\mathfrak{g}$ get sent to an upper-triangular matrix.*
(*iii*) *There a filtration of V such that successive quotient are one-dimensional.*

Theorem 6.6.6 (Engel's theorem). *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if for every $x \in \mathfrak{g}$, $\operatorname{ad} x \in \mathsf{End}(\mathfrak{g})$ is nilpotent.*

Remark 6.6.7. It is natural to ask if there's an analog of Lie's theorem for nilpotent Lie algebras with upper-triangular replaced by strictly upper-triangular. The answer is no. For example, one can take a commutative Lie algebra acting diagonally on a vector space.

Note that this is very different from the case of groups. $\mathbb{G}_\mathrm{m}$ and $\mathbb{G}_\mathrm{a}$ have very different representation categories. However, $\mathsf{Lie}(\mathbb{G}_\mathrm{m}) \simeq \mathsf{Lie}(\mathbb{G}_\mathrm{a})$.

Back to the study of category $\mathcal{O}$.

Lemma 6.6.8. *The action of $\mathfrak{n}$ on any object of $\mathcal{O}$ is locally nilpotent.*

Proof. Let M be an object in the category $\mathcal{O}$ and $v \in M$. It suffices to show that there exists a finite dimensional $\mathfrak{n}$-stable subspace W containing v admitting an $\mathfrak{n}$-stable filtration whose successive quotients are trivial $\mathfrak{n}$-modules.

Indeed, consider $W = U(\mathfrak{b})v$. Then, we know that W is a finite dimensional $\mathfrak{n}$-stable subspace of M. By Lie's theorem, as a $U(\mathfrak{b})$-module, W admits a filtration whose associated graded are one-dimensional. The action of $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ on these subquotients are necessarily trivial. And hence, we are done. □

Remark 6.6.9. In the last step of the proof above, we show that the actions of $\mathfrak{b}$ on all the successive quotients factors through $\mathfrak{b} \twoheadrightarrow \mathfrak{h} = \mathfrak{b}/\mathfrak{n} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$.

We are now ready to show that the category $\mathcal{O}$ is generated by Verma modules.

Proposition 6.6.10. *Every object in the category $\mathcal{O}$ is a quotient of a finite successive extension of Verma modules.*

Proof. Let M be an object of $\mathcal{O}$ and W a finite dimensional subspace of M that generates M. Consider $W' = U(\mathfrak{b})W$, then by assumption, W' is also finite dimensional.

Consider the $\mathfrak{g}$-module

$$M' = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} W',$$

then we have a surjection of $\mathfrak{g}$-module $M' \twoheadrightarrow M$. It suffices to show that M' is finite successive extension of Verma modules. By Remark 6.6.9, we see that as a $\mathfrak{b}$-module, W' admits a finite filtration whose associated are of the form $\mathbb{C}^\lambda$. Thus, M' also admits a filtration whose associated graded are Verma modules. □

Corollary 6.6.11. *For $M \in \mathcal{O}$, and $M = \oplus_\mu M(\mu)$ be the decomposition of M into weight spaces. Then, M is finite dimensional.*

Proof. This follows directly from Proposition 6.6.10 and Corollary 6.3.3. □

Corollary 6.6.12. *Any object of $\mathcal{O}$ admits a non-zero map from some M_λ.*

Corollary 6.6.13. *All irreducible objects of $\mathcal{O}$ are of the form L_λ for some $\lambda \in \mathfrak{h}^*$.*

Proof. Let L be an irreducible object of $\mathcal{O}$. Then by the corollary above L admits a non-zero map from a Verma module $M_\lambda \twoheadrightarrow L$. This map is necessarily surjective since L is irreducible. Thus, $L \simeq L_\lambda$ by Proposition 6.5.1. □

7. Center of $U(\mathfrak{g})$ and blocks in $\mathcal{O}$

Let A be an algebra, and $x \in Z(A)$ a central element. For an A-module M, assume that x is semi-simple on M and has finite-dimensional eigenspaces M_λ, $\lambda \in \Lambda$. Then $M = \bigoplus_\lambda M_\lambda$ as an A-module, and $\mathsf{Hom}(M_\lambda, M_{\lambda'}) = 0$ for $\lambda \neq \lambda'$. We see that the category A-mod' (of A-modules such that the action of x is semi-simple) is thus broken up into "blocks" $A/\langle x-\lambda\rangle$-$\mathsf{mod}$, which do not interact with each other. Thus, in order to understand the structure of category $\mathcal{O}$, we first need to compute the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$.

7.1. The Chevalley isomorphism.

We begin with the commutative version of this question to warm up. Namely, consider the space $\mathsf{Fun}(\mathfrak{g}) = \mathsf{Sym}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}$. The Lie group G acts on $\mathsf{Fun}(\mathfrak{g})$ by conjugation, and $\mathfrak{g}$ acts on it by the dual of Lie bracket. Since we assume G to be connected, the space of invariants $\mathsf{Fun}(\mathfrak{g})^G$ is the same as the space of $\mathfrak{g}$-invariants $\mathsf{Fun}(\mathfrak{g})^{\mathfrak{g}}$.

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. Under this identification, we have $\mathfrak{b}^* \simeq \mathfrak{g}/\mathfrak{n} \simeq \mathfrak{b}^-$, and the projection $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$ corresponds to the embedding of algebras $\mathsf{Sym}(\mathfrak{h}) \hookrightarrow \mathsf{Sym}(\mathfrak{b}^-)$.

Let $a \in \mathsf{Fun}(\mathfrak{g})^G$, and consider its restriction to $\mathfrak{b}$. Since $\mathsf{Sym}(\mathfrak{b}^-) \simeq \mathsf{Sym}(\mathfrak{h}) \otimes \mathsf{Sym}(\mathfrak{n}^-)$ and a is H-invariant, we see that $a|_{\mathfrak{b}}$ belongs to $\mathsf{Sym}(\mathfrak{h})$. We thus obtain a homomorphism $\mathsf{Fun}(\mathfrak{g})^G \to \mathsf{Fun}(\mathfrak{h})$, which sends a to its restriction to $\mathfrak{h}$.

Recall that the *Weyl group* of G is defined as $W = N_G(H)/H$. The action of $N_G(H)$ on $\mathsf{Fun}(\mathfrak{g})$ restricts to $\mathsf{Fun}(\mathfrak{h})$. Moreover, since H is commutative the action of H on $\mathsf{Fun}(\mathfrak{h})$ by conjugation is trivial, so that the action of $N_G(H)$ factors through W. In particular, the restriction $\mathsf{Fun}(\mathfrak{g})^G \to \mathsf{Fun}(\mathfrak{h})$ factors as follows:

$$\begin{array}{ccc} \mathsf{Fun}(\mathfrak{g})^G & \longrightarrow & \mathsf{Fun}(\mathfrak{h}) \\ \big\downarrow & & \big\uparrow \\ \mathsf{Fun}(\mathfrak{g})^{N_G(H)} & \longrightarrow & \mathsf{Fun}(\mathfrak{h})^W \end{array}$$

We have therefore obtained a map $\phi_{cl} : \mathsf{Fun}(\mathfrak{g})^G \to \mathsf{Fun}(\mathfrak{h})^W$.

Theorem 7.1.1 (Chevalley isomorphism)**.** *The map ϕ_{cl} is an isomorphism.*

Proof. Suppose $a \in \mathsf{Fun}(\mathfrak{g})^G$ vanishes on $\mathfrak{h}$. Then a also vanishes on $G.\mathfrak{h}$, which is Zariski dense in $\mathfrak{g}$. Thus $a = 0$, which proves that ϕ_{cl} is injective.

For surjectivity, let us consider two families of functions. First, for any $\lambda \in P^+$ consider the corresponding finite dimensional representation L_λ with highest weight λ, and let

$$a_{\lambda,n} \in \mathsf{Fun}(\mathfrak{g})^G, \qquad a_{\lambda,n}(x) = \mathsf{Tr}(x^n, L_\lambda).$$

Second, let

$$b_{\lambda,n} \in \mathsf{Fun}(\mathfrak{h})^W, \qquad b_{\lambda,n}(x) = \sum_{w \in W} w.\lambda(x)^n.$$

It is clear that the functions $b_{\lambda,n}$ span $\mathsf{Sym}^n(\mathfrak{h}^*)^W$. In effect, since W is finite, it suffices to prove that the elements $\lambda(-)^n$ span $\mathsf{Sym}^n(\mathfrak{h}^*)$. This in turn follows from the following easy fact:

Question 7.1.2. Any homogeneous polynomial $p(x) \in \mathbb{C}[x_1, \dots x_m]$ of degree n can be expressed as a linear combination of functions $(\sum c_i x_i)^n$, where c_i's are non-negative integers.

The surjectivity of ϕ_{cl} then follows from the following lemma:

Lemma 7.1.3. *For some scalars c', we have*

$$\phi_{cl}(a_{\lambda,n}) = \frac{1}{|\,\mathsf{Stab}_W(\lambda)|} b_{\lambda,n} + \sum_{\lambda' < \lambda} c' b_{\lambda',n}.$$

Proof. For $x \in \mathfrak{h}$, we have $a_{\lambda,n}(x) = \sum_\mu \mu^n(x)$, where the sum is taken over the set of weights of L_λ, taking into account multiplicities. Since L_λ can be integrated to a representation of G, this set is W-invariant. Furthermore, all weights λ' of L_λ satisfy $\lambda' \le \lambda$, with λ appearing with multiplicity 1. The claim follows. □

□

The map ϕ_{cl} can be interpreted as a map of varieties $\mathfrak{g} \to \mathfrak{h}/W$, where we write $\mathfrak{h}/W := \mathsf{Spec}(\mathsf{Fun}(\mathfrak{h})^W)$.

Example 7.1.4. Let $\mathfrak{g} = \mathfrak{sl}_n$. In this case $\mathsf{Fun}(\mathfrak{h})^W$ is freely generated by $n-1$ elementary symmetric functions

$$a_i : x \mapsto \mathsf{Tr}(x^i), \quad 2 \le i \le n.$$

The map $\mathfrak{sl}_n \to \mathsf{Spec}\,\mathbb{C}[a_2, \dots, a_n]$ assigns to a matrix x its characteristic polynomial.

In general, $\mathsf{Fun}(\mathfrak{h})^W$ is a polynomial algebra on $\dim \mathfrak{h}$ generators.

7.2. **The Harish-Chandra homomorphism.** Let us return to the study of $Z(\mathfrak{g})$. It is easy to see that

$$Z(\mathfrak{g}) = \{u \in U(\mathfrak{g}) \mid [x,u] = 0, \forall x \in \mathfrak{g}\} = U(\mathfrak{g})^{\mathfrak{g}}.$$

The PBW filtration on $\mathfrak{g}$ produces the following short exact sequences of finite-dimensional $\mathfrak{g}$-modules:

$$0 \to U(\mathfrak{g})_{i-1} \to U(\mathfrak{g})_i \to \mathsf{Sym}^i(\mathfrak{g}) \to 0.$$

Let $Z(\mathfrak{g})_i = Z(\mathfrak{g}) \cap U(\mathfrak{g})_i$. Since the category $\mathsf{Rep}(G)$ is semi-simple, the short exact sequence above gives rise to isomorphisms $Z(\mathfrak{g})_i/Z(\mathfrak{g})_{i-1} \simeq \mathsf{Sym}^i(\mathfrak{g})^{\mathfrak{g}}$. In particular, we have an isomorphism of algebras $\mathsf{gr}\, Z(\mathfrak{g}) \simeq \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}$.

Thanks to the triangular decomposition (6.1.1), we have a map

$$U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) = U(\mathfrak{h}).$$

Let us denote its restriction $Z(\mathfrak{g}) \to U(\mathfrak{h})$ to the center of $U(\mathfrak{g})$ by ϕ.

Proposition 7.2.1. *The map ϕ is a homomorphism of algebras.*

Proof. Let $h + n_- x + y n_+$, $h' + n'_- x' + y' n'_+$ be two elements in $Z(\mathfrak{g})$, where $h, h' \in U(\mathfrak{h})$, $n_-, n'_- \in \mathfrak{n}^-$, $n_+, n'_+ \in \mathfrak{n}^+$, and $x, y, x', y \in U(\mathfrak{g})$. We have the following chain of equalities in $U(\mathfrak{g})/(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$:

$$\begin{aligned}(h + n_- x + y n_+)(h' + n'_- x' + y' n'_+) &= (h + n_- x + y n_+)(h' + n'_- x') \\ &= (h + n_- x + y n_+)h' = hh' + y n_+ h' = hh' + y\tilde{h}' n_+ = hh',\end{aligned}$$

where $\tilde{h}' \in U(\mathfrak{h})$. This proves our claim. □

Lemma 7.2.2. *For any $a \in Z(\mathfrak{g})$, $\lambda \in P^+$, we have $a.v = \phi(a)(\lambda)v$ for any $v \in M_\lambda$.*

Proof. Consider the composition $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}} \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}$. Since $[a, \mathfrak{h}] = 0$, the image of a lies in $U(\mathfrak{h}) = U(\mathfrak{b})/U(\mathfrak{b})\mathfrak{n} \subset U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}$.

Let $v_\lambda \in M_\lambda$ be the highest weight vector. By definition of Verma modules, the map

$$U(\mathfrak{g}) \to M_\lambda, \qquad x \to x v_\lambda$$

factors through $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}$. In particular, $a v_\lambda = \phi(a) v_\lambda = \phi(a)(\lambda) v_\lambda$. Since M_λ is generated by v_λ as a $U(\mathfrak{g})$-module and a is central, we conclude that a acts by the scalar $\phi(a)(\lambda)$ on the whole M_λ. □

7.3. **The dot action of Weyl group.** By Chevalley isomorphism, we know that $\operatorname{im}\phi_{cl} = \mathrm{Sym}(\mathfrak{h})^W$. However, observe that while ϕ_{cl} preserves the natural gradings, $Z(\mathfrak{g})$ is not a graded algebra, so that $\operatorname{im}\phi$ is not necessarily a graded submodule of $U(\mathfrak{h})$.

Example 7.3.1. Let $\mathfrak{g} = \mathfrak{sl}_2$, and $C = ef + fe + h^2/2 \in Z(\mathfrak{sl}_2)$. Then $\phi(C) = \phi(2fe + h + h^2/2) = h + h^2/2$ is not a homogeneous element.

Thus we see that $\operatorname{im}\phi \neq U(\mathfrak{h})^W$. In a sense this is not surprising, because there exist many W-actions on $U(\mathfrak{h})$, which recover the usual one after passing to the associated graded. Let us consider one in particular.

Definition 7.3.2. The *dot action* of W on $U(\mathfrak{h})$ is given by

$$w \cdot x = w(x) + \langle \rho, w(x) - x \rangle, \quad x \in U(\mathfrak{h}).$$

Note that the dot action is not additive.

Remark 7.3.3. The dot action is intertwined with the usual action by means of the automorphism of algebras

$$U(\mathfrak{h}) \to U(\mathfrak{h}), \qquad h \mapsto h - \langle \rho, h \rangle, \quad h \in \mathfrak{h}.$$

Geometrically, the dot action on $\mathfrak{h}^* = \operatorname{Spec} U(\mathfrak{h})$ is given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Theorem 7.3.4 (Harish-Chandra isomorphism)**.** *The map ϕ defines an isomorphism*

$$Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{h})^{W,\cdot}.$$

Proof. Let us first show that ϕ lands in $\mathsf{Sym}(\mathfrak{h})^{W,\cdot}$. Let $a \in Z(\mathfrak{g})$. By Lemma 7.2.2, it suffices to show that for any simple reflection $s_i \in W$, the action of a on M_λ and $M_{s_i\cdot\lambda}$ is given by the same scalar. It is enough to prove this for a Zariski dense subset of weights, which we take to be P^+. Our assertion then follows from the lemma below.

Lemma 7.3.5. *Let $\lambda \in P^+$, α_i a simple root, and assume that $\langle \lambda + \rho, \check{\alpha}_i \rangle \in \mathbb{Z}^+$. Then M_λ contains $M_{s_i\cdot\lambda}$ as a submodule.*

Proof. Set $n = \langle \lambda + \rho, \check{\alpha}_i \rangle$, and note that $s_i \cdot \lambda = \lambda - n\alpha_i$. The proof is a straightforward generalization of the computation in Proposition 6.4.1.

Consider the vector $f_i^n v_\lambda \in M_\lambda$. It is non-zero and has weight $\lambda - n\alpha_i$. It remains to check that $\mathfrak{n}(f_i^n v_\lambda) = 0$. Since $\mathfrak{n}$ is generated as a Lie algebra by simple coroots, it is enough to show that $e_j f_i^n v_\lambda = 0$. For $i \neq j$ we have $[e_j, f_i] = 0$, so this is obvious. For $i = j$, we conclude by the same computation as in Proposition 6.4.1.[7] □

From Definition 7.3.2, it is easy to see that the dotted action of W on $U(\mathfrak{h})$ is compatible with the PBW filtration, and hence, it induces an action on the associated graded. It is easy to see that this induced action is the usual action of W on $\mathsf{Sym}\,\mathfrak{h}$. In fact, the image of the map

$$\phi : Z(\mathfrak{g}) \to U(\mathfrak{h})^{W,\cdot}$$

under the functor of taking associated graded is simply

$$\phi_{cl} : \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{h})^W,$$

which is an isomorphism. Since taking associated-graded is conservative, this implies that ϕ itself is also an isomorphism and the proof concludes. □

8. Further properties of $\mathcal{O}$

8.1. **Decomposition of the category $\mathcal{O}$.** With a description of $Z(\mathfrak{g})$ in place, we can now break the category $\mathcal{O}$ into smaller pieces.

Lemma 8.1.1. *Let $M \in \mathcal{O}$. The action of $Z(\mathfrak{g})$ on M factors through a finite-dimensional quotient.*

Proof. Thanks to Proposition 6.6.10, it suffices to assume that $M = M_\lambda$ is a Verma module for some $\lambda \in \mathfrak{h}^*$. But in this case, the action of $Z(\mathfrak{g})$ factors through λ:

$$Z(\mathfrak{g}) \simeq \mathsf{Sym}(\mathfrak{h})^{W,\cdot} \hookrightarrow \mathsf{Sym}(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}.$$

□

Corollary 8.1.2. *Every object $M \in \mathcal{O}$ splits into a direct sum*

$$M \simeq \bigoplus_{\chi \in \mathsf{Spec}\, Z(\mathfrak{g})} M_\chi,$$

[7] one can say that we apply the proof of Proposition 6.4.1 to the $\mathfrak{sl}_2$-triple e_i, f_i, h_i

where for each χ the action of $Z(\mathfrak{g})$ on M_χ factors through $Z(\mathfrak{g})/\chi^n$ for some n. Moreover, for any $M, N \in \mathcal{O}$ we have $\mathrm{Hom}(M, N) = \bigoplus_\chi \mathrm{Hom}(M_\chi, N_\chi)$.

Thus the category $\mathcal{O}$ splits into a direct sum of *blocks* $\mathcal{O}_\chi$, parameterized by points of $\mathrm{Spec}\, Z(\mathfrak{g})$. Here, $\mathcal{O}_\chi \subset \mathcal{O}$ is the full subcategory consisting of modules, on which the center $Z(\mathfrak{g})$ acts by the generalized character χ.

Let us denote the algebraic variety $\mathrm{Spec}\left(\mathrm{Sym}(\mathfrak{h})^{W,\cdot}\right) \simeq \mathrm{Spec}\, Z(\mathfrak{g})$ by $\mathfrak{h}/\!/W$, and let $\varpi : \mathfrak{h} \to \mathfrak{h}/\!/W$ be the natural map. Recall that the $\mathbb{C}$-points of $\mathfrak{h}/\!/W$ are in bijection with $(W, \cdot)$-orbits in $\mathfrak{h}$ [8] , and the map ϖ is finite. Note that for any $\lambda \in \mathfrak{h}$ we have $M_\lambda \in \mathcal{O}_{\varpi(\lambda)}$.

Lemma 8.1.3. *If L_μ is isomorphic to a subquotient of M_λ, then $\mu = w \cdot \lambda$ for some $w \in W$.*

Proof. L_μ is a subquotient of both M_λ and M_μ, and therefore lies in $\mathcal{O}_{\varpi(\lambda)} \cap \mathcal{O}_{\varpi(\mu)}$. This implies that $\varpi(\lambda) = \varpi(\mu)$. □

Now we are ready to prove that $\mathcal{O}$ is an Artinian category.

Theorem 8.1.4. *Every object of $\mathcal{O}$ has finite length.*

Proof. By Proposition 6.6.10, it suffices to prove the claim for Verma modules M_λ. By Lemma 8.1.3, all irreducible subquotients of M_λ are of the form $L_{w\cdot\lambda}$ for $w \in W$. Hence, it is enough to show that for each μ and each filtration M_λ^i of M_λ, the number of indices such that L_μ is a subquotient of $M_\lambda^i / M_\lambda^{i-1}$ is bounded. In effect, $L_\mu(\mu) \simeq \mathbb{C}$, so that the multiplicity above cannot exceed $\dim M_\lambda(\mu)$, which is finite by Corollary 6.3.3. □

8.2. Dominant and anti-dominant weights.

Definition 8.2.1. A weight $\lambda \in \mathfrak{h}^*$ is said to be *dominant* if $w(\lambda) - \lambda \notin Q^+ \setminus \{0\}$ for any $w \in W$. λ is said to be *anti-dominant* if $-\lambda$ is dominant.

Theorem 8.2.2. *A weight λ is dominant if and only if for all $\alpha \in \Delta^+$, we have $\langle \lambda, \check{\alpha} \rangle \neq -1, -2, \dots$ I.e. $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}_{<0}$.*

Note that there are certain (non-)integrality built into the definition.

Example 8.2.3. Consider the case of $\mathfrak{sl}_2$. In this case, $Q = 2\mathbb{Z}$, $Q^+ = 2\mathbb{Z}^+$. The set of dominant weights is thus

$$\mathbb{C} \setminus \{-1, -2, \dots\} \subset \mathbb{C} \simeq \mathfrak{h}^*.$$

We will only give some indication for why Theorem 8.2.2 is true.

First, let λ be a dominant weight. We want to show that $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}_{<0}$ for any $\alpha \in \Delta^+$. Indeed, suppose there exists $\alpha \in \Delta^+$ such that $\langle \lambda, \check{\alpha} \rangle = -n$ for some $n \in \mathbb{Z}_{>0}$. Then

$$s_\alpha(\lambda) = \lambda + n\alpha$$

and hence

$$s_\alpha(\lambda) - \lambda = n\alpha \in Q^+ \setminus \{0\},$$

which contradicts the dominant condition. Thus, $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}_{<0}$.

[8] if W were a reductive group of positive dimension, this set would parameterize *closed* orbits in $\mathfrak{h}$.

The main point of Theorem 8.2.2 is thus that dominance can be checked by only looking at reflections in W. Let us give some indication for why this is true.

Suppose that λ is integral such that $\langle\lambda,\check{\alpha}\rangle \notin \mathbb{Z}_{<0}, \forall\alpha\in\Delta^+$. Then λ belongs to the dominant Weyl chamber, i.e. $\langle\lambda,\check{\alpha}\rangle \geq 0, \forall\alpha\in\Delta^+$. We will show that in this case $w(\lambda)-\lambda \notin Q^+\setminus\{0\}$.

We will prove this by contradiction. Suppose otherwise that there exists $w\in W$ such that $w(\lambda)=\lambda+\mu$ with $\mu\in Q^+$. Let $(-,-)$ denote an W-invariant inner product on $\mathfrak{h}$. Then

$$(\lambda,\lambda)=(w(\lambda),w(\lambda))=(\lambda+\mu,\lambda+\mu)=(\lambda,\lambda)+(\mu,\mu)+2(\lambda,\mu),$$

and hence

$$(\mu,\mu)+2(\lambda,\mu)=0.$$

Since $(\mu,\mu)>0$, we will derive a contradiction by showing that $(\lambda,\mu)\geq 0$. Since μ is a positive combination of positive roots, it suffices to show that $(\lambda,\alpha)\geq 0, \forall\alpha\in\Delta^+$. But

$$\langle\lambda,\check{\alpha}\rangle = 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}, \tag{Lemma 8.2.4}$$

Thus, $(\lambda,\alpha)\geq 0$ since $\langle\lambda,\check{\alpha}\rangle\geq 0$ by assumption. We thus get the desired contradiction.

Lemma 8.2.4. *Let $(-,-)$ be a W-invariant inner product on $\mathfrak{h}$. Then, for any $\alpha\in\Delta$, we have*

$$\langle\lambda,\check{\alpha}\rangle = 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}.$$

Proof. By W-invariant, we have

$$(\lambda,\alpha)=(s_\alpha(\lambda),s_\alpha(\alpha))=(\lambda-\langle\lambda,\check{\alpha}\rangle\alpha,-\alpha)=-(\lambda,\alpha)+\langle\lambda,\check{\alpha}\rangle(\alpha,\alpha),$$

from which the desired assertion is clear. □

Finally, to get a hint for what the proof of the general case of Theorem 8.2.2 looks like, we assume that $\langle\lambda,\check{\alpha}\rangle\notin\mathbb{Z}$. We claim that in this case, $w(\lambda)-\lambda\notin Q$ for any $w\in W$.

We will now study how the behavior of M_λ depends on λ.

Proposition 8.2.5. *Let $\lambda\in\mathfrak{h}^*$.*

(*i*) *Assume that $\lambda+\rho$ is anti-dominant. Then the Verma module M_λ is irreducible.*

(*ii*) *Assume that $\lambda+\rho$ is dominant. Then the Verma module M_λ is a projective object of $\mathcal{O}$.*

Proof. First, suppose that $\lambda+\rho$ is anti-dominant, i.e. for all $w\in W$, $w(\lambda+\rho)-(\lambda+\rho)\notin -(Q^+\setminus\{0\})$. We will show that M_λ is irreducible. Suppose otherwise, M_λ has at least one irreducible submodule L_μ for some $\mu\neq\lambda$. Then, we know that $\mu=w\cdot\lambda$ for some $w\in W$. In other words

$$\mu=w(\lambda+\rho)-\rho$$

an and hence

$$\mu+\rho=w(\lambda+\rho).$$

By construction $\mu \in \lambda - (Q^+ \setminus \{0\})$, and hence,

$$w(\lambda + \rho) - (\lambda + \rho) = \mu + \rho - (\lambda - \rho) \in -(Q^+ \setminus \{0\}),$$

which contradicts the fact that $\lambda + \rho$ is anti-dominant. This concludes the proof of (i).

Now, suppose that $\lambda+\rho$ is dominant, i.e. for all $w \in W$, $w(\lambda+\rho)-(\lambda+\rho) \notin Q^+ \setminus \{0\}$. We will show that M_λ is projective, i.e. we will prove that any surjection $\mathcal{M} \twoheadrightarrow M_\lambda$ splits, where $\mathcal{M} \in \mathcal{O}$. To show that this map splits, it suffices to show that there exists a vector v of weight λ inside $\mathcal{M}$ that is annihilated by $\mathfrak{n}^+$ and that has image v_λ in M_λ.

Since the action of $\mathfrak{h}$ on $\mathcal{M}$ is semi-simple, there exists a vector $v \in \mathcal{M}$ of weight λ whose image in M_λ is v_λ. We claim that v is automatically annihilated by $\mathfrak{n}^+$. By the block decomposition of $\mathcal{O}$, without loss of generality, we can assume that $Z(\mathfrak{g})$ acts on $\mathcal{M}$ by the same generalized central character as that of M_λ, i.e. $\mathcal{M} \in \mathcal{O}_{\varpi(\lambda)}$.

Suppose otherwise that $\mathfrak{n}^+$ doesn't annihilate v. Then, since $U(\mathfrak{n}^+)$ acts on $\mathcal{M}$ locally nilpotently, $U(\mathfrak{n}^+)v$ as a vector v' of weight $\mu > \lambda$ that is annilhilate by $\mathfrak{n}^+$. We thus obtain a non-zero map $M_\mu \to \mathcal{M}$. But this means that $\varpi(\mu) = \varphi(\lambda)$, i.e. $\mu = w \cdot \lambda$ for some $w \in W$ and hence, $\mu + \rho = w(\lambda + \rho)$. But then, from the above, we have $w(\lambda + \rho) = \mu + \rho \in \lambda + \rho + (Q^+ \setminus \{0\})$, which contradicts the fact that $\lambda + \rho$ is dominant. □

Example 8.2.6. Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Then $\lambda + \rho$ is anti-dominant iff

$$\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}.$$

Moreover, $\lambda + \rho$ is dominant iff

$$\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leq -2}.$$

8.3. Behavior of $\mathcal{O}_\chi$ for various χ.

8.3.1. *The case* $\chi = \varpi(-\rho)$. Observe that $-\rho + \rho = 0$ is both dominant and anti-dominant. Thus $M_{-\rho}$ is both irreducible and projective. Moreover, $W \cdot (-\rho) = -\rho$, which means that $M_{-\rho}$ is the only Verma module of $\mathcal{O}_\chi$ in this case. Thus, it's also the only irreducible module. As a result, $\mathcal{O}_{\varpi(-\rho)}$ is equivalent to the category of finite dimensional vector spaces. Note that under this equivalence, $M_{-\rho}$ corresponds to $\mathbb{C}$.

8.3.2. *The case* $\chi = \varpi(\lambda)$ *with* $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}, \forall \alpha \in \Delta^+$. Let $\mu \in W \cdot \lambda$. We will show that $\langle \mu, \check{\alpha} \rangle \notin \mathbb{Z}, \forall \alpha \in \Delta^+$. Since W is generated by reflection, it suffices to prove it for the case where $\mu = s_i \cdot \lambda$ where s_i is the (simple) reflection associated to the simple root α_i.

Recall the following lemma from the theory of root system.

Lemma 8.3.3. *Let α_i be a simple root and s_i the associated reflection. Then s_i sends α_i to $-\alpha_i$ (as usual) and permutes $\Delta^+ \setminus \{\alpha_i\}$.*

Corollary 8.3.4. *Let α_i be a simple root and s_i the associated reflection. Then*

$$s_i(\rho) = \rho - \alpha_i.$$

Equivalently, $\langle \rho, \check{\alpha}_i \rangle = 1$.

Thus,

$$\langle s_i \cdot \lambda, \check{\alpha} \rangle = \langle s_i(\lambda+\rho)-\rho, \check{\alpha}\rangle = \langle s_i(\lambda) - \alpha_i, \check{\alpha}\rangle = \langle \lambda, s_i(\check{\alpha})\rangle - \langle \alpha_i, \check{\alpha}\rangle \notin \mathbb{Z}.$$

In particular, all of these weights are distinct and moreover, $\mu+\rho$ are all dominant and anti-dominant for all $\mu \in W \cdot \lambda$. Thus, M_μ are all irreducible and projective. As a result, $\mathcal{O}_{\varpi(\lambda)}$ is equivalent to a direct sum of $|W|$ copies of the category of finite dimensional vector spaces.

8.3.5. *The case* $\chi = \varpi(\lambda)$ *where* $\langle \lambda, \check{\alpha}\rangle \in \mathbb{Z}_{\geq 0}, \forall \alpha \in \Delta^+$. In this case, $\lambda+\rho$ is a regular dominant weight, where regular means that the statbilizer of $\lambda+\rho$ in W is trivial. In particular, L_μ are all distinct for all $\mu \in W\cdot\lambda$.

Proposition 8.3.6. *Under the above circumstances, the category* $\mathcal{O}_\chi$ *is indecomposible, i.e. it cannot be non-trivially written as a direct sum of abelian subcategories.*

Proof. In general, if $\mathcal{C}$ is an abelian category such that $\mathcal{C} \simeq \mathcal{C}_1 \oplus \mathcal{C}_2$, then $\mathsf{Irr}(\mathcal{C}) = \mathsf{Irr}(\mathcal{C}_1) \sqcup \mathsf{Irr}(\mathcal{C}_2)$, where $\mathsf{Irr}(\mathcal{C})$ denotes the set of irreducible objects. Suppose $\mathcal{C}$ is Artinian, then any object admits a finite filtration with irreducible associated graded subquotients. Moreover, if $M \in \mathcal{C}$ is an indecomposable object, the irreducible subquotients belong to either $\mathsf{Irr}(\mathcal{C}_1)$ or $\mathsf{Irr}(\mathcal{C}_2)$, but not both.

Now, suppose that $\mathcal{O}_\chi \simeq \mathcal{C}_1 \oplus \mathcal{C}_2$. Note that M_λ is indecomposable since it has a unique irreducible quotient L_λ. Suppose that $L_\lambda \in \mathcal{C}_1$, then so are L_μ for any $\mu \in W\cdot\lambda$, by the remark above and Lemma 8.3.7. Thus $\mathsf{Irr}(\mathcal{C}_1) = \mathsf{Irr}(\mathcal{C})$ and hence, $\mathcal{C} \simeq \mathcal{C}_1$. □

Lemma 8.3.7. (*i*) *The object* M_λ *contains every other* M_μ *as a sub-module.*
(*ii*) *Every* M_μ *contains* $M_{w_0\lambda}$ *as a sub-module, where* w_0 *is the longest element of the Weyl group.*

Proof. See Lemma 7.3.5. Induct on length of w. □

8.4. **Contragredient duality.** Let $\mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss}}$ be the full-subcategory of $\mathfrak{g}$-mod consisting of objects, on which the action of $\mathfrak{h}$ is semi-simple. If $\mathcal{M}$ is such a module then, we write $\mathcal{M} = \bigoplus_\mu \mathcal{M}(\mu)$ its decomposition into weight spaces. We will use $\mathcal{M}^*$ to denote the linear dual of $\mathcal{M}$. It is naturally a $\mathfrak{g}$-module.

Lemma 8.4.1. (*i*) *For any* $\mathcal{M} \in \mathfrak{g}$-mod *and* $\mathfrak{l} \subseteq \mathfrak{g}$, *the maximal subspace of* $\mathcal{M}$, *on which the action of* $\mathfrak{l}$ *is locally finite, is* $\mathfrak{g}$-*stable.*
(*ii*) *For* $\mathcal{M} \in \mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss}}$, *the maximal subspace of* $\mathcal{M}^*$ *on which the action of* $\mathfrak{h}$ *is locally finite is* $\oplus_\mu M(\mu)^*$.

Proof. For the first part, it suffices to show that if $V \subseteq \mathcal{M}$ is a finite dimensional $\mathfrak{l}$-stable subspace of $\mathcal{M}$, then $\mathfrak{l}$ acts on $U(\mathfrak{g})V$ locally finitely.[9] Since $U(\mathfrak{g})_i V$ is finite dimensional for each i, it suffices to show that it is $\mathfrak{l}$-stable for each i. This is a straightforward computation using commutation relations.

The second part is left as an exercise. □

[9]Note that it is obviously $\mathfrak{l}$-stable, since $U(\mathfrak{g})V$ is even $\mathfrak{g}$-stable.

Let τ be the Cartan involution of $\mathfrak{g}$. This is a unique automorphism of $\mathfrak{g}$, which acts as -1 on $\mathfrak{h}$, and maps $\mathfrak{b}$ to $\mathfrak{b}^-$. For example, when $\mathfrak{g} = \mathfrak{sl}_n$, $\tau(M) = -M^T$.

For $\mathcal{M} \in \mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss}}$, we define $\mathcal{M}^\vee = \bigoplus_\mu \mathcal{M}(\mu)^* \in \mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss}}$ with the action of $\mathfrak{g}$ twisted by τ. Note that the $\mathfrak{h}$-weight of $\mathcal{M}(\mu)^*$ is still μ, i.e. $M^\vee(\mu) \simeq M(\mu)^*$. Moreover, it is clear that $\mathcal{M} \mapsto \mathcal{M}^\vee$ is an exact contravariant functor from $\mathfrak{g}\text{-mod}$ to itself.

Theorem 8.4.2. *If $\mathcal{M}$ belongs to $\mathcal{O}$, then so does $\mathcal{M}^\vee$.*

Consider the following full-subcategory $\mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss,fd}} \subset \mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss}}$ spanned by $\mathcal{M}$ such that $\mathcal{M}(\mu)$'s are finite dimensional. Clearly, this sub-category is preserved by $(-)^\vee$ and moreover $((-)^\vee)^\vee \simeq \mathrm{id}$ on $\mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss,fd}}$.

Since $\mathcal{O} \subset \mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss,fd}}$, the theorem above implies that $(-)^\vee$ is an auto-equivalence of $\mathcal{O}$.

The proof of Theorem 8.4.2 makes use of the following result.

Proposition 8.4.3. *For any $\lambda \in \mathfrak{h}^*$, there exists an isomorphism $L_\lambda^\vee \simeq L_\lambda$.*

Proof. Since $(-)^\vee$ is an auto-equivalence, $L_\lambda^\vee$ is irreducible. Thus, it suffices to show that there is a surjection $M_\lambda \twoheadrightarrow L_\lambda^\vee$. This is equivalent to finding a vector v_λ inside $L_\lambda^\vee$ of weight λ that is annihilated by $\mathfrak{n}$. But recall that $L_\lambda^\vee(\lambda) \simeq L_\lambda(\lambda)^* \simeq \mathbb{C}$. Let $v_\lambda \in L_\lambda^\vee(\lambda) \setminus \{0\}$. Since the weights that appear in $L_\lambda^\vee$ are the same as those appearing in L_λ, λ is also the highest weight of $L_\lambda^\vee$ and hence, v_λ must be annihilated by $\mathfrak{n}$ and we are done. □

Proof of Theorem 8.4.2. Let $\mathcal{M} \in \mathcal{O}$. Then, we've seen above that the action of $\mathfrak{h}$ on $\mathcal{M}^\vee$ is locally finite and semi-simple. It remains to show that $\mathcal{M}^\vee$ is finite generated as a $\mathfrak{g}$-module and moreover, the action of $\mathfrak{n}$ on it is locally finite. Both of these properties are preserved under extensions. Thus, we are done by the proposition above and the fact that any element in $\mathcal{O}$ is a finite filtration whose associated graded pieces are of the form L_λ. □

8.5. **Dual Verma modules.**

8.5.1. *Invariant and co-invariant.* For any Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module M, we define

$$M^{\mathfrak{g}} = \mathsf{Hom}_{\mathfrak{g}}(\mathbb{C}, M) \qquad \text{and} \qquad M_{\mathfrak{g}} = \mathbb{C} \otimes_{U(\mathfrak{g})} M \simeq M/\mathfrak{g}M$$

to be the space of $\mathfrak{g}$-invariants (resp. $\mathfrak{g}$-coinvariants) of M. Note that these are right (resp. left) adjoint to the functor

$$\mathsf{triv} : \mathsf{Vect} \longrightarrow \mathfrak{g}\text{-mod}$$

obtained by restriction of scalars along $U(\mathfrak{g}) \to \mathbb{C}$.

Question 8.5.2. Let $\mathfrak{g}$ and M be as above. Show that

$$(M_{\mathfrak{g}})^* \simeq (M^*)^{\mathfrak{g}}.$$

It is easy to see that $(V \otimes U(\mathfrak{g}))_{\mathfrak{g}} \simeq V$.

8.5.3. *Functorial description of $M_\lambda^\vee$.*

Lemma 8.5.4. *Let $\mathcal{M} \in \mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss}}$. Then the space $\mathsf{Hom}_{\mathfrak{g}\text{-mod}}(\mathcal{M}, M_\lambda^\vee)$ is canonically isomorphic to the space of functionals $\mathcal{M}_{\mathfrak{n}^-}(\lambda) \to \mathbb{C}$. In other words, this space is isomorphic to the space of functionals $\mathcal{M} \to \mathbb{C}^\lambda$ which are $\mathfrak{b}^-$-invariant.*

Proof. We have,

$$\mathsf{Hom}_{\mathfrak{g}\text{-mod}}(\mathcal{M}, M_\lambda^\vee) \simeq \mathsf{Hom}_{\mathfrak{g}}(M_\lambda, \mathcal{M}^\vee) \simeq (\mathcal{M}^\vee)^{\mathfrak{n}^+}(\lambda) \simeq \mathcal{M}_{\mathfrak{n}^-}(\lambda)^*.$$

□

Theorem 8.5.5. (*i*) *The module $M_\lambda^\vee$ has a unique irreducible sub-module.*

(*ii*) $\mathsf{Hom}_{\mathfrak{g}\text{-mod}}(M_\lambda, M_\lambda^\vee) \simeq \mathbb{C}$ *such that* $1 \in \mathbb{C}$ *corresponds to the decomposition*

$$M_\lambda \twoheadrightarrow L_\lambda \hookrightarrow M_\lambda^\vee.$$

(*iii*) *For $\lambda \neq \mu$,*

$$\mathsf{Hom}_{\mathfrak{g}\text{-mod}}(M_\lambda, M_\mu^\vee) \simeq 0.$$

(*iv*) $\mathsf{Ext}^1_{\mathcal{O}}(M_\lambda, M_\mu^\vee) \simeq 0, \forall \lambda, \mu.$

Proof. The first part follows from the fact that $(-)^\vee$ is a contravariant autoequivalence of $\mathfrak{g}\text{-mod}^{\mathfrak{h}\text{-ss},\text{fd}}$. Note that $(M_\lambda)_{\mathfrak{n}^-} \simeq M_\lambda(\lambda)$, and when $\lambda \neq \mu$, $(M_\lambda)_{\mathfrak{n}^-}(\mu) = 0$. The lemma above implies that

$$\mathsf{Hom}_{\mathfrak{g}\text{-mod}}(M_\lambda, M_\mu^\vee) \simeq ((M_\lambda)_{\mathfrak{n}^-}(\mu))^* \simeq \begin{cases} M_\lambda(\lambda)^* \simeq \mathbb{C}, & \mu = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

This finishes (iii) and the first part of (ii). In the case $\mu = \lambda$, let $M_\lambda \to M_\lambda^\vee$ denote the map correspoding to 1. This map is determined by the image of the highest weight vector v_λ, which is also the highest weight vector of $v'_\lambda \in M_\lambda^\vee$. From (i), we know that $U(\mathfrak{g})v'_\lambda = L_\lambda$, and hence, the non-zero map necessarily factors through

$$M_\lambda \twoheadrightarrow L_\lambda \hookrightarrow M_\lambda^\vee.$$

For (iv), suppose we have a short exact sequence of objects in $\mathcal{O}$

$$0 \to M_\mu^\vee \xrightarrow{i_0} \mathcal{N} \to M_\lambda \to 0.$$

We want to show that this sequence splits. By Lemma 8.5.4, the identity map $\mathrm{id}_{M_\mu^\vee}$ induces a linear functional $M_\mu^\vee \to \mathbb{C}^\mu$ that is $\mathfrak{b}^-$-invariant. Let $\mathcal{N}'$ denote its kernel. We obtain the following exact sequence of $\mathfrak{b}^-$-modules

$$0 \to \mathbb{C}^\mu \xrightarrow{i} \mathcal{N}/\mathcal{N}' \xrightarrow{p} M_\lambda \to 0. \tag{8.5.6}$$

Note that by Lemma 8.5.4, a retraction of $\mathcal{N}/\mathcal{N}'$ onto $\mathbb{C}^\mu$ is equivalent to a retraction of $\mathcal{N}$ onto $M_\mu^\vee$. It thus suffices to split (8.5.6), i.e. find a section to p. But now, such a section is given by a vector of weight λ in $\mathcal{N}/\mathcal{N}'$. This is possible since the action of $\mathfrak{h}$ on everything is semi-simple. □

We will now describe how $M_\lambda^\vee$ looks like as a vector space with $\mathfrak{n}^+$-action, parallel to the description of M_λ as $U(\mathfrak{n}^-)$ as an $\mathfrak{n}^-$-module.

Let N be the algebraic group corresponding to $\mathfrak{n}$. The category of N-representations is equivalent to the full subcategory of $\mathfrak{n}$-modules consisting of locally nilpotent representations. Let $\mathsf{Fun}(N) = \mathcal{O}_N$ be the space of regular functions on N; this is an N-representation by translation. For any $V \in \mathsf{Rep}(N)$, we have

$$\mathsf{Hom}_{N^+}(V, \mathsf{Fun}(N^+)) \simeq \mathsf{Hom}_{\mathsf{Vect}}(V, \mathbb{C}).$$

Proposition 8.5.7. *For any λ, we have an isomorphism of $\mathfrak{n}^+$-modules*

$$M_\lambda^\vee \simeq \mathsf{Fun}(N).$$

8.6. **Projective objects in $\mathcal{O}$.** Recall that a category $\mathcal{C}$ is said to have *enough projectives*, if every object of $\mathcal{C}$ admits a surjection from a projective one.

Let $\chi \in \mathfrak{h}/\!/W$, and consider the functor

$$F_{\mu,\chi} : \mathcal{O}_\chi \longrightarrow \mathsf{Vect}, \qquad \mathcal{M} \mapsto \mathcal{M}(\mu).$$

Proposition 8.6.1. *The functor $F_{\mu,\chi}$ is representable.*

Proof. The proof is similar to the proof of Proposition 8.2.5.

Let $U(\mathfrak{n})^+ \subset U(\mathfrak{n})$ be the augmentation ideal, and let K_μ be the kernel of the map $U(\mathfrak{h}) \longrightarrow \mathbb{C}$, induced by the character μ. We define

$$M_{\mu,n} = U(\mathfrak{g})/U(\mathfrak{g})I_n, \quad I_n = (U(\mathfrak{n})^+)^n K_\mu.$$

It is easy to see that $M_{\mu,n} \in \mathcal{O}$, and moreover $M_{\mu,1} = M_\mu$. For a given χ, let $M_{\mu,n,\chi}$ be the image of $M_{\mu,n}$ in $\mathcal{O}_\chi$. We claim that for n large enough, the module $M_{\mu,n,\chi}$ represents $F_{\mu,\chi}$.

Indeed, for any $\mathcal{M} \in \mathcal{O}$ the set $\mathsf{Hom}(M_{\mu,n}, \mathcal{M})$ is isomorphic to the set of elements in $\mathcal{M}(\mu)$, which are annihilated by any monomial $x_1 \dots x_n$, $x_i \in \mathfrak{n}$. We claim that this recovers the whole $\mathcal{M}(\mu)$, provided that n is big enough.

Let $\{\lambda_1, \dots, \lambda_k\}$ be the set of weights with $\varpi(\lambda_i) = \chi$, and pick a number n such that for any i we have

$$n > \langle \lambda_i - \mu, \check{\rho} \rangle.$$

Suppose that $v' = x_1 \dots x_n v \neq 0$ for some $v \in \mathcal{M}(\mu)$. Then there exists $v'' \in U(\mathfrak{n})v'$, which is annihilated by $\mathfrak{n}$. Let μ', μ'' be the weight of v', v'' respectively. We have $\mu'' - \mu' \in Q^+ \setminus 0$, and so $\langle \mu'' - \mu, \check{\rho} \rangle > \langle \mu' - \mu, \check{\rho} \rangle \geq n$. However, we have $\varpi(\mu'') = \chi$, so we have arrived at a contradiction. □

Corollary 8.6.2. *The category $\mathcal{O}$ has enough projectives.*

Proof. It is enough to prove the claim for each of $\mathcal{O}_\chi$, so let us fix χ. Since the functor $F_{\mu,\chi}$ is exact, the object $\mathcal{P}_{\mu,\chi}$ representing it is projective. For any $\mathcal{M} \in \mathcal{O}_\chi$, we have a surjection of $\mathfrak{g}$-modules $\bigoplus_\mu \mathcal{P}_{\mu,\chi} \otimes \mathcal{M}(\mu) \twoheadrightarrow \mathcal{M}$ by definition of $\mathcal{P}_{\mu,\chi}$. However, since $\mathcal{M}$ is finitely generated, this map will remain a surjection after restricting it to a certain finite subset of direct summands. □

Let P_λ be a *projective cover* of L_λ, that is a projective object together with an epimorphism $\pi_\lambda : P_\lambda \twoheadrightarrow L_\lambda$, such that no proper submodule of P_λ is mapped onto L_λ. Since $\mathcal{O}$ has enough projectives, such an object exists, and is unique up to a (non-unique) isomorphism. The definition easily implies that P_λ has a unique maximal submodule, namely $\ker \pi_\lambda$. In particular, P_λ is indecomposable, and

$$\mathsf{Hom}(P_\lambda, L_\mu) = 0, \text{ when } \mu \neq \lambda. \tag{8.6.1}$$

Corollary 8.6.3. *The category $\mathcal{O}_\chi$ is equivalent to the category of finite-dimensional modules over a finite-dimensional associative algebra.*

Proof. The direct sum $\mathcal{P} = \bigoplus_{\varpi(\lambda)=\chi} P_\lambda$ is a projective generator of $\mathcal{O}_\chi$. Thus $\mathcal{O}_\chi \simeq \mathsf{End}(\mathcal{P})\text{-}\mathsf{mod}$, and the latter algebra is finite-dimensional since $\mathcal{O}_\chi$ is Artinian. □

Recall that by Proposition 6.6.10, every object of $\mathcal{O}$ is a quotient of a successive extension of Verma modules. We say that $\mathcal{M} \in \mathcal{O}$ admits a *standard filtration*, if $\mathcal{M}$ itself is isomorphic to a successive extension of Verma modules.

Theorem 8.6.4. *Every projective object of $\mathcal{O}$ admits a standard filtration.*

Proof. Recall the modules $M_{\mu,n}$. They are not in general projective; however, any projective object in $\mathcal{O}$ can be realized as a direct summand of a direct sum of some $M_{\mu,n}$. [10]

Note that $M_{\mu,n}$ admits a standard filtration. In effect, we have a sequence of surjections

$$M_{\mu,n} \twoheadrightarrow M_{\mu,n-1} \twoheadrightarrow \cdots \twoheadrightarrow M_{\mu,1} = M_\mu,$$

whose kernels are given by $U(\mathfrak{g})(I_{k-1}/I_k)$. By definition of I_k we see that I_{k-1}/I_k is annihilated by $U(\mathfrak{n})$, and $U(\mathfrak{h})$ acts on it semisimply. In particular, $\ker(M_{\mu,k} \twoheadrightarrow M_{\mu,k-1})$ is a direct sum of Verma modules. Thus, it remains to show that if $\mathcal{M}_1 \oplus \mathcal{M}_2$ admits a standard filtration, then both $\mathcal{M}_1$ and $\mathcal{M}_2$ do as well.

Let us begin with an auxiliary assertion. Namely, let $\mathcal{M} \in \mathcal{O}$ admit a standard filtration, let $\lambda \in \mathfrak{h}^*$ be a maximal weight of $\mathcal{M}$, and pick $v \in \mathcal{M}(\lambda)$. We claim that the corresponding map $M_\lambda \to \mathcal{M}$ is injective, and the quotient $\mathcal{M}/M_\lambda$ admits a standard filtration. Indeed, let $(\mathcal{M}_i)$ be a standard filtration of $\mathcal{M}$, and let i be the minimal index for which the image of M_λ belongs to $\mathcal{M}_i$. This implies that the map $M_\lambda \to \mathcal{M}_i/\mathcal{M}_{i-1}$ is non-zero. However, $M_i/\mathcal{M}_{i-1}$ is a Verma module itself, thus the map above is necessarily an isomorphism. Hence, we get a short exact sequence

$$0 \to \mathcal{M}_{i-1} \to \mathcal{M}/M_\lambda \to \mathcal{M}/\mathcal{M}_i \to 0,$$

which implies our assertion.

Now, suppose that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ admits a standard filtration. We argue by a decreasing induction on the length of $\mathcal{M}$. Let λ be a maximal weight of $\mathcal{M}$. Without loss of generality $\mathcal{M}_1(\lambda) \neq 0$, and we have a non-zero map $M_\lambda \to \mathcal{M}_1$. By the assertion above this map is injective, and $\mathcal{M}/M_\lambda \simeq \mathcal{M}_1/M_\lambda \oplus \mathcal{M}_2$ admits a standard filtration. This completes the induction step. □

Corollary 8.6.5. *Let $\mathcal{M} \to M_\mu$ be a surjection, and $\mathcal{M}$ admits a standard filtration. The kernel of this map admits a standard filtration as well.*

[10] This can be deduced, for example, from Corollary 8.6.3

Proof. Let λ be a maximal weight of $\mathcal{M}$, and let $M_\lambda \subset \mathcal{M}$ be an embedding. If $\lambda \neq \mu$, then the composition $M_\lambda \to \mathcal{M} \to M_\mu$ is zero. Thus we have a surjection $\mathcal{M}/M_\lambda \to M_\mu$, and we argue by induction. If $\lambda = \mu$, then $\ker(\mathcal{M} \to M_\mu) = \mathcal{M}/M_\lambda$, and we are done by the assertion in the proof above. □

Corollary 8.6.6. $\mathsf{Ext}^i(M_\lambda, M_\mu^\vee) = 0$ *for all* $i > 0$ *and* λ, μ.

Proof. We argue by induction on i. By Theorem 8.5.5(iv), the assertion holds for $i = 1$. Let $\mathcal{P}$ be a projective module which surjects to M_λ, and let $\mathcal{M}$ be the kernel. We have a long exact sequence

$$\ldots \to \mathsf{Ext}^i(\mathcal{M}, M_\mu^\vee) \to \mathsf{Ext}^{i+1}(M_\lambda, M_\mu^\vee) \to \mathsf{Ext}^{i+1}(\mathcal{P}, M_\mu^\vee) \to \ldots$$

The module $\mathcal{M}$ has a standard filtration by Corollary 8.6.5. In particular, $\mathsf{Ext}^i(\mathcal{M}, M_\mu^\vee) = 0$ by the induction hypothesis. Moreover, $\mathsf{Ext}^{i+1}(\mathcal{P}, M_\mu^\vee) = 0$ since $\mathcal{P}$ is projective. Hence, $\mathsf{Ext}^{i+1}(M_\lambda, M_\mu^\vee) = 0$. □

The property of an object in $\mathcal{O}$ admitting a standard filtration can be formulated intrinsically.

Proposition 8.6.7. *Let* $\mathcal{M} \in \mathcal{O}$. *The following conditions are equivalent*:

(*i*) $\mathcal{M}$ *admits a standard filtration*;

(*ii*) $\mathsf{Ext}^i(M_\lambda, M_\mu^\vee) = 0$ *for any* $i > 0$ *and* μ;

(*iii*) $\mathsf{Ext}^1(M_\lambda, M_\mu^\vee) = 0$ *for any* μ.

Proof. Omitted. □

8.7. **BGG reciprocity.** By Theorem 8.6.4, the projective cover P_λ admits a filtration, whose subquotients are isomorphic to Verma modules. Let $\mathsf{mult}(M_\mu, P_\lambda)$ denote the number of occurences of M_μ for any such filtration. Similarly, let $[L_\lambda : \mathcal{M}]$ denote the multiplicity of L_λ in the Jordan-Hölder series of a module $\mathcal{M}$.

Theorem 8.7.1 (BGG reciprocity)**.** *We have* $\mathsf{mult}(M_\mu, P_\lambda) = [L_\lambda : M_\mu^\vee]$.

Proof. Consider the vector space $\mathsf{Hom}(P_\lambda, M_\mu^\vee)$. On one hand, its dimension is equal to $[L_\lambda : M_\mu^\vee]$ by (8.6.1). On the other hand, the lemma below implies that $\mathsf{mult}(M_\mu, P_\lambda) = \dim \mathsf{Hom}(P_\lambda, M_\mu^\vee)$. □

Lemma 8.7.2. *If* $\mathcal{M}$ *has a standard filtration, then* $\mathsf{mult}(M_\lambda, \mathcal{M}) = \dim \mathsf{Hom}(\mathcal{M}, M_\lambda^\vee)$.

Proof. The proof is by induction on the filtration length. When $\mathcal{M} = M_\mu$, this follows from Theorem 8.5.5(ii). For the induction step, consider the long exact sequence associated with a short exact sequence $0 \to \mathcal{N} \to \mathcal{M} \to M_\mu \to 0$:

$$0 \to \mathsf{Hom}(M_\mu, M_\lambda^\vee) \to \mathsf{Hom}(\mathcal{M}, M_\lambda^\vee) \to \mathsf{Hom}(\mathcal{N}, M_\lambda^\vee) \to \mathsf{Ext}^1(M_\mu, M_\lambda^\vee) \to \ldots$$

The Ext-term vanishes by Corollary 8.6.6, and $\dim \mathsf{Hom}(M_\mu, M_\lambda^\vee) = \delta_{\lambda\mu}$. Since

$$\mathsf{mult}(M_\lambda, \mathcal{M}) = \mathsf{mult}(M_\lambda, \mathcal{N}) + \delta_{\lambda\mu}$$

by the choice of μ, we can conclude by induction. □

Remark 8.7.3. Consider the Grothendieck group of the category $\mathcal{O}$. It has a basis, given by the classes of irreducibles L_λ. Another basis is given by the classes of Verma modules M_λ, and the transition matrix is triangular. BGG reciprocity then implies that the classes of projectives P_λ form yet another basis, and the transition matrix from $[P_\lambda]$ to $[M_\lambda]$ can be expressed in terms of the transition matrix from $[M_\lambda]$ to $[L_\lambda]$.

The following result, which we state without proof, gives a necessary and sufficient condition for $[L_\mu, M_\lambda]$ to be non-zero.

Theorem 8.7.4. *The following conditions are equivalent*:

(*i*) M_λ *contains* M_μ *as a submodule*;

(*ii*) M_λ *contains* L_μ *as a subquotient*;

(*iii*) *there exists a sequence of weights* $\lambda = \mu_0, \mu_1, \dots, \mu_n = \mu$, *such that* $\mu_{i+1} = s_{\beta_i} \cdot \mu_i$ *for some* $\beta \in \Delta^+$, *and* $\langle \mu_i, \check{\beta}_i \rangle \in \mathbb{Z}_{\geq 0}$.

Question 8.7.5. Deduce that the conditions of Proposition 8.2.5 are necessary and sufficient.

The precise values of the numbers $[L_\mu, M_\lambda]$ are highly non-trvial, and constitute the main object of *Kazhdan-Lusztig conjecture*.

8.8. **Translation functors.** Let v be a finite-dimensional $\mathfrak{g}$-module. We can consider the functor

$$T_V : \mathfrak{g}\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}, \qquad \mathcal{M} \mapsto \mathcal{M} \otimes V.$$

This functor is exact, preserves $\mathcal{O}$, and has an adjoint (both left and right) given by T_{V^*}.

For $\chi_1, \chi_2 \in \mathfrak{h}/\!/W$, consider the composition

$$T_{\chi_1, V, \chi_2} : \mathcal{O}_{\chi_1} \hookrightarrow \mathcal{O} \xrightarrow{T_V} \mathcal{O} \twoheadrightarrow \mathcal{O}_{\chi_2}.$$

This functor is also exact, and its adjoint is given by T_{χ_2, V^*, χ_1}.

Lemma 8.8.1. *Let* $\chi_i = \varpi(\lambda_i)$. *Then* $T_{\chi_1, V, \chi_2} = 0$ *unless there exists* $w_1, w_2 \in W$ *and* $\mu \in \mathfrak{h}^*$ *with* $V(\mu) \neq 0$, *such that* $w_1 \cdot \lambda_1 = w_2 \cdot \lambda_2 + \mu$.

Let λ be dominant, and μ a dominant integral weight. Set $\chi_1 = \varpi(\lambda)$, $\chi_2 = \varpi(\lambda + \mu)$, and consider the irreducible finite-dimensional $\mathfrak{g}$-module L_μ.

Theorem 8.8.2. *The translation functors define mutually quasi-inverse equivalences*

$$T_{\chi_1, L_\mu, \chi_2} : \mathcal{O}_{\chi_1} \rightleftarrows \mathcal{O}_{\chi_2} : T_{\chi_2, (L_\mu)^*, \chi_1}.$$

9. Localization

The theory of $\mathcal{D}$-modules is intimately related to the theory of representations of Lie algebras. More precisely, let X be a variety with an action of an algebraic group G. Then, we have a map of algebras $U(\mathfrak{g}) \to \mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$. In particular, for any $\mathcal{D}_X$-module M, $\Gamma(X, M)$ acquires the structure of a $\mathcal{D}_X$-module, and hence, also a $U(\mathfrak{g})$-module.

Localization is the other direction. Namely, if N is a $\mathfrak{g}$-module, we can induce up to obtain a $\mathcal{D}_X$-module

$$\mathcal{D}_X \otimes_{U(\mathfrak{g})} M,$$

where $p : X \to \operatorname{Spec}\mathbb{C}$ is the structure map.

The goal of this section is to explain the above in more details.

9.1. $\mathcal{D}_X$ **and Lie algebras revisited.** We start with the case of Lie algebras. Let G be an affine algebraic group[11] with identity $1 \in G$. Let $\mathcal{O}(G)$ be the Hopf algebra associated to G and $\mathfrak{m}_1$ the maximal ideal associated to 1. Then, for any positive integer n, $\mathcal{O}(G)/\mathfrak{m}_1^n$ is a finite dimensional Hopf algebra and hence, so is its dual $(\mathcal{O}_G/\mathfrak{m}_1^n)^*$. Note that $(\mathcal{O}_G/\mathfrak{m}_1^n)^*$ is co-commutative.

Proposition 9.1.1. *The map* $\mathfrak{g} = (\mathfrak{m}_1/\mathfrak{m}_1^2)^* \to (\mathcal{O}_G/\mathfrak{m}_1^2)^*$ *is a map of Lie algebras. Note that here, we view the RHS as an associative algebra, and hence, also as a Lie algebra with the Lie bracket given by commutators.*

Proof (sketch). Since any finite dimensional affine algebraic group has a finite dimensional faithful representation, we can embed $G \to \mathrm{GL}_d$ for some d. Thus, we immediately reduce to the case where $G = \mathrm{GL}_d$.

In general, $\mathfrak{g} = \operatorname{Lie} G = \ker(G(\mathbb{C}[\varepsilon]/\varepsilon^2) \to G(\mathbb{C}))$. For GL_d, an element in $\mathfrak{gl}_d = \operatorname{Lie}\mathrm{GL}_d$ has the form $1 + \epsilon M$ and Lie bracket given by the commutators. Chasing the definitions, we see that the multiplication structure used in these commutators coincide with the ring structure of $(\mathcal{O}_G/\mathfrak{m}_1^2)^*$. Thus, we are done. ◻

Let $\mathrm{Dist}(G) = \operatorname{colim}_n(\mathcal{O}_G/\mathfrak{m}_1^n)^*$, then $\mathrm{Dist}(G)$ has a natural structure of a Hopf algebra. $\mathrm{Dist}(G)$ is called the Hopf algebra of distributions on G (supported at 1).

Corollary 9.1.2. *We have a natural isomorphism of Hopf algebras*

$$U(\mathfrak{g}) \simeq \mathrm{Dist}(G).$$

Proof (sketch). The map of Lie algebras above gives a map of Lie algebras $\mathfrak{g} \to \mathrm{Dist}(G)$ and hence, a homomorphism $U(\mathfrak{g}) \to \mathrm{Dist}(G)$.

By construction $\mathrm{Dist}(G)$ has a natural filtration whose associated graded is $\operatorname{Sym}\mathfrak{g}$. $U(\mathfrak{g})$ has a PBW filtration whose associated graded is also $\operatorname{Sym}\mathfrak{g}$. It is easy to see that the map $U(\mathfrak{g}) \to \mathrm{Dist}(G)$ is compatible with these filtrations. Upon taking associated graded, the resulting map is an isomorphism. Thus we are done. ◻

We will now move to the case of the ring of differential operators $\mathcal{D}_X$ on a smooth variety X. Since we work locally, we will assume that X is affine, i.e. $X = \operatorname{Spec}\mathcal{O}(X)$. Consider the diagonal embedding $\Delta : X \to X \times X$, which realizes X as a closed subscheme of $X \times X$ given by an ideal I_Δ. Note that I_Δ is generated by elements of the form $a \otimes 1 - 1 \otimes a$ for $a \in \mathcal{O}(X)$.

The role of $\mathfrak{g}$ is given by $(I_\Delta/I_\Delta^2)^\vee$, where $(-)^\vee$ is to be understood as $\mathcal{O}(X)$-linear dual. The role of infinitesimal neighborhoods around $1 \in G$ is now played by $\mathcal{O}(X \times X)/I_\Delta^n$ as well as its dual $(\mathcal{O}(X \times X)/I_\Delta^n)^\vee$.

Proposition 9.1.3. *We have a non-degenerate* $\mathcal{O}_X$*-bilinear pairing*

$$\mathcal{D}_X^{\leq n} \times \mathcal{O}(X \times X)/I_\Delta^{n+1} \to \mathcal{O}_X$$

given by

$$A \times (f \otimes g \mod I_\Delta^{n+1}) \mapsto fAg.$$

[11] The statements proved here should hold more generally.

Proof. Exercise. □

Corollary 9.1.4. *For any closed point $x \in X$, we have*

$$\delta_x = \mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq (\mathcal{D}_X)_1 \simeq \operatorname{colim}_n (\mathcal{O}(X)/\mathfrak{m}_x^n)^*.$$

Proof. This follows directly from the fact that $\mathcal{O}(X \times X)/(I_\Delta^n, \mathfrak{m}_x) \simeq \mathcal{O}(X)/\mathfrak{m}_x^n$. □

Corollary 9.1.5. *We have*

$$U(\mathfrak{g}) \simeq \mathsf{Dist}(G) \simeq \mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

9.2. **Equivariant quasi-coherent sheaves.** Throughout, as above, we let G be an affine algebraic group and X a scheme equipped with a G-action, i.e. we have a map $G \times X \to X$.

9.2.1. *The case of $\mathcal{O}_X$.* Differentiating this map, we obtain $TG \times TX \to TX$ and hence, $\mathfrak{g} \times X \to TX$. This is the same as giving $\mathfrak{g} \to \Gamma(X, TX) = \mathsf{Vect}(X)$, the space of vector fields on X. In particular, we can associate to each element of $\mathfrak{g}$ a degree 1 differential operator of $\mathcal{O}(X)$, i.e. we get a map $\mathfrak{g} \to \mathcal{D}_X$.

To talk about algebra structures, we will take the point of view of the preceeding subsection. For each non-negative integer n, consider $G^{(n)} = \mathsf{Spec}\, \mathcal{O}(G)/\mathfrak{m}_1^n$, the n-th infinitesimal neighborhood around $1 \in G$. The action of G on X induces an action of $G^{(n)}$ on X. In particular we obtain a map of sheaves on X

$$\mathcal{O}_X \to \mathcal{O}(G)/\mathfrak{m}_1^n \otimes_{\mathbb{C}} \mathcal{O}_X.$$

Dualizing, we obtain

$$(\mathcal{O}_G/\mathfrak{m}_1^n)^* \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X,$$

and hence,

$$U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X,$$

by passing to the colimit. Thus, we obtain a map of algebras

$$U(\mathfrak{g}) \to \mathsf{End}_{\mathbb{C}}(\mathcal{O}_X).$$

Since $\mathfrak{g} \mapsto \mathsf{Vect}(X)$, we see that this map factors through an algebra homomorphism

$$U(\mathfrak{g}) \to \mathcal{D}_X \subset \mathsf{End}_{\mathbb{C}}(\mathcal{O}_X).$$

9.2.2. *The general case.* Let X and G be as above. A quasi-coherent sheaf $M \in \mathsf{QCoh}(X)$ is called G-equivariant if we are given an isomorphism

$$\phi_M : \mathrm{act}^*(M) \simeq p_2^*(M)$$

that satisfies the following conditions

(i) the restriction of ϕ_M to $1 \times X \subset G \times X$ is the identity map $M \to M$;

(ii) the following diagram of sheaves on $G \times G \times X$ commutes

$$\begin{array}{ccccc} (\mathrm{id}_G \times \mathrm{act})^* \mathrm{act}^* M & \xrightarrow{\simeq} & (\mathrm{id}_G \times \mathrm{act})^* p_2^* M & \xrightarrow{\simeq} & p_3^* M \\ \downarrow \simeq & & & & \downarrow \simeq \\ (m \times \mathrm{id}_X)^* \mathrm{act}^* M & \xrightarrow{\simeq} & (m \times \mathrm{id}_X)^* p_2^* X & \xrightarrow{\simeq} & p_3^* M \end{array}$$

Here, it is helpful to keep in mind the following commutative diagram

$$\begin{array}{ccc} G\times G\times X & \xrightarrow{\mathrm{id}_G\times \mathrm{act}} & G\times X \\ \downarrow{\scriptstyle m\times \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{act}} \\ G\times X & \xrightarrow{\mathrm{act}} & X \end{array}$$

Remark 9.2.3. The map ϕ_M captures the following idea: for any $x \in X$ and $g \in G$, we are given an isomorphism $M_{gx} \simeq M_x$. From this perspective, the second condition above then says that for any $x \in X$ and $g, h \in G$, the following diagram commutes

$$\begin{array}{ccc} M_{g(hx)} & \xrightarrow{\simeq} & M_{gx} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ M_{(gh)x} & \xrightarrow{\simeq} & M_x \end{array}$$

Question 9.2.4. Show that the category of G-equivariant sheaves on G is equivalent to the category of vector spaces.

Let M be a G-equivariant sheaf on X. We say that a global section $m \in \Gamma(X, M)$ is G-invariant if its image under

$$\Gamma(X, M) \to \Gamma(G\times X, \mathrm{act}^* M) \simeq \Gamma(G, \mathcal{O}_G) \otimes \Gamma(X, M)$$

equals $1 \otimes m$.

Question 9.2.5. Show that for $X = G$, the functor $M \mapsto M_1$ is equivalent to the functor of taking G-invariant global sections.

Given a G-equivariant quasi-coherent sheaf M on X, our goal now is to differentiate the G action. First, note that M is also a $G^{(n)} = \mathrm{Spec}\, \mathcal{O}(G)/\mathfrak{m}_1^n$-equivariant sheaf on X. WLOG, we will assume that X is affine. We have the following sequence of morphisms (by abuse of notation, we will use the same notation as those used for G here)

$$\Gamma(X, M) \to \Gamma(G^{(n)} \times X, \mathrm{act}^* M) \simeq \mathcal{O}(G)/\mathfrak{m}_1^n \otimes_{\mathbb{C}} \Gamma(X, M)$$

and hence

$$(\mathcal{O}(G)/\mathfrak{m}_1^n)^* \to \mathrm{End}_{\mathbb{C}}(M).$$

Passing to the colimit, we get a map of algebras

$$U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(M),$$

or equivalently, a map of Lie algebras

$$a_M : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(M).$$

Note that the target for a_M is $\mathrm{End}_{\mathbb{C}}(M)$ and not $\mathrm{End}_{\mathcal{O}_X}(M)$, i.e. we don't get $\mathcal{O}_X$-linear map. To see how a_M interacts with $\mathcal{O}_X$-linear structure of M, consider the following map on $\mathcal{O}_{X\times X}$

$$\mathcal{O}_X \boxtimes M \to \Delta_* M.$$

It is easy to see that both terms are G-equivariant with respect to the action of G on $X \times X$.

Functoriality of the construction above then say that this map is a map of $\mathfrak{g}$-modules. In particular, for each $f \in \mathcal{O}_X$, $m \in M$, and $\xi \in \mathfrak{g}$, we have (note that we still assume that X is affine here)

$$a_M(\xi)(fm) = a_{\mathcal{O}}(\xi)(f)m + f a_M(\xi)(m).$$

In particular,

$$a_M(\xi)(fm) - f a_M(\xi)(m) = (a_{\mathcal{O}}(\xi)(f))m$$

which means that $a_M(\xi) \in \mathcal{D}_X(M, M)$ is a twisted differential operator of degree 1. Equivalently, we have a map of algebras

$$U(\mathfrak{g}) \to \mathcal{D}_X(M, M).$$

When $M = \mathcal{O}_X$ we recover the map $U(\mathfrak{g}) \to \mathcal{D}_X$ constructed above.

9.2.6. *The case of* $\mathcal{D}_X$. We claim, without proving, that the sheaf of differential operators $\mathcal{D}_X$ itself is G-equivariant. The discussion above then implies that $\mathfrak{g}$ acts on $\mathcal{D}_X$. Moreover, for each $\xi \in \mathfrak{g}$ and for each local section D of $\mathcal{D}_X$, we have

$$a_{\mathcal{D}}(\xi)(D) = a_{\mathcal{O}}(\xi)D - Da_{\mathcal{O}}(\xi).$$

9.3. **Localization.** Let X be a smooth variety. Then, we have the following pair of adjoint functors

$$\mathcal{D}_X \otimes_{\mathcal{D}(X)} - : \mathcal{D}(X)\text{-mod} \rightleftarrows \mathcal{D}_X\text{-mod} : \Gamma(X, -).$$

Now, let X be a smooth variety equipped with an action of an affine algebraic group G. Then, we've seen above that we have a map of algebras $U(\mathfrak{g}) \to \mathcal{D}(X)$. In particular, we have a pair of adjoint functors

$$\mathcal{D}(X) \otimes_{U(\mathfrak{g})} - : \mathfrak{g}\text{-mod} \rightleftarrows \mathcal{D}(X)\text{-mod} : \mathrm{res}_{U(\mathfrak{g}) \to \mathcal{D}(X)}.$$

Combining these two pairs of adjoint functors, we obtain the following pair of adjoint functors

$$\mathrm{Loc} = \mathcal{D}_X \otimes_{U(\mathfrak{g})} - : \mathfrak{g}\text{-mod} \rightleftarrows \mathcal{D}_X\text{-mod} : \Gamma(X, -)$$

9.4. **The flag variety.** We will now specialize to the case where G is a semi-simple affine algebraic group and $X = G/B$ its flag variety. Our goal is to prove the following theorem.

Theorem 9.4.1.

(*i*) *The homomorphism* $Z(\mathfrak{g}) \to \mathcal{D}(G/B)$ *factors through the character* χ_0, *corresponding to the trivial* $\mathfrak{g}$*-module.*

(*ii*) *The resulting homomorphism* $U(\mathfrak{g})_{\chi_0} = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathbb{C}_{\chi_0} \to \mathcal{D}(G/B)$ *is an isomorphism.*

(*iii*) *The functor* $\Gamma : \mathcal{D}_{G/B}\text{-mod} \to U(\mathfrak{g})_{\chi_0}\text{-mod}$ *is exact and faithful.*

(*iv*) *The functor* Γ *and its adjoint* Loc *are mutually inverse equivalences of categories.*

From Proposition 5.3.2, we see that (ii) and (iii) imply (iv). The rest of this section will be dedicated to the proofs of (i) and (ii). The proof of part (iii) will need input from the theory of category $\mathcal{O}$.

9.5. **Fibers of localization.** Let X be a smooth variety with a transitive action of G. Let M be a $\mathfrak{g}$-module and $x \in X$. We want to understand the fiber $\mathsf{Loc}(M)_x$ of $\mathsf{Loc}(M)$ at x. Let $\mathfrak{g}_x$ denote the (infinitesimal) stabilizer of x, i.e.

$$\mathfrak{g}_x = \ker(\mathfrak{g} \to T_x X).$$

Proposition 9.5.1. *For $M \in \mathfrak{g}\text{-mod}$, we have a canonical isomorphism*

$$\mathsf{Loc}(M)_x \simeq M_{\mathfrak{g}_x},$$

where, by definition, $M_{\mathfrak{g}_x} := M \otimes_{U(\mathfrak{g}_x)} \mathbb{C}$ is the $\mathfrak{g}_x$-coinvariant part of M.

Proof. Observe that

$$\mathsf{Loc}(M)_x \simeq \mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{U(\mathfrak{g})} M \simeq \delta_x \otimes_{U(\mathfrak{g})} M,$$

which commutes with colimits in the variable M. It's also clear that the construction $M_{\mathfrak{g}_x}$ also commutes with colimits in the variable M. Thus, resolving M by free $U(\mathfrak{g})$-modules, we reduce to the case where $M = U(\mathfrak{g})$. Namely, it remains to show that we have a canonical isomorphism

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} \mathbb{C} \xrightarrow{\simeq} \Gamma(X, \delta_x).$$

We have the following commutative diagram

$$\begin{array}{ccccccc}
U(\mathfrak{g}) & \longrightarrow & \mathcal{D}(X) & \longrightarrow & \mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{D}_X & \longrightarrow & \Gamma(X, \delta_x) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g} & \longrightarrow & \mathsf{Vect}(X) & \longrightarrow & T_x X & \xrightarrow{\simeq} & T_x X \\
\uparrow & & & & & & \uparrow \\
\mathfrak{g}_x & & & \longrightarrow & & & 0
\end{array}$$

which implies that the map $U(\mathfrak{g}) \to \Gamma(X, \delta_x)$ factors through $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} \mathbb{C} \to \Gamma(X, \delta_x)$.

Both sides of this morphism have natural filtrations and the map above is compatible with the filtration. Thus, to show that this map is an isomorphism, it suffices to show that it is so after taking associated graded. But after taking associated graded, the resulting map is simply

$$\mathsf{Sym}(\mathfrak{g}/\mathfrak{g}_x) \to \mathsf{Sym}(T_x X),$$

which is an isomorphism since $\mathfrak{g}$ surjects to $T_x X$ with kernel $\mathfrak{g}_x$. □

We will need the following result from commutative algebra.

Proposition 9.5.2. *Let A be a reduced ring, which is a finitely generated k-algebra, and M a locally free A-module. Let $m \in M$ such that $\overline{m}_x \in M/\mathfrak{m}_x M$ vanishes for all closed point $x \in \operatorname{Spec} A$. Then, $m = 0$.*

Proof. By gluing, without loss of generality, we can assume that M is free, i.e. $M \simeq A^I$ for indexing set I. By projecting onto the summands, we immediately reduce to the case where $|I| = 1$, i.e. $M \simeq A$.

Now, suppose $a \in A$ such that $\overline{a}_x = 0$ for all closed point $x \in \operatorname{Spec} A$. Then, $a \in \bigcap \mathfrak{m}_x = \mathsf{Nil}(A) = 0$. Here, the second equality is a consequence of Hilbert's nullstellensatz and the third is by the assumption that A is reduced. Thus we are done. □

Proof of Theorem 9.4.1 (i). By Proposition 9.5.2, it suffices to show that for any $u \in \ker(\chi_0)$, the image of u in the fiber of $\mathcal{D}_{G/B}$ at any point $x \in G/B$ is 0. This is applicable since $\mathcal{D}_{G/B}$ a locally free sheaf on G/B.

For any $x \in G/B$, Proposition 9.5.1 says that the map $U(\mathfrak{g}) \to \Gamma(X, \delta_x)$ is identified with the quotient map $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} \mathbb{C}$. Note that $\mathfrak{g}_x = x(\mathfrak{b})x^{-1}$, so for this choice of the Borel subalgebra, $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} \mathbb{C} \simeq M_0$, the Verma module associated to the zero character. But the action of $Z(\mathfrak{g})$ on M_0 is given by the same character as on $\mathbb{C}$, which is χ_0. Thus we are done. □

9.6. **Proof of Theorem 9.4.1 (ii).** The map $U(\mathfrak{g})_{\chi_0} \to \mathcal{D}(G/B)$ is evidently compatible with filtrations on both sides. We will start by analyzing the associated graded level.

Lemma 9.6.1. *For any smooth algebraic variety X, there is a natural embedding*

$$\operatorname{gr} \mathcal{D}(X) \hookrightarrow \Gamma(X, \operatorname{Sym}_{\mathcal{O}_X} \operatorname{Vect}_X) \simeq \Gamma(T^*(X), \mathcal{O}_{T^*X}).$$

Proof. Since taking global section is left exact, the following exact sequence of sheaves

$$0 \to (\mathcal{D}_X)^{\leq n-1} \to (\mathcal{D}_X)^{\leq n} \to \operatorname{Sym}^n_{\mathcal{O}_X} \operatorname{Vect}_X \to 0$$

induces a left exact sequence

$$0 \to \mathcal{D}(X)^{\leq n-1} \to \mathcal{D}(X)^{\leq n} \to \Gamma(X, \operatorname{Sym}^n_{\mathcal{O}_X} \operatorname{Vect}_X.$$

This, in turn induces an injection $\operatorname{gr}^n \mathcal{D}(X) \to \Gamma(X, \operatorname{Sym}^n_{\mathcal{O}_X} \operatorname{Vect}_X)$. □

Recall that $\operatorname{gr}(Z(\mathfrak{g})) \simeq \operatorname{Sym}(\mathfrak{g})^G$. Let

$$\operatorname{Sym}(\mathfrak{g})^G_+ = \ker(\operatorname{Sym}(\mathfrak{g})^G \hookrightarrow \operatorname{Sym}(\mathfrak{g}) \to \mathbb{C}).$$

The map

$$\operatorname{Sym}(\mathfrak{g}) \to \Gamma(X, \operatorname{Sym}_{\mathcal{O}_X} \operatorname{Vect}_X) \tag{9.6.2}$$

induced by $\mathfrak{g} \to \operatorname{Vect}(X)$ is the associated graded form of of the map $U(\mathfrak{g}) \to \mathcal{D}(X)$ considered above.

Theorem 9.6.3 (Kostant)**.** *When $X = G/B$, the map (9.6.2) annihilates $\operatorname{Sym}(\mathfrak{g})^G_+$ and the resulting map*

$$\operatorname{Sym}(\mathfrak{g})/\operatorname{Sym}(\mathfrak{g})\operatorname{Sym}(\mathfrak{g})^G_+ \to \Gamma(G/B, \operatorname{Sym}_{\mathcal{O}_{G/B}} \operatorname{Vect}_{G/B})$$

is an isomorphism.

This is a non-trivial result, which we will prove in the next section. Now, we will see how this implies Theorem 9.4.1 (ii).

Proof of Theorem 9.4.1 (ii). It suffices to show that $\operatorname{gr}(U(\mathfrak{g})_{\chi_0}) \to \operatorname{gr}(\mathcal{D}(G/B))$ is an isomorphism. We have the following sequence of morphisms

$$\operatorname{Sym}(\mathfrak{g})/\operatorname{Sym}(\mathfrak{g})\operatorname{Sym}(\mathfrak{g})^G_+ \xrightarrow{\simeq} \operatorname{gr}(U(\mathfrak{g})_{\chi_0}) \to \operatorname{gr}(\mathcal{D}(G/B)) \hookrightarrow \Gamma(G/B, \operatorname{Sym}_{\mathcal{O}_{G/B}} \operatorname{Vect}_{G/B}).$$

The composition is an isomorphism and hence, each of the map is an isomorphism. Hence we are done by Theorem 9.6.3. □

10. Kostant's theorem

In this section, we will prove Kostant's theorem, Theorem 9.6.3.

10.1. **Chevalley map.**

Proposition 10.1.1. (*i*) *The variety $\mathfrak{h}/W$ is smooth.*
(*ii*) *The map $\varpi : \mathfrak{h} \to \mathfrak{h}/W$ is flat.*

Proof. Note that part (ii) follows from part (i) by the following result: any finite map between smooth varieties of the same dimension is flat. This is called miracle flatness.

Part (i) is a general result and holds for any finite group action where the action is generated by reflections. □

Corollary 10.1.2. *The map $\phi_{cl} : \mathfrak{g} \to \mathfrak{h}/W$ is flat.*

Proof. We need to prove that $\mathsf{Sym}\,\mathfrak{g}$ is flat over $\mathsf{Sym}(\mathfrak{g})^G = \mathsf{Sym}(\mathfrak{h})^W$. We will in fact show that $\mathsf{Sym}\,\mathfrak{g}$ is in fact flat over $\mathsf{Sym}\,\mathfrak{n} \otimes_{\mathbb{C}} (\mathsf{Sym}\,\mathfrak{g})^G$. But this is equivalent to $\mathsf{Sym}\,\mathfrak{g}/\mathfrak{n}$ being flat over $(\mathsf{Sym}\,\mathfrak{g})^G \simeq (\mathsf{Sym}\,\mathfrak{h})^W$. Now, $\mathsf{Sym}\,\mathfrak{g}/\mathfrak{n}$ is free (hence flat) over $\mathsf{Sym}\,\mathfrak{h}$. Since $\mathsf{Sym}\,\mathfrak{h} \to (\mathsf{Sym}\,\mathfrak{h})^W$ is flat, by the preivous proposition, we are done. □

Corollary 10.1.3. *$U(\mathfrak{g})$ is flat over $Z(\mathfrak{g})$.*

10.2. **Grothendieck's alteration.**

Consider the adjoint representations $\mathfrak{b}, \mathfrak{g}$ of B, G respectively. Since the inclusion $\mathfrak{b} \subset \mathfrak{g}$ is clearly B-equivariant, we have a proper map

$$\mu : \widetilde{\mathfrak{g}} := G \times_B \mathfrak{b} \to \mathfrak{g}.$$

In other words, μ sends a pair $(x, \mathfrak{b}')$, where $\mathfrak{b}' \in G/B$ is a Borel and $x \in \mathfrak{b}'$, to $x \in \mathfrak{g}$.

Definition 10.2.1. The map μ is called *Grothendieck's simultaneous resolution*, or *Grothendieck's alteration*.

Let us study the fibers of μ, that is the sets $\mu^{-1}(x) = \{\mathfrak{b}' \in G/B : \mathfrak{b}' \ni x\}$, $x \in \mathfrak{g}$.

10.2.2. *Regular semisimple.* Let $\mathfrak{g}_{rs} \subset \mathfrak{g}$ denote the locus of regular semisimple elements, and let $x \in \mathfrak{g}_{rs}$. The centralizer of x is a Cartan subalgebra $\mathfrak{h}_x \subset \mathfrak{g}$. In particular, any Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ containing x has to contain $\mathfrak{h}_x$ as well. The set of such Borels consists of $|W|$ elements, on which the Weyl group W acts transitively. In other words, the restriction $\mu^{-1}(\mathfrak{g}_{rs}) \to \mathfrak{g}_{rs}$ is an étale cover with Galois group W.

10.2.3. *Nilpotent.* Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone, and consider the preimage $\mu^{-1}(\mathcal{N})$. Since $\mathfrak{n}$ consists precisely of nilpotent elements in $\mathfrak{b}$, we have $\mu^{-1}(\mathcal{N}) \simeq G \times_B \mathfrak{n}$. On the other hand,

$$T^*_e G/B = (\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n}.$$

Since G/B is a homogeneous G-space and T^*G/B is G-equivariant, we see that

$$T^*G/B \simeq G \times_B \mathfrak{n} \simeq \mu^{-1}(\mathcal{N}).$$

The map $\widetilde{\mathcal{N}} := T^*G/B \to \mathcal{N}$ is called the *Springer resolution*. Later we will show that generically it is an isomorphism.

Remark 10.2.4. The map $\mu : T^*G/B \to \mathfrak{g}$ can be thought of as a semi-classical limit of the homomorphism $U(\mathfrak{g}) \to \mathcal{D}(G/B)$.

Since the adjoint action of B on $\mathfrak{h} \simeq \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is trivial, we have the following short exact sequence of vector bundles on G/B:

$$0 \to G \times_B \mathfrak{n} \to G \times_B \mathfrak{b} \to G/B \times \mathfrak{h} \to 0.$$

In particular, we obtain a smooth map $q : \widetilde{\mathfrak{g}} \simeq G \times_B \mathfrak{b} \to \mathfrak{h}$. Recall also that Chevalley isomorphism provides us with a map $\phi_{cl} : \mathfrak{g} \to \mathfrak{g}/G \simeq \mathfrak{h}/W$.

Lemma 10.2.5. *The following square is commutative (but not Cartesian):*

(10.2.1)
$$\begin{array}{ccc} \widetilde{\mathfrak{g}} & \xrightarrow{q} & \mathfrak{h} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \pi} \\ \mathfrak{g} & \xrightarrow{\phi_{cl}} & \mathfrak{h}/W \end{array}$$

Proof. The definition of the map ϕ_{cl} in Section 7.1 can be summarized by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{g}] & \xrightarrow{\mathsf{res}} & \mathbb{C}[\mathfrak{b}] \\ {\scriptstyle i}\uparrow & & {\scriptstyle j}\uparrow \\ \mathbb{C}[\mathfrak{g}]^G & = & \mathbb{C}[\mathfrak{h}]^W \end{array}$$

We can read it in another way. Namely, any B-invariant function f on $\mathfrak{b}$ produces a G-invariant function f_{ind} on $G \times_B \mathfrak{b} = \widetilde{\mathfrak{g}}$ by induction. In particular, if we start with a function $f \in \mathbb{C}[\mathfrak{h}]^W$, the diagram above tells us that

$$\mu^* \circ i(f) = (j(f))_{\mathsf{ind}}.$$

Note that $i = \phi_{cl}^*$, and $(\pi \circ q)^*$ coincides with $(j(-))_{\mathsf{ind}}$. This proves the statement of lemma on rings of functions; since all varieties in the square except for $\widetilde{\mathfrak{g}}$ are affine, we may conclude. □

Recall that Theorem 9.6.3 claims that

$$\mathsf{Sym}(\mathfrak{g})/\mathsf{Sym}(\mathfrak{g})\,\mathsf{Sym}(\mathfrak{g})^G_+ \to \Gamma(G/B, \mathsf{Sym}_{\mathcal{O}_{G/B}} \mathsf{Vect}_{G/B}).$$

Thanks to the diagram (10.2.1), LHS is equal to $\mathbb{C}[\phi_{cl}^{-1}(0)] = \mathcal{O}_{\mathcal{N}}$, and RHS is equal to $\mu_* \mathcal{O}_{\widetilde{\mathcal{N}}}$. We therefore need to prove that the natural map $\mathcal{O}_{\mathcal{N}} \to \mu_* \mathcal{O}_{\widetilde{\mathcal{N}}}$ is an isomorphism.

10.3. **Nilpotent orbits.** In order to proceed, we need to recall some properties of G-orbits in $\mathcal{N}$.

Question 10.3.1. Let $\xi \in \mathfrak{g}^*$, and denote the orbit of ξ under the coadjoint G-action by $\mathbb{O}_\xi$. Define a 2-form ω on $\mathbb{O}_\xi$ by

$$\omega_\eta(\hat{X}, \hat{Y}) = -\eta([X, Y]),$$

where $\hat{X}$ is the tangent vector at $\eta \in \mathbb{O}_\xi$ given by the infinitesimal action of $X \in \mathfrak{g}$. Show that ω is a symplectic form.

In particular, any coadjoint orbit in $\mathfrak{g}$ has even complex dimension.

Proposition 10.3.2. *For any coadjoint orbit* $\mathbb{O} \subset \mathfrak{g}$, *we have* $\dim(\mathbb{O} \cap \mathfrak{b}) \geq (\dim \mathbb{O})/2$.

Proof. We have the following Cartesian diagram:

$$\begin{array}{ccccc}
\mathbb{O} \cap \mathfrak{b} & \longrightarrow & 0 & \hookrightarrow & \mathfrak{b} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{O} & \xrightarrow{m} & \mathfrak{n}^* & \hookrightarrow & \mathfrak{g}^*
\end{array}$$

Here, the map m can be interpreted as the moment map for the Hamiltonian action $N \curvearrowright \mathbb{O}$. However, it is known that for solvable groups, the preimage $m^{-1}(0)$ is always a coisotropic subvariety; see [CG, Theorem 1.5.7]. This implies the desired inequality. □

Let us consider the *Steinberg variety* $\mathcal{Z} := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$.

Lemma 10.3.3. *Each irreducible component of* $\mathcal{Z}$ *has dimension* $2\dim(G/B)$.

Proof. Recall Bruhat's decomposition $G = \bigsqcup_{w \in W} BwB$. It induces a stratification of $G/B \times G/B$ into $|W|$ strata, where for any $w \in W$ the stratum Ω_w parameterizes the pairs of Borel subalgebras in relative position w. We have a natural map $\mathcal{Z} \to G/B \times G/B$; let us denote the preimage of Ω_w by $\mathcal{Z}_w$. It is clear that $\mathcal{Z}_w \to \Omega_w$ is a vector bundle with fiber $\mathfrak{n} \cap w.\mathfrak{n}$. An easy computation concludes that $\dim \mathcal{Z}_w = 2\dim G/B$. □

Theorem 10.3.4. *The nilpotent cone* $\mathcal{N}$ *has finitely many* G*-orbits.*

Proof. Given a G-orbit $\mathbb{O} \subset \mathcal{N}$, consider its preimage $\widetilde{\mathbb{O}} = G \times_B (\mathbb{O} \cap \mathfrak{b}) \subset \widetilde{\mathcal{N}}$ under μ. Its dimension is $\dim G/B + \dim(\mathbb{O} \cap \mathfrak{b})$. In particular,

$$\dim(\widetilde{\mathbb{O}} \times_{\mathbb{O}} \widetilde{\mathbb{O}}) \geq 2(\dim G/B + \dim(\mathbb{O} \cap \mathfrak{b})) - \dim \mathbb{O} \geq 2\dim G/B$$

by Proposition 10.3.2. Hence, $\widetilde{\mathbb{O}} \times_{\mathbb{O}} \widetilde{\mathbb{O}}$ is a union of irreducible components of $\mathcal{Z}$; in particular, the inequality above turns into an equality. Since $\mathcal{Z}$ has finitely many irreducible components, we conclude that $\mathcal{N}$ has finitely many G-orbits. □

Note that

$$2\dim \mathfrak{n} = \dim \widetilde{\mathcal{N}} \geq \dim \mathcal{N} \geq \dim \mathfrak{g} - \dim \mathfrak{h} = 2\dim \mathfrak{n}$$

Thus $\dim \mathcal{N} = 2\dim \mathfrak{n}$.

Corollary 10.3.5. *The Springer resolution is a resolution of singularities.*

Proof. It remains to prove that $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ is birational.

Since $\mathcal{N}$ is irreducible and a union of finitely many G-orbits, it contains a unique open dense G-orbit $\mathbb{O}$. Let $x \in \mathbb{O}$. We have

$$\dim Z(x) = \dim G - \dim \mathcal{N} = \dim \mathfrak{g} - 2\dim \mathfrak{n} = \operatorname{rk} \mathfrak{g}.$$

Thus x is regular, and all regular nilpotent elements are conjugate. Therefore it suffices to prove that $\mu^{-1}(x)$ consists of one point for one specific regular x.

Let $x = E_1 + \ldots + E_{\operatorname{rk}\mathfrak{g}}$. One can easily write down a regular semisimple element $h \in \mathfrak{h}$ with $[h, x] = x$ (exercise). This implies that each point of $\mu^{-1}(x)$ is a Borel containing x and h. Using relations in $U(\mathfrak{g})$, one can deduce that this Borel can to contain all Chevalley generators E_i, and thus has to be the standard Borel. □

10.4. **Proof of Kostant's theorem.** Let $\mathfrak{g}_r \subset \mathfrak{g}$ denote the locus of regular elements, and write $\widetilde{\mathfrak{g}}_r = \mu^{-1}(\mathfrak{g}_r)$. We have the following refinement of Lemma 10.2.5.

Proposition 10.4.1. *The following square is cartesian*:

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_r & \xrightarrow{q} & \mathfrak{h} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \pi} \\ \mathfrak{g}_r & \xrightarrow{\phi_{cl}} & \mathfrak{h}/W \end{array}$$

Proof. Since the map $\widetilde{\mathfrak{g}}_r \longrightarrow \mathfrak{g}_r \times_{\mathfrak{h}/W} \mathfrak{h}$ is proper, it suffices to show that the tangent spaces to its fibers vanish. One can check that for $(\mathfrak{b}', \xi) \in \widetilde{\mathfrak{g}}_r$, the tangent space to the fiber is given by elements $\eta \in \mathfrak{g}$ modulo $\mathfrak{b}$, such that $[\eta, \xi] \in \mathfrak{n}$.

Using an induction on $\operatorname{rk}\mathfrak{g}$, one can assume that ξ in regular nilpotent. As in Corollary 10.3.5, we can assume that $\xi = \sum_i E_i$, and $[h, \xi] = \xi$. Then the subspaces $\mathfrak{n}^-$, $\mathfrak{h}$, $\mathfrak{n}$ of $\mathfrak{g}$ are precisely the sums of negative, zero, positive eigenspaces of h respectively. This shows that if $[\eta, \xi] \in \mathfrak{n}$, then $\eta \in \mathfrak{b}$. □

Corollary 10.4.2. *The map* $\phi_{cl} : \mathfrak{g}_r \longrightarrow \mathfrak{h}/W$ *is smooth.*

Proof. q is smooth, π is flat, and smoothness is a local property. □

Let us state a couple of classical results (in very weak forms) without proofs.

Theorem 10.4.3 (Serre's criterion)**.** *Let X be a scheme. Suppose that X is Cohen-Macaulay, and* $\operatorname{codim}(X \setminus X^{\mathrm{sm}}) > 1$. *Then X is normal.*

Theorem 10.4.4 (Zariski's main theorem)**.** *Let $f : X \longrightarrow Y$ be a birational proper morphism of integral schemes, where Y is normal. Then the fibers $f^{-1}(y)$, $y \in Y$ are connected.*

Question 10.4.5. Let Y be affine. Show that under the conditions of Zariski's main theorem, one has $f_* \mathcal{O}_X = \mathcal{O}_Y$.

Proof of Theorem 9.6.3. By Corollary 10.1.2 and Proposition 10.1.1, the scheme $\mathcal{N}$ is a complete intersection, and hence Cohen-Macaulay. Corollary 10.4.2 implies that $\mathcal{N}_{\mathrm{reg}} = \mathfrak{g}_{\mathrm{reg}} \times_{\mathfrak{h}/W} 0$ is smooth, so that $\mathcal{N}$ is reduced. Moreover, by Question 10.3.1 and Theorem 10.3.4 we have $\operatorname{codim}(\mathcal{N} \setminus \mathcal{N}_{\mathrm{reg}}) \geq 2$. By Serre's criterion, the nilpotent cone $\mathcal{N}$ is normal. We conclude by applying Question 10.4.5 and Corollary 10.3.5. □

Remark 10.4.6. Passing through another proof, which relies on Borel-Weil-Bott theorem, one can show that $\mu_* \mathcal{O}_{\widetilde{\mathcal{N}}} = R\mu_* \mathcal{O}_{\widetilde{\mathcal{N}}}$.

11. Equivariant $\mathcal{D}$-modules

In order to continue with the proof of Theorem 9.4.1 (and to generalize it later on), we need to introduce another point of view on $\mathcal{D}$-modules on G/B.

11.1. **Definitions.** Let X be a variety, equipped with an action of an algebraic group B.

Definition 11.1.1. A *weakly B-equivariant* $\mathcal{D}$-module on X is a $\mathcal{D}$-module $F \in \mathcal{D}_X\text{-mod}$, equipped with a B-equivariant structure as a quasi-coherent sheaf, such that the action map

$$\mathcal{D}_Y \otimes_{\mathcal{O}_Y} F \to F$$

is B-equivariant.

It is clear that weakly equivariant $\mathcal{D}$-modules form a category.

Recall (see Section 9.2) that for any B-equivariant quasi-coherent sheaf M we have a map of Lie algebras

$$a_M : \mathfrak{b} \to \mathrm{End}_{\mathbb{C}}(M).$$

In addition, the map $U(\mathfrak{b}) \to \mathcal{D}_X$ induces another map of Lie algebras $a_{\mathcal{O}} : \mathfrak{b} \to \mathrm{End}_{\mathbb{C}}(M)$. We denote $a_\natural = a_M - a_{\mathcal{O}}$.

Lemma 11.1.2. *$a_\natural$ defines a homomorphism of Lie algebras $\mathfrak{b} \to \mathrm{End}_{\mathcal{D}_X}(M)$.*

Proof. We have

$$\begin{aligned} a_\natural(\xi)(Dm) &= a_M(\xi)(Dm) - \xi Dm \\ &= a_{\mathcal{D}_X}(\xi)(D)m + Da_M(\xi)(m) - \xi Dm \\ &= ([\xi, D] - \xi D)m + Da_M(\xi)(m) \\ &= D(a_M(\xi) - \xi)m = Da_\natural(\xi)m. \end{aligned}$$

□

Definition 11.1.3. A weakly B-equivariant $\mathcal{D}$-module M is *strongly B-equivariant* if $a_\natural = 0$.

Strongly equivariant $\mathcal{D}$-modules form a full subcategory in weakly equivariant $\mathcal{D}$-modules. We will denote this subcategory by $\mathcal{D}_X\text{-mod}^B$.

Example 11.1.4.
(1) Let $M = \mathcal{O}_X$. Then by definition $a_M = a_{\mathcal{O}}$, and so $\mathcal{O}_X$ is strongly B-equivariant.
(2) Let $M = \mathcal{D}_X$. It has a natural equivariant structure, compatible with product; thus, $\mathcal{D}_X$ is weakly B-equivariant. On the other hand,

$$\begin{aligned} a_\natural(\xi)(D) &= a_{\mathcal{D}_X}(\xi)(D) - a_{\mathcal{O}}(\xi)(D) = (a_{\mathcal{O}}(\xi)D - Da_{\mathcal{O}}(\xi)) - a_{\mathcal{O}}(\xi)D \\ &= -Da_{\mathcal{O}}(\xi), \end{aligned}$$

and so $\mathcal{D}_X$ is not strongly equivariant.

11.2. **Equivalence.** Let us specialize the discussion above to the case when $\pi : X \to Y$ is a principal B-bundle.

Proposition 11.2.1. *The pullback functor $M \mapsto \pi^*(M)$ defines an equivalence of categories $\mathcal{D}_Y\text{-mod} \xrightarrow{\sim} \mathcal{D}_X\text{-mod}^B$.*

Proof. Note that the following diagram commutes:

$$\begin{array}{ccc} B \times X & \xrightarrow{\text{act}} & X \\ \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y \end{array}$$

In particular, we have a natural isomorphism $\text{act}^*(\pi^*(M)) \simeq p_2^*(\pi^*(M))$, which equips $\pi^*(M)$ with B-equivariant structure.

Let us prove that this equivariant structure is compatible with $\mathcal{D}_X$-action. This amounts to checking that the following diagram commutes:

$$\begin{array}{ccc} \text{act}^*(\mathcal{D}_X \otimes_{\mathcal{O}_X} \pi^*(M)) & \longrightarrow & p_2^*(\mathcal{D}_X \otimes_{\mathcal{O}_X} \pi^*(M)) \\ \Big\downarrow & & \Big\downarrow \\ \text{act}^*(\pi^*(M)) & \longrightarrow & p_2^*(\pi^*(M)) \end{array}$$

Since the question is local in Y, we can assume that $X \simeq Y \times B$. In this case, we have $\pi^*(M) \simeq M \boxtimes \mathcal{O}_B$, where the first factor has trivial B-equivariant structure. Thus, we are in the situation of the first part of Example 11.1.4. In particular, $\pi^*(M)$ is strongly equivariant.

We have constructed a functor $\mathcal{D}_Y\text{-mod} \to \mathcal{D}_X\text{-mod}^B$. Let us prove that it is fully faithful. This question is again local in Y, so it suffices to show that

$$\text{Hom}_{\mathcal{D}_Y}(M_1, M_2) \simeq \text{Hom}_{\mathcal{D}_Y}(M_1, M_2) \otimes \text{Hom}_{\mathcal{D}_B\text{-mod}^B}(\mathcal{O}_B, \mathcal{O}_B),$$

which is clear.

It remains to show that π^* is essentially surjective. Since $\mathcal{D}$-modules can be glued, it suffices to show it for $X \simeq Y \times B$. Let N be a weakly B-equivariant $\mathcal{D}_X$-module. As a quasi-coherent sheaf, V is isomorphic to $M \boxtimes \mathcal{O}_B$, where $M = \Gamma(X, V)^B$; see Question 9.2.4. Since $\mathcal{D}_Y \subset (\mathcal{D}_Y \otimes \mathcal{D}_B)^B$, we see that M is naturally a $\mathcal{D}_Y$-module.

In order to prove that $N \simeq M \boxtimes \mathcal{O}_B$ as B-equivariant $\mathcal{D}$-modules, it remains to analyze the $\mathcal{D}_B$-module structure on $\mathcal{O}_B$. It suffices to compute the action of left-invariant vector fields on B, which are identified with $\xi \in \mathfrak{b}$. We have

$$a_{\mathcal{O}_B}(\xi)(f) = \xi(f) + a_\natural(\xi)(f).$$

Thus we get the correct action if and only if V is strongly B-equivariant. □

The proof above implies that on the level of $\mathcal{O}$-modules, the inverse equivalence $\mathcal{D}_X\text{-mod}^B \to \mathcal{D}_Y\text{-mod}$ is given by taking invariants $M \mapsto M^B$. Let us describe the action of $\mathcal{D}_Y$ on M^B. Consider the $\mathcal{D}_X$-module

$$\mathcal{D}_{X,\mathfrak{b}} := \mathcal{D}_X / \mathcal{D}_X \cdot \mathfrak{b}.$$

It clearly inherits weakly B-equivariant structure. Moreover, the second part of Example 11.1.4 implies that it is strongly equivariant.

Lemma 11.2.2. *We have $\mathcal{D}_{X,\mathfrak{b}} \simeq \pi^*\mathcal{D}_Y$. Moreover, we have natural isomorphisms*

$$\mathcal{D}_Y^{\text{op}} \simeq \text{End}_{\mathcal{D}_X\text{-mod}^B}(\mathcal{D}_{X,\mathfrak{b}}) \simeq (\mathcal{D}_X / \mathcal{D}_X \cdot \mathfrak{b})^B.$$

Proof. We have seen that $\pi^*\mathcal{D}_Y \simeq \mathcal{D}_X/\mathcal{D}_X \cdot T(X/Y)$, where $T(X/Y)$ is the sheaf of vertical vector fields. This implies the first isomorphism. The second one follows from Proposition 11.2.1. □

Note that at the end of the proof of Proposition 11.2.1, the failure of a weakly equivariant $\mathcal{D}$-module to be strongly equivariant is measured by

$$a_\natural : \mathfrak{b} \longrightarrow \mathsf{End}_{\mathcal{D}_B}(\mathcal{O}_B) \simeq \mathbb{C},$$

that is a character of $\mathfrak{b}$. Inspired by the lemma above, let us define for any such character λ

$$\mathcal{D}^\lambda_{X,\mathfrak{b}} := \mathcal{D}_X/\mathcal{D}_X \cdot (b - \lambda(b), b \in \mathfrak{b}), \quad \mathcal{D}^\lambda_Y \simeq \mathsf{End}_{\mathcal{D}_X\text{-mod}^{B,\lambda}}(\mathcal{D}^\lambda_{X,\mathfrak{b}})^{\mathrm{op}},$$

where $\mathcal{D}_X\text{-mod}^{B,\lambda}$ is the category of weakly B-equivariant $\mathcal{D}_X$-modules, with $a_\natural = -\lambda$. The algebra $\mathcal{D}^\lambda_Y$ will play an important role in formulating the general case of Beilinson-Bernstein localization.

11.3. **Exactness.** Recall that in order to finish the proof of Theorem 9.4.1, we need to prove that the functor

$$\Gamma : \mathcal{D}_{G/B}\text{-mod} \longrightarrow U(\mathfrak{g})_{\chi_0}\text{-mod}$$

is exact and faithful.

Proposition 11.3.1. Γ *is exact.*

Proof. For the purposes of exactness, we can replace the target of Γ by Vect. In view of Proposition 11.2.1, Γ decomposes as

$$\mathcal{D}_{G/B}\text{-mod} \xrightarrow{\pi^*} \mathcal{D}(G)\text{-mod}^B \xrightarrow{B\text{-inv}} \mathsf{Vect}.$$

For a $\mathcal{D}(G)$-module M, let us regard the space $\Gamma(G, M)$ as a $\mathfrak{g}$-module via the map $a_\mathcal{O} : \mathfrak{g} \longrightarrow \mathcal{D}(G)$. By Proposition 11.2.1, for any $M' \in \mathcal{D}_{G/B}\text{-mod}$ the action of $\mathfrak{g}$ on $\Gamma(G, \pi^*(M'))$ is such that its restriction to $\mathfrak{b}$ comes from an action of the algebraic group B. In particular, this assures that first two axioms of category $\mathcal{O}$ are satisfied for $\Gamma(G, \pi^*(M'))$.

Let us denote by $\overline{\mathcal{O}}$ the category of $\mathfrak{g}$-modules which are unions of modules in the category $\mathcal{O}$. Thus the functor Γ factors further as

$$\mathcal{D}_{G/B}\text{-mod} \xrightarrow{\pi^*} \mathcal{D}(G)\text{-mod}^B \xrightarrow{\Gamma} \overline{\mathcal{O}} \xrightarrow{\mathfrak{b}\text{-inv}} \mathsf{Vect},$$

and the first two functors are exact. Furthermore, the functor $M \mapsto M^\mathfrak{b}$ on $\overline{\mathcal{O}}$ can be also described as

$$M \mapsto \mathsf{Hom}(\mathcal{M}_0, M).$$

Since $\mathcal{M}_0$ is projective in $\mathcal{O}$ by Proposition 8.2.5(ii), the functor $\mathfrak{b}$-inv is exact. □

11.4. **Faithfulness.** As in the proof of Theorem 5.3.3, since Γ is already exact, its faithfulness is equivalent to conservativity. It thus remains to show that Γ is conservative. We will, in fact, sketch two proofs, the first one being more geometric and the second one more algebraic.

11.4.1. *An approach via* Lie*-cohomology.* First, recall that N-orbits of the flag variety $X = G/B$ are in canonical bijection with elements of the Weyl group W. For each $w \in W$, we denote by $\iota_W : X_w \to X$ the embedding of the corresponding locally closed subvariety into X. Recall that we also have the following pair of adjoint functors

$$\mathsf{triv} : D(\mathsf{Vect}) \rightleftarrows D(\mathfrak{g}\text{-}\mathsf{mod}) : (-)^{\mathfrak{g}}$$

where $(-)^{\mathfrak{g}}$ is the derived functor of taking $\mathfrak{g}$-invariants. For $M \in \mathfrak{g}\text{-}\mathsf{mod}$, it is also customary to write $\mathsf{H}^*(\mathfrak{g}, M)$ instead of $\mathsf{H}^*(M^{\mathfrak{g}})$, and call it the *Lie cohomology* of $\mathfrak{g}$ with coefficients in M. This cohomology can be computed using an explicit chain complex, called the Chevalley complex. However, we won't need this for our purposes.

Given a $\mathfrak{b}$-module M, we obviously have $M \in \mathfrak{n}\text{-}\mathsf{mod}$. Moreover, the cohomology $\mathsf{H}^*(\mathfrak{n}, M)$ acquires a natural structure of an $\mathfrak{h}$-module.

Lemma 11.4.2. (*i*) *For any $\mathcal{D}$-module $\mathcal{F}$ on X_w, there exists a canonical isomorphism*

$$\mathsf{H}^*(\mathfrak{n}, \Gamma(X, \iota_{w*,\mathrm{dR}}\mathcal{F})) \simeq \mathsf{H}^{*+\ell(w)}_{\mathrm{dR}}(X_w, \mathcal{F}).$$

(*ii*) *The $\mathfrak{h}$-action on the LHS is given by the character $-w(\rho) - \rho$.*

We will now give a proof of conservativity using this lemma. But first, we need to recall a certain construction for filtered varieties. Let X be a filtered algebraic variety, that is

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where X_k's are closed subvarieties of X. For each k, let $\iota_k : X_k \setminus X_{k-1} \to X$ and $i_k : X_k \to X$ denote the embeddings. Then any $\mathcal{F}$ in the derived category of $\mathcal{D}$-modules on X is equipped with a filtration

$$0 \to (i_0)_*(i_0)^!\mathcal{F} \to (i_1)_*(i_1)^!\mathcal{F} \to \cdots \to (i_{n-1})_*(i_{n-1})^!\mathcal{F} \to \mathcal{F}, \tag{11.4.3}$$

whose associated graded (obtained by taking the cone of each map) is given by $\bigoplus_k (\iota_k)_*(\iota_k)^!\mathcal{F}$.

Unless $\mathcal{F} = 0$, there exists a (not necessarily closed) point $x \in X$ such that $\iota_x^!\mathcal{F} \neq 0$. From now until the rest of the proof, our zoo of Borel subgroup B, $\mathfrak{b} = \mathsf{Lie}\, B$, Cartan subalgebra $\mathfrak{h}$, and the nilradical $\mathfrak{n} \subset \mathfrak{b}$ will be the one associated to x.

The flag variety X has Bruhat stratification $X = \bigsqcup_{w \in W} X_w$, and hence a filtration given by

$$\overline{X}_k = \bigsqcup_{\ell(w) \leq k} X_w; \qquad \overline{X}_k \setminus \overline{X}_{k-1} = \bigsqcup_{\ell(w)=k} X_w.$$

Applying $(-)^{\mathfrak{n}} \circ \Gamma$ to the filtration (11.4.3), we get a filtration on $\Gamma(X, \mathcal{F})^{\mathfrak{n}}$, whose associated graded pieces are given by

$$\bigoplus_{\ell(w)=k} \Gamma(X, \iota_{w*,\mathrm{dR}}\iota_w^!\mathcal{F})^{\mathfrak{n}}.$$

Thus, we obtain a spectral sequence converging to $\mathsf{H}^i(\mathfrak{n}, \Gamma(X, \mathcal{F}))$, with second page is given by

$$E_2^{jk} = \bigoplus_{\ell(w)=k} \mathsf{H}^j(\mathfrak{n}, \Gamma(X, \iota_{w*,\mathrm{dR}}\iota_w^!\mathcal{F})).$$

Now, this spectral sequence is compatible with the $\mathfrak{h}$-action. By weight considerations, we see that all differentials vanish, and we obtain a direct sum decomposition

$$\mathsf{H}^i(\mathfrak{n}, \Gamma(X, \mathcal{F})) \simeq \bigoplus_{w \in W} \mathsf{H}_{\mathrm{dR}}^{i+\ell(w)}(X_w, \iota_w^! \mathcal{F}).$$

The term corresponding to $w = 1$ is simply $\iota_x^! \mathcal{F} \neq 0$. Thus the LHS, and hence $\Gamma(X, \mathcal{F})$ is non-trivial. This concludes the proof of Theorem 9.4.1.

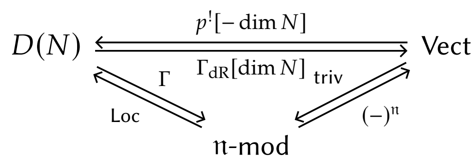
11.4.4. *A remark on Lemma 11.4.2.* We will now give some simple computations that illustrate part (i) of Lemma 11.4.2.

Consider the open cell of the flag variety. In this case, we have the following

Lemma 11.4.5. *Let $\mathcal{F}$ be a $\mathcal{D}$-module on N. Then,*

$$\mathsf{H}^*(\mathfrak{n}, \Gamma(N, \mathcal{F})) \simeq \mathsf{H}_{\mathrm{dR}}^{*+\dim N}(N, \mathcal{F}).$$

Proof. We have the following diagram of adjoint functors



where $p : N \longrightarrow \mathsf{Spec}\,\mathbb{C}$. Note that the underlying functor of $p^![-\dim N]$ at the level of quasi-coherent sheaves is just the pullback of quasi-coherent sheaves. It is easy to see that $\mathsf{Loc} \circ \mathsf{triv} \simeq p^*$. Thus, $\Gamma_{\mathrm{dR}}[\dim N] \simeq (-)^{\mathfrak{n}} \circ \Gamma$ and we are done. □

Now, let us consider the opposite case, the smallest cell $X_1 \simeq \mathrm{pt}$ of the flag variety. From the proof of Proposition 9.5.1, we see that

$$\Gamma(X, \iota_{*,\mathrm{dR}}\mathbb{C}) \simeq M_0,$$

where M_0 is the Verma module corresponding to the trivial character of $\mathfrak{b}$. The claim is thus that

$$(M_0)^{\mathfrak{n}} \simeq \mathbb{C}.$$

Let us compute explicitly why this holds for SL_2, $X = \mathsf{SL}_2/B \simeq \mathbb{P}^1$. In this case, $\iota_{*,\mathrm{dR}}\mathbb{C} = \mathbb{C}\langle t, \partial\rangle / \mathbb{C}\langle t, \partial\rangle t \simeq \mathbb{C}[\partial]$. Moreover, the generator of $\mathfrak{n} = \langle e \rangle$ acts by $t^2\partial$. A quick computation shows that this is given precisely (up to rescaling) by the derivative ∂. It is easy to check that the Lie cohomology of this object is indeed $\mathbb{C}$.

11.4.6. *An approach via block decomposition of the category $\mathcal{O}$.* Recall that Borel–Weil–Bott theorem starts with the following commutative diagram for each character λ of B:

$$\begin{array}{ccccc} G/B & \xrightarrow{q} & BB & \xrightarrow{\lambda} & B\,\mathbb{G}_{\mathrm{m}} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p'} & & \\ \mathrm{pt} & \xrightarrow{q'} & BG & & \end{array}$$

where the square is a pull-back square. The line bundle $\mathcal{L}^\lambda$ on G/B is defined by pulling back the universal line bundle $\mathcal{L}_{\mathrm{univ}}$ from $B\,\mathbb{G}_{\mathrm{m}}$.

From now on we assume that λ is anti-dominant (i.e. $-\lambda$ is dominant). In this case, we use V_λ to denote the irreducible representation of G with lowest weight λ. By Borel–Weil, we have

$$V_\lambda = \Gamma(G/B, \mathcal{L}^\lambda) = q^* p_* \lambda^* \mathcal{L}_{\text{univ}} \simeq q'^* p'_* \lambda^* \mathcal{L}_{\text{univ}}.$$

One can show that when λ is anti-dominant, $\mathcal{L}^\lambda$ is ample.

Let $\mathcal{V}_\lambda = p^* V_\lambda$. It has a natural G-equivariant structure. To see this, first note that a quasi-coherent G-equivariant sheaf on G/B is the same as a quasi-coherent sheaf on BB, or equivalently, a B-representation. Now, $\mathcal{V}_\lambda$ could be alternatively constructed as follows:

$$\mathcal{V}_\lambda = q^* p'^* p'_* \lambda^* \mathcal{L}^\lambda \simeq q^* p'^* V_\lambda.$$

In the last term, we view V_λ as a quasi-coherent sheaf on BG.

Lemma 11.4.7. (*i*) *There exists a natural G-equivariant morphism*

$$\mathcal{V}_\lambda \to \mathcal{L}^\lambda.$$

(*ii*) *The G-equivariant coherent sheaf $\mathcal{V}_\lambda$ admits a filtration whose sub-quotients are isomorphic to $\mathcal{L}^{\lambda'}$, with multiplicity given by* $\dim(V_\lambda)(\lambda')$. *Moreover, the map in the previous part is projecting to the last quotient.*

Proof. The first part follows from adjunction.

For the second part, we operate at the level of BB. As a B-representation, $p'^* V_\lambda$ is just V_λ with the B-action given by $B \to G$. Now, any finite dimensional B-representation has a finite filtration with 1-dimensional associated graded pieces essentially by Lie's theorem. We conclude by pulling this filtration back from BB to G/B. □

For any $\mathcal{D}$-module $\mathcal{F}$ on the flag variety X, we have a map of quasi-coherent sheaves

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{V}_\lambda \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^\lambda. \tag{11.4.8}$$

Lemma 11.4.9. *The map* (11.4.8) *admits a $\mathbb{C}$-linear splitting.*

Proof. The LHS admits a filtration whose associated graded is of the form $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\lambda'}$, where λ' is a weight of $\mathcal{V}_\lambda$. The center $Z(\mathfrak{g})$ acts on $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\lambda'}$ via the character $\varpi(\lambda')$. It remains to show that

$$\varpi(\lambda') \neq \varpi(w_0\lambda)$$

when $\lambda' \neq \lambda$.[12] Indeed, this means that $\lambda' = w \cdot \lambda = w(w_0\lambda + \rho) - \rho$ for some $w \in W$, or equivalently

$$w_0\lambda - w^{-1}\lambda' = w^{-1}\rho - \rho.$$

Thus, $w_0\lambda - w^{-1}\lambda' \in Q^+$. Since ρ is regular dominant, this implies that $w = 1$. In particular, $\lambda = \lambda'$ and so we are done. □

[12]Note that $w_0\lambda$ appears on the RHS since V_λ is a representation of lowest weight λ, and hence, of highest weight $w_0\lambda$.

Second proof of conservativity. Showing that $\Gamma(X, \mathcal{F})$ does not vanish is equivalent to showing that $\Gamma(X, \mathcal{F}) \otimes V_\lambda$ does not vanish for some λ. By projection formula, we have

$$\Gamma(X, \mathcal{F}) \otimes V_\lambda \simeq \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{V}_\lambda).$$

By the lemma above, the RHS surjects to $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^\lambda)$. Pick a weight λ such that $-\lambda$ is sufficiently positive. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^\lambda$ is generated by global sections, and hence $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^\lambda) \neq 0$. □

11.5. **General case of Beilinson–Bernstein.** For any character λ of $\mathfrak{h}$, we have previously defined

$$\mathcal{D}^\lambda_{G,\mathfrak{b}} = \mathcal{D}_G / \mathcal{D}_G(b - \lambda(b), b \in \mathfrak{b})$$

and

$$\mathcal{D}^\lambda_{G/B} = \mathsf{End}_{\mathcal{D}_G\text{-}\mathsf{mod}^{B,\lambda}}(\mathcal{D}^\lambda_{X,\mathfrak{b}})^{\mathrm{op}}.$$

Theorem 11.5.1 (Beilinson–Bernstein localization)**.**
(*i*) *The map $U(\mathfrak{g})_\chi \to \Gamma(X, \mathcal{D}^\lambda_X)$ is an isomorphism, where $\chi = \varpi(\lambda)$.*
(*ii*) *If $\lambda + \rho$ is dominant, then the functor $\Gamma : \mathcal{D}^\lambda_X\text{-}\mathsf{mod} \to \mathsf{Vect}$ is exact.*
(*iii*) *If λ is dominant, then Γ is conservative (and hence, faithful).*
(*iv*) *Under the assumption of (iii), Γ induces an equivalence of categories $\mathcal{D}^\lambda\text{-}\mathsf{mod} \to U(\mathfrak{g})_\chi\text{-}\mathsf{mod}$.*

In terms of techniques, essentially the same proof as for $\chi = \chi_0$ goes through.

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