EQUIVARIANT METHODS IN REPRESENTATION THEORY

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1. Introduction

Let us begin by having a glimpse at the end goal of this course. If we had to distill representation theory to one meaningless phrase, it would be something like "the study of homomorphisms from an algebra A to $\operatorname{End} V$, where V is a vector space". Adding the adjective "geometric" would then translate to ending with "where V is the vector space of invariants of some geometric object X" instead. How can one produce such homomorphisms? The easiest setting is when we take X to be a finite set, and $V = \mathsf{Fun}(X, \mathbb{C})$ to be the set of $\mathbb{C}$-valued functions on X. It is an easy exercise to see that $\operatorname{End} V \simeq \mathsf{Fun}(X \times X, \mathbb{C})$, where the product is given by convolution:

$$(f * g)(x_1, x_2) = \sum_{y \in X} f(x_1, y) g(y, x_2).$$

If we have a group Γ acting on X, then we immediately have a homomorphism

$$\Gamma \to \operatorname{End} V, \quad g \mapsto \mathbf{1}_{\mathsf{Graph}(g)},$$

where $\mathsf{Graph}(g) = \{(x, gx) : x \in X\} \subset X \times X$. However, it is a very old observation that interesting algebras (e.g. Hecke algebras) appear inside such convolution algebras, but usually don't come from symmetries of the set X.

A souped up version of this pictuUnder some geometric assumptions, one can show that a subvariety $Z \subset X \times X$ defines an element $[Z] \in \operatorname{End} H^*(X)$. Thus a very natural thing to consider, given a collection of subvarieties $Z_i \subset X \times X$, the subalgebra of $\operatorname{End} H^*(X)$ generated by all $[Z_i]$'s. We are met with a question: how to compute such things?

A helping hand comes from symmetries. It turns out that the spaces one wants to consider often come equipped with an action of some Lie group G. Therefore it makes sense to consider a cohomology theory which takes into account the G-action; this is achieved by *equivariant* cohomology. It shares many properties of singular cohomology: it is functorial (for equivariant maps), has Chern classes (for equivariant vector bundles), and fundamental classes (of G-invariant subvarieties). However, one difference is that it is highly non-trivial even for $X = \mathrm{pt}$; in general, $\Lambda_G := H^*_G(\mathrm{pt}) \neq \mathbb{C}$! In particular, the pullback map $\Lambda_G := H^*_G(\mathrm{pt}) \to H^*_G(X)$ endows G-equivariant cohomology of any X with the richer structure of a Λ_G-algebra.

We can then try to reduce our computations of convolution algebras by restricting to the fixed points X^G. This works best when $G = T$ is a torus. In nice situations, we have:

- The map $H^*_T(X) \to H^*(X)$ is *surjective*, and its kernel is generated by the kernel of $\Lambda_T \to \mathbb{C} = H^*(\mathrm{pt})$;
- The pullback $H^*_T(X) \to H^*_T(X^T)$ is *injective*, and becomes an isomorphism after inverting enough elements of Λ_T. Its image can be explicitly characterized;
- Pushforward along a proper T-equivariant map $X \to Y$ can be computed via restriction to fixed points.

All these properties can, and do, sometimes fail; for instance, the second property makes little sense when the fixed point set is empty. Nevertheless, all of them hold true in many common situations; for example, when X is a nonsingular projective variety and X^T is finite. Theorems about when these properties hold constitute the *localization package*.

Returning to convolution algebras, going from X to X^T at a first glance brings us back to the simple situation of functions on finite sets. However, the presence of Λ_G-module structure unleashes combinatorial mayhem. Symmetric polynomials come into play, diagrammatics naturally appear, and representation theory becomes infinitely richer. We will explore some of these topics in the second half of the course.

2. Equivariant cohomology

Througout this course, we will work with algebraic varieties over $\mathbb{C}$. The original definition of equivariant cohomology by Borel requires using infinite dimensional topological spaces; because of that, we will slightly modify it in order to remain firmly in the algebraic realm. The advantage is that the same construction can be used verbatim with other cohomology theories. We consider cohomology with $\mathbb{Z}$-coefficients, unless otherwise stated.

2.1. G-torsors.

Let E be a complex vector bundle of rank n on a space Y. To it, we can associate the *frame bundle* $\mathsf{Fr}(E) \to Y$, whose fiber over a point $y \in Y$ is the set of all ordered bases $(v_1, \ldots, v_n)$ of E_y.

There is a natural right GL_n-action on $\mathsf{Fr}(E)$:

$$(v_1, \ldots, v_n) \cdot g = (w_1, \ldots, w_n), \qquad w_j = \sum_i g_{ij} v_i,$$

which is transitive and free on each fiber $\mathsf{Fr}(E)_y$. Moreover, $\mathsf{Fr}(E)$ is an open of $E^{\oplus n}$, and thus a locally trivial fibration over Y.

Definition 2.1. Let $\mathbb{B}$ be a space, and G a Lie group. A *(right) G-torsor* over $\mathbb{B}$ is a map $p : \mathbb{E} \to \mathbb{B}$ with a free right G-action on $\mathbb{E}$, such that $\mathbb{B}$ is covered by opens U with G-equivariant isomorphisms $p^{-1}(U) \simeq U \times G$.

In particular, the frame bundle $\mathsf{Fr}(E) \to Y$ is a G-torsor, called the *associated principal bundle* (or torsor) to E.

Given a right G-action on Y, and a left G-action on X, we denote by $Y \times^G X$ the quotient of $Y \times X$ by the relation $(yg, x) \sim (y, gx)$. When Y is a G-torsor over B, this quotient is locally on B isomorphic to $U \times X$, and is therefore "nice" (separated etc) whenever B is.

Example 2.2. Consider the natural action of GL_n on $\mathbb{C}^n$. We have an isomorphism

$$\mathsf{Fr}(E) \times^{GL_n} \mathbb{C}^n \xrightarrow{\sim} E, \qquad (v_1, \ldots, v_n) \times (z_1, \ldots, z_n) \mapsto \sum_i z_i v_i.$$

In a similar way, we have

$$\mathsf{Fr}(E) \times^{GL_n} (\mathbb{C}^n)^\vee \simeq E^\vee, \quad \mathsf{Fr}(E) \times^{GL_n} \wedge^d \mathbb{C}^n \simeq \wedge^d E, \quad \mathsf{Fr}(E) \times^{GL_n} \mathsf{Sym}^d\, \mathbb{C}^n \simeq \mathsf{Sym}^d\, E.$$

Exercise 2.3. Let $d \le n$, and consider the fiber bundle $\mathsf{Fr}(d, E) \to Y$, with

$$\mathsf{Fr}(d, E)_y = \left\{ (v_1, \ldots, v_d) \,:\, v_i\text{'s are linearly independent in } E_y \right\}.$$

Show that $\mathsf{Fr}(d, E) \times^{GL_d} \mathbb{C}^d$ is naturally identified with the tautological rank d bundle S on $\mathsf{Gr}(d, E) = \mathsf{Fr}(E) \times^{GL_n} \mathsf{Gr}(d, \mathbb{C}^n)$.

2.2. Borel construction.

A naive definition of equivariant cohomology would be simply $H^*(X/G)$. This has two immediate issues:

- It is not homotopy invariant. For example, compare $\mathrm{pt}/\mathbb{Z}$ with $\mathbb{R}/\mathbb{Z}$;
- The quotient X/G is typically very nasty; e.g. the quotient of $\mathbb{C}^2 \setminus \{0\}$ by $\mathbb{C}^*$, $t \cdot (x, y) = (tx, t^{-1} y)$ is not separated.

Both or these issues can be resolved by picking a G-torsor $\mathbb{E} \to \mathbb{B}$ with $\mathbb{E}$ contractible, and replacing X/G by $\mathbb{E} \times^G X$. The issue is that one cannot typically choose $\mathbb{E}$ to be algebraic, the classic example being $\mathbb{C}^\infty \setminus \{0\} \to \mathbb{P}^\infty$ for $G = \mathbb{C}^*$.

Exercise 2.4. Check that $\mathbb{C}^\infty \setminus \{0\}$ is contractible.

We sidestep this by approximating $\mathbb{E}$ by a sequence of G-torsors $\mathbb{E}_N \to \mathbb{B}_N$, such that $\mathbb{E}_N$ is path-connected and $H^i(\mathbb{E}_N) = 0$ for $0 < i < N$.

Definition 2.5. Let G be a Lie group, and X a G-variety. We define

$$H^i_G(X) := H^i(\mathbb{E}_N \times^G X) \qquad \text{for } i < N.$$

Of course, in order for this to make sense, we need to construct $\mathbb{E}_N$'s, and show that the definition is independent of choices. Existence is taken care of by the following lemma, which we will prove in Section 2.4:

Lemma 2.6. *Let G be a complex linear algebraic group, and $N > 0$. We have a G-torsor $\mathbb{E} \to \mathbb{B}$ on a smooth algebraic variety $\mathbb{B}$, such that $\mathbb{E}_N$ is path-connected and $H^i(\mathbb{E}_N) = 0$ for $0 < i < N$.*

Onto the independence from choices:

Lemma 2.7. *If $\mathbb{E} \to \mathbb{B}$, $\mathbb{E}' \to \mathbb{B}'$ are two path-connected G-torsors with $H^i(\mathbb{E}) = H^i(\mathbb{E}') = 0$ for $0 < i < N$, then there are canonical isomorphisms*

$$H^i(\mathbb{E} \times^G X) \simeq H^i(\mathbb{E}' \times^G X)$$

for all $i < N$, compatible with cup product in this range.

Proof. Consider the product $\mathbb{E} \times \mathbb{E}'$ with the diagonal G-action. We have a commuting diagram

$$\begin{array}{ccccc}
\mathbb{E} \times X & \longleftarrow & \mathbb{E} \times \mathbb{E}' \times X & \longrightarrow & \mathbb{E} \times X \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{E} \times^G X & \longleftarrow & (\mathbb{E} \times \mathbb{E}') \times^G X & \longrightarrow & \mathbb{E}' \times^G X
\end{array}$$

The horizontal maps are locally trivial fibrations, with fibers $\mathbb{E}$ and $\mathbb{E}'$. Recall Leray spectral sequence:

$$H^p(X, H^q(F)) \Rightarrow H^{p+q}(Y) \text{ for a fibration } Y \to X \text{ with fiber } F.$$

As easy consequence is that when F is path-connected and $H^i(F) = 0$ for $0 < i < N$, we have $H^i(X) \xrightarrow{\sim} H^i(Y)$, with the map being pullback. We obtain

$$H^i(\mathbb{E} \times^G X) \xrightarrow{\sim} H^i((\mathbb{E} \times \mathbb{E}') \times^G X) \xleftarrow{\sim} H^i(\mathbb{E}' \times^G X)$$

for $i < N$, and these maps are compatible with cup product because pullbacks are. □

Any G-equivariant map $f : X \to Y$ determines a map $\mathbb{E} \times^G X \to \mathbb{E} \times^G Y$, so we get pullbacks

$$f^* : H^i_G(Y) \to H^i_G(X).$$

In particular, the pullback along $X \to \mathrm{pt}$ defines a ring homomorphism

$$\Lambda_G := H^*_G(\mathrm{pt}) \to H^*_G(X),$$

which makes $H^*_G(X)$ into a graded-commutative Λ_G-algebra.

Exercise 2.8. Check that the isomorphisms from Lemma 2.7 are functorial, i.e. commute with pullbacks.

Summarizing, we have constructed a functor

$$H_G^* : G\text{-spaces} \to \Lambda_G\text{-algebras}.$$

Let us consider the simple, and most important for us, example of $G = \mathbb{C}^*$. Take $\mathbb{E}_N = \mathbb{C}^N \setminus \{0\}$, with $\mathbb{C}^*$-action by scaling. This action is free, and the quotient is $\mathbb{B}_N = \mathbb{P}^{N-1}$. Since $\mathbb{E}_N$ is homotopic to $\mathbb{S}^{2N-1}$, and $H^i(\mathbb{S}^{2N-1}) = 0$ unless $i = 0, 2N-1$, for any $\mathbb{C}^*$-space X we have

$$H^i_{\mathbb{C}^*}(X) = H^i((\mathbb{C}^N \setminus \{0\}) \times^{\mathbb{C}^*} X) \quad \text{for } i < 2N-1.$$

In particular, $H^i_{\mathbb{C}^*}(\mathrm{pt}) = H^i(\mathbb{P}^{N-1})$ for $i < 2N-1$. Since we have a ring isomorphism $H^*(\mathbb{P}^{N-1}) \simeq \mathbb{Z}[t]/(t^N)$, $\deg t = 2$ for all N, this shows us that

$$\Lambda_{\mathbb{C}^*} \simeq \mathbb{Z}[t].$$

Similarly, for an algebraic torus $G = T = (\mathbb{C}^*)^m$, taking products of everything above we get $\Lambda_T \simeq \mathbb{Z}[t_1, \dots, t_m]$.

2.3. **Chern classes and fundamental classes.** Let G be an algebraic group, and X a G-space. A *G-equivariant vector bundle* on X is a vector bundle $E \to X$ with a G-action making the projection equivariant, such that for any $g \in G$, $x \in X$ the induced maps $g : E_x \to E_{gx}$ are linear. An equivariant vector bundle gives rise to an ordinary vector bundle $\mathbb{E} \times^G E \to \mathbb{E} \times^G X$. Choosing $\mathbb{E}$ appropriately, we can then define the equivariant Chern classes of E:

$$c_k^G(E) := c_k(\mathbb{E} \times^G X) \text{ in } H_G^{2k}(X) = H^{2k}(\mathbb{E} \times^G X).$$

Assume X is smooth. Then similarly, a G-invariant subvariety $V \subset X$ of codimension d gives rise to a subvariety $\mathbb{E} \times^G V \subset \mathbb{E} \times^G X$ of codimension d. We define the equivariant fundamental class by

$$[V]^G := [\mathbb{E} \times^G V] \text{ in } H_G^{2d}(X) = H^{2d}(\mathbb{E} \times^G X).$$

In the future, we will often drop the superscripts G.

Exercise 2.9. Show that these definitions are independent of $\mathbb{E}$.

Let us summarize some useful properties of Chern classes and fundamental classes; they are proved exactly in the way you can guess they are.

- Additivity: $c_1(L \otimes M) = c_1(L) \oplus c_1(M)$ for line bundles L, M;
- Whitney formula: $c(E) = c(E')c(E'')$ for an exact sequence $0 \to E' \to E \to E'' \to 0$;
- Let $E \to X$ be a vector bundle of rank r, $s : X \to E$ a G-equivariant section, and consider the zero locus $Z(s) \subset X$. If $\operatorname{codim} Z(s) = r$, then $[Z(s)] = c_r(E)$;
- Let $V, W \subset X$ be two invariant subvarieties with proper intersection. If $V \cdot W = \sum m_i Z_i$ as cycles, then all Z_i's are invariant as long as G is connected. Then $[V][W] = \sum m_i [Z_i]$ in $H_G^*(X)$. In particular, if the intersection is empty, then $[V][W] = 0$.

Exercise 2.10. Let $\mathbb{C}^*$ act on $\mathbb{C}$ in a standard way, and let $o \subset \mathbb{C}$ be the origin. Check that $[o]^2 \neq 0$ in $H^*_{\mathbb{C}^*}(\mathbb{C})$.

Let us now look at some examples, beginning with $X = \mathrm{pt}$. In this case a G-equivariant vector bundle is nothing else than a representation of G, and so each representation V has Chern classes $c_i(V) \in \Lambda_G^{2i}$.

Example 2.11. Let $G = \mathbb{C}^*$, and consider the 1-dimensional representations $\mathbb{C}_a$, $a \in \mathbb{Z}$. We have the isomorphisms

$$\begin{array}{ccc} (\mathbb{C}^* \setminus 0) \times^{\mathbb{C}^*} \mathbb{C}_1 & \xrightarrow{\sim} & \mathcal{O}(-1) \\ \downarrow & & \downarrow \\ (\mathbb{C}^* \setminus 0) \times^{\mathbb{C}^*} \mathrm{pt} & \xrightarrow{\sim} & \mathbb{P}^{N-1} \end{array}$$

and so, taking $t = c_1(\mathbb{C}_1)$, we see that $\Lambda_{\mathbb{C}^*}$ is generated by the Chern class of the standard representation. This gives us a canonical choice of a generator for $\Lambda_{\mathbb{C}^*}$. Note that we have $c_1(\mathbb{C}_a) = at$ by additivity.

Example 2.12. Similarly, let $T = (\mathbb{C}^*)^n$ act on $V = \mathbb{C}^n$ by scaling coordinates. For $1 \leq i \leq n$, we have a 1-dimensional representation $\mathbb{C}_{t_i}$ of T, which only remembers the i-th component of T. Let us denote $t_i = c_1(\mathbb{C}_{t_i})$; then we have $\Lambda_T = \mathbb{Z}[t_1, \dots, t_n]$. By Whitney formula,

$$c_i(V) = e_i(t_1, \dots, t_n),$$

where e_i is the i-th elementary symmetric polynomial.

Now let V be a representation of G of dimension n. Then G acts on $\mathbb{P}(V)$, the tautological subbundle $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$. Let $\zeta = c_1(\mathcal{O}(1)) \in H^2_G(\mathbb{P}(V))$.

Proposition 2.13. *We have a ring isomorphism*

$$H^*_G(\mathbb{P}(V)) = \Lambda_G[\zeta]/(\zeta^n + c_1\zeta^{n-1} + \dots + c_n),$$

where $c_i = c_i(V) \in \Lambda_G$ are the Chern classes.

Proof. Note that $\mathbb{E} \times^G \mathbb{P}(V)$ can be identified with the projective bundle $\mathbb{P}(\mathbb{E} \times^G V)$, in such a way that $\mathcal{O}(1)$ goes to $\mathcal{O}(1)$. Then the claim results from a general formula for cohomology of projective bundle in terms of cohomology of the base. □

For example, let $T = (\mathbb{C}^*)^n$ act on $V = \mathbb{C}^n$ in the standard way. Then

$$H^*_T(\mathbb{P}(V)) = \mathbb{Z}[t_1, \dots, t_n, \zeta]/\prod_i(\zeta + t_i).$$

Similarly, let $G = GL(V)$ act on V. We will see in the next section that $\Lambda_G = \mathbb{Z}[c_1, \dots, c_n]$, and so we get

$$H^*_G(\mathbb{P}(V)) = \mathbb{Z}[c_1, \dots, c_n, \zeta]/(\zeta^n + c_1\zeta^{n-1} + \dots + c_n).$$

2.4. **The general linear group.** Let $G = GL(V)$ act on V, $\dim V = n$, and consider the Chern classes $c_i = c_i(V) \in \Lambda_G^{2i}$.

Proposition 2.14. *We have $\Lambda_G = \mathbb{Z}[c_1, \dots, c_n]$.*

Proof. We begin by constructing an explicit collection of approximating varieties $\mathbb{E}_N$ for $GL(V)$; by restriction, this will also show their existence for any linear group. Let $N > n$, and consider $\mathbb{E}_N := \mathsf{Emb}(V, \mathbb{C}^N)$, the space of embeddings $V \hookrightarrow \mathbb{C}^N$. It is clearly open in the vector space of all linear maps $\mathsf{Hom}(V, \mathbb{C}^N)$, with the complement being the locus Z_{n-1} of all maps with non-trivial kernel. Consider the vector bundle $K \to \mathbb{P}(V)$, whose fiber at a line $L \subset V$ is the space of maps $A \in \mathsf{Hom}(V, \mathbb{C}^N)$ such that $L \subset \operatorname{Ker} A$. We have a surjective map

$$K \to Z_{n-1}, \qquad (L, A) \mapsto A,$$

which is one-to-one on the generic locus of Z_{n-1} of maps of rank $n-1$. Therefore Z_{n-1} is irreducible, and $\dim_{\mathbb{C}} Z_{n-1} = (n-1) + (n-1)N = nN - (N-n+1)$.

Lemma 2.15. *We have $H^i(\mathbb{E}_N) = 0$ for $0 < i \leq 2(N-n)$.*

Proof. From the long exact sequence in cohomology, we have

$$H^i(\mathbb{E}_N) = H^{i+1}(\mathsf{Hom}(V, \mathbb{C}^N), \mathbb{E}_N).$$

Recall that Z_{n-1} has real codimension $2(N-n+1)$. When $i \leq 2(N-n)$, we have $i+1 < 2(N-n+1)$, so that the relative cohomology group above vanishes. □

Observe that $\mathbb{B}_N = \mathbb{E}_N / GL_n = \mathsf{Gr}(n, \mathbb{C}^N)$. Moreover, the map $\mathbb{E}_N \to \mathbb{B}_N$ is the frame bundle $\mathsf{Fr}(S)$ associated to the tautological bundle S on $\mathsf{Gr}(n, \mathbb{C}^N)$. Therefore the vector bundle $\mathbb{E}_N \times^{GL(V)} V$ identifies with S by Exercise 2.3, and so the Chern classes $c_i(V)$ identify with Chern classes $c_i(S)$.

Exercise 2.16. Let S be the tautological rank n vector bundle on $\mathsf{Gr}(n, \mathbb{C}^N)$, and let Q be the quotient bundle $\mathbb{C}^N / S$. Then

$$H^*(\mathsf{Gr}(n, \mathbb{C}^N)) = \mathbb{Z}[c_1(S), \ldots, c_n(S), c_1(Q), \ldots, c_{N-n}(Q)] / (c(S)c(Q) = 1).$$

Hint: use the variety of partial flags $\{\mathbb{C}^n \subset \mathbb{C}^{n+1} \subset \mathbb{C}^N\}$, and proceed by induction on n.

Unraveling the relations, we have one relation in each (even) degree. However, the first $N-n$ relations simply express $c_i(Q)$'s in terms of $c_i(S)$'s. Thus we can remove these relations together with $c_i(Q)$'s, and the remaining relations between $c_i(S)$'s have degree at least $2(N-n+1)$. Choosing N big enough, we see that each Λ_G^{2i} is freely generated by $c_i(S)$'s. □

Note that for $T = (\mathbb{C}^*)^n$, we now have two choices of approximating spaces. On one hand, we have $\mathbb{E}_N = (\mathbb{C}^N \setminus 0)^n$, which is identified with $N \times n$ matrices with non-zero columns. On the other hand, we can embed $T \subset GL_n$, and consider $\mathbb{E}'_N = \mathsf{Emb}(V, \mathbb{C}^N)$, which is identified with $N \times n$ matrices with linearly independent columns. We have a commutative diagram

$$\begin{tikzcd} \mathbb{E}'_N \arrow[r, hook] \arrow[d] & \mathbb{E}_N \arrow[d] \\ \mathsf{Gr}^{\mathrm{split}}(n, \mathbb{C}^N) \arrow[r] & (\mathbb{P}^{N-1})^n \end{tikzcd}$$

where $\mathsf{Gr}^{\mathrm{split}}(n, \mathbb{C}^N)$ parameterizes subspaces $V \subset \mathbb{C}^N$ together with a decomposition into lines $V = L_1 \oplus \ldots \oplus L_n$. We have an obvious forgetful map $\pi : \mathsf{Gr}^{\mathrm{split}}(n, \mathbb{C}^N) \to \mathsf{Gr}(n, \mathbb{C}^N)$.

Exercise 2.17. The pullback π^* induces the inclusion

$$\Lambda_{GL_n} = \mathbb{Z}[c_1, \ldots, c_n] \to \mathbb{Z}[t_1, \ldots, t_n] = \Lambda_T,$$

defined by $c_i \mapsto e_i(t_1, \dots, t_n)$.

Remark 2.18. The map π can be understood as the "universal map for the splitting principle". Namely, given a vector bundle $E \to X$ of rank n and $d \le n$, we can consider

$$\mathsf{Gr}^{\text{split}}(d, E) = \mathsf{Fr}(E) \times^{GL_n} \mathsf{Gr}^{\text{split}}(d, \mathbb{C}^n).$$

When $d = n$, the pullback of E along $X' := \mathsf{Gr}^{\text{split}}(n, E) \to X$ splits into line bundles, and embeds $H^*(X)$ into $H^*(X')$.

Remark 2.19. Let G be a reductive group. We always have a map $\Lambda_G \to \Lambda_T^W$; however, it is neither surjective nor injective in general. For example, Λ_{PGL_n} is not completely known! Nevertheless, we will see that this map is an isomorphism over $\mathbb{Q}$, and return to its description over $\mathbb{Z}$ later in the course.

Example 2.20. We have $\Lambda_{SL_n} = \mathbb{Z}[c_2, \dots, c_n]$, and $\Lambda_{Sp_{2n}} = \mathbb{Z}[c_2, c_4, \dots, c_{2n}]$. Λ_{SO_n} is typically *not* generated by the Chern classes of the standard representation.

2.5. **Changing G.** We have already shown that a G-equivariant map $f : X \to X'$ induces a pullback $f^* : H_G^*(X') \to H_G^*(X)$. Let us extends this slightly by allowing G to vary. Namely, let $\varphi : G \to G'$ be a group homomorphism, and suppose G acts on X, G' acts on X'.

Definition 2.21. A map $f : X \to X'$ is *equvariant with respect to φ* if

$$f(gx) = \varphi(g)f(x) \qquad \text{for all } g \in G,\, x \in X.$$

Given such a map, we define a pullback $f^* : H_{G'}^*(X') \to H_G^*(X)$. More precisely, let $\mathbb{E} \to \mathbb{B}$, $\mathbb{E}' \to \mathbb{B}'$ be approximation bundles with $H^i(\mathbb{E}) = H^i(\mathbb{E}') = 0$ for $0 < i < N$. Then G acts on $\mathbb{E} \times \mathbb{E}'$ diagonally: $(e, e')g = (eg, e'\varphi(g))$. The projection to $\mathbb{E}'$ is equivariant, and so we have maps

$$(\mathbb{E} \times \mathbb{E}') \times^G X \to (\mathbb{E} \times \mathbb{E}') \times^G X' \to \mathbb{E}' \times^{G'} X'.$$

The action of G on $\mathbb{E} \times \mathbb{E}'$ is free, $H^i(\mathbb{E} \times \mathbb{E}') = 0$ for $0 < i < N$, and so $H_G^*(X) = H^*((\mathbb{E} \times \mathbb{E}') \times^G X)$ for $i < N$. Taking pullbacks, we get the desired map $f^* : H_{G'}^*(X') \to H_G^*(X)$.

Exercise 2.22. Check that when $G = G'$, $\varphi = \mathrm{id}$ this map agrees with our old pullback. Check that this pullback is functorial.

If $E' \to X'$ is a G'-equivariant vector bundle, its pullback f^*E' is a G-equivariant vector bundle on X, and we have $f^*(c_k(E')) = c_k(E)$.

Example 2.23. Let $X = X' = \mathrm{pt}$, and $\varphi : G \to G'$. Pullback gives us a homomorphism $\Lambda_{G'} \to \Lambda_G$. In particular, for $G = T \subset GL_n = G'$ we recover the map from Exercise 2.17.

Example 2.24. Let G act freely on X, $X' := G\backslash X$. The quotient map is equivariant with respect to $\varphi : G \to \{e\}$, and the pullback $H_G^i(X) \to H^i(G\backslash X)$ is an isomorphism. Indeed, the pullback along $\mathbb{E} \times^G X \to G\backslash X$ is a locally trivial fibration with fiber $\mathbb{E}$. Choosing $\mathbb{E}$ acyclic enough, Leray spectral sequence shows this is an isomorphism on H^i.

Example 2.25. Let G act trivially on X. If Λ_G is free over $\mathbb{Z}$, we have $H_G^*(X) = \Lambda_G \otimes H^*(X)$. Indeed, we have $\mathbb{E} \times^G X \simeq \mathbb{B} \times X$, and so $H_G^*(X) \simeq H^*(B \times X)$. We conclude by Künneth isomorphism, which holds under our assumptions. In particular, $\Lambda_G \otimes \Lambda_{G'} \simeq \Lambda_{G \times G'}$ when Λ_G is free over $\mathbb{Z}$.

We have *Mayer-Vietoris sequence* in equivariant cohomology:

$$\cdots \to H^k_G(U \cup V) \to H^k_G(U) \oplus H^k_G(V) \to H^k_G(U \cap V) \to H^{k+1}_G(U \cup V) \to \cdots$$

Exercise 2.26. Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by $z[a:b] = [a:zb]$. Glue together the two fixed points 0 and ∞ to obtain a nodal curve X. Show that

$$H^*_{\mathbb{C}^*}(X) \simeq \Lambda_{\mathbb{C}^*}[\alpha]/(\alpha^2, t\alpha),$$

where $\deg \alpha = 1$, and $\Lambda_{\mathbb{C}^*} = \mathbb{Z}[t]$.

Proposition 2.27. *Let $f : X \to X'$, $\varphi : G \to G'$ as before. Assume that the pullbacks $H^i(G') \to H^i(G)$, $H^i(X') \to H^i(X)$ are isomorphisms for $i < N$. Then $f^* : H^i_{G'}(X) \to H^i_G(X)$ is an isomorphism for $i < N$.*

Proof. First, let us consider the case $G = G'$, $\varphi = \mathrm{id}$. In this case we need to prove $H^i(\mathbb{E} \times^G X) \simeq H^i(\mathbb{E} \times^G X')$. Both spaces are locally trivial fibrations over $\mathbb{B}$, which is proper by our construction in Section 2.4. If the fibrations are trivial, the statement is obvious; otherwise we pick a finite trivializing cover of $\mathbb{B}$ and conclude by Mayer-Vietoris.

Let us apply this to any approximation space $\mathbb{E} \to \mathbb{B}$ for G. We get that $H^i(\mathbb{E} \times^G G') \to H^i(\mathbb{E})$ is an isomorphism for $i < N$, and so $\mathbb{E} \times^G G'$ is an approximation space for G'. Finally, in the general case we get

$$H^i_{G'}(X') \simeq H^i((\mathbb{E} \times^G G') \times^{G'} X') \simeq H^i(\mathbb{E} \times^G X') \simeq H^i_G(X') \simeq H^i_G(X),$$

and so we're done. □

Corollary 2.28. *If f, φ are homotopy equivalences, then $H^*_{G'}(X') \simeq H^*_G(X)$.*

Example 2.29. Let G be a linear algebraic group acting on X. It has a maximal unipotent subgroup, called the *unipotent radical* $R_u(G) \subset G$. We have an exact sequence

$$1 \to R_u(G) \to G \to G^{\mathrm{red}} \to 1,$$

where G^{red} is reductive. Any unipotent group is isomorphic to an affine space, so that $G \to G^{\mathrm{red}}$ is a homotopy equivalence, and $H^G(X) \simeq H^{G^{\mathrm{red}}}(X)$. For instance, this applies to $G = B \subset GL_n$ standard Borel, and $G^{\mathrm{red}} = T$ maximal torus.

2.6. **Gysin pushforward.** Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties over $\mathbb{C}$. Recall that in this situation, we have pullback in compactly supported cohomology, and so pushforward in cohomology by Poincaré duality:

$$H^*(X) \simeq H_c^{2\dim X - *} X^\vee \xrightarrow{(f^*)^\vee} H_c^{2\dim X - *} Y^\vee \simeq H^{*+2(\dim Y - \dim X)}.$$

Note that the sign of this morphism in principle depends on the choice of orientations on X, Y. Here we induce them from the complex structure.

Using our usual strategy of replacing $H^*_G(X)$ with $H^*(\mathbb{E} \times^G X)$, we obtain a pushforward $f_* : H^*_G(X) \to H^*_G(Y)$ when f is G-equivariant. Let us recall the usual properties:

(1) Functoriality: $(gf)_* = g_* f_*$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$;
(2) Projection formula: $f_*(f^*(\beta) \cup \alpha) = \beta \cup f_*(\alpha)$. In particular, f_* is Λ_G-linear (set $\beta \in \Lambda_G$);

(3) Base change: for a fiber square of equivariant maps

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f proper and $\dim Y - \dim X = \dim Y' - \dim X'$, we have $g^* f_* = f'_*(g')^*$;

(4) Self-intersection: if $\iota : X \hookrightarrow Y$ is a closed immersion of codimension d, we have $\iota^*\iota_*(-) = c_d(N_X Y) \cup -$;

(5) Image of subvarieties: let $V \subset X$ closed irreducible, and $W = f(V) \subset Y$. Then $f_*([V]) = \deg(V/W)[W]$ if $\dim V = \dim W$, and zero otherwise.

Exercise 2.30. Let $f : X \to Y$ proper G-equivariant, and $\varphi : G' \to G$ a group homomorphism. Then the following diagram commutes:

$$\begin{array}{ccc} H^*_G(X) & \longrightarrow & H^*_{G'}(X) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ H^*_G(Y) & \longrightarrow & H^*_{G'}(Y) \end{array}$$

Example 2.31. Let a torus T act on $\mathbb{P}^1$ via characters $\chi_1 \neq \chi_2$, that is

$$z[a:b] = [\chi_1(z)a : \chi_2(z)b].$$

Denote $\Lambda = \Lambda_T = \mathbb{Z}[T^\vee]$. Recall that

$$H^*_T(\mathbb{P}^1) \simeq \Lambda[\zeta]/(\zeta + \chi_1)(\zeta + \chi_2),$$

where $\zeta = c_1(\mathcal{O}(1))$. Let us denote the fixed points $0 = [1:0]$, $\infty = [0:1]$, and $\chi = \chi_2 - \chi_1$, and the inclusion $0 \cup \infty \subset \mathbb{P}^1$ by $\iota = (\iota_0, \iota_\infty)$. Since $z[1:b] = [\chi_1(z) : \chi_2(z)b] = [1 : \chi(z)b]$, the tangent space $T_0\mathbb{P}^1$ has weight χ, and similarly $T_\infty \mathbb{P}_1$ has weight $-\chi$. Since $\mathcal{O}(1)$ is the dual of tautological line bundle, we have

$$\iota_0^*\zeta = -\chi_1, \qquad \iota_\infty^*\zeta = -\chi_2.$$

In particular, the pullback ι^* is injective. By self-intersection, $\iota_0^*[0] = \chi$ and $\iota_\infty^*[\infty] = -\chi$. Furthermore, by base change $\iota_0^*[\infty] = \iota_\infty^*[0] = 0$. Putting everything together, we have $\iota^*[0] = \iota^*\zeta + \chi_2$, $\iota^*[\infty] = \iota^*\zeta + \chi_1$, and so by injectivity of ι^*

$$[0] = \zeta + \chi_2, \qquad [\infty] = \zeta + \chi_1.$$

What have we learned? Let us write the Λ-linear maps ι^*, ι_* explicitly:

$$\begin{aligned} \iota^* &: \Lambda[\zeta]/(\zeta + \chi_1)(\zeta + \chi_2) \to \Lambda \oplus \Lambda, \quad \zeta \mapsto (-\chi_1, -\chi_2); \\ \iota_* &: \Lambda \oplus \Lambda \to \Lambda[\zeta]/(\zeta + \chi_1)(\zeta + \chi_2), \quad (a, b) \mapsto a\chi_2 + b\chi_1 + (a+b)\zeta. \end{aligned}$$

Both these maps are injective, and both become isomorphisms as soon as we invert $\chi = \chi_1 - \chi_2$. Moreover, the image of ι^* is precisely the pairs (a, b) where the difference $a - b$ is divisible by χ.

3. Equivariant localization

Our main goal for now is to prove *localization theorem*, that is an isomorphism

$$S^{-1}H_T^*(X) \simeq S^{-1}H_T^*(X^T)$$

for some multiplicative subset $S \subset \Lambda_T$. We will first deal with the simpler case of a smooth variety X, where we have stronger results. But first, we will need some preparation.

3.1. **Local structure of group actions.**

In differential geometry, we have the following simple result:

Lemma 3.1. *Let K be a compact Lie group acting on a smooth manifold X, and $\mathcal{O} \subset X$ a K-orbit. There exist K-invariant neighborhoods U, V of $\mathcal{O}$ in X, $N_{\mathcal{O}}X$ respectively, together with a K-equivariant isomorphism $U \simeq V$, which restricts to identity on $\mathcal{O}$.*

Proof. Choose a K-invariant metric on X (such metric always exists by averaging, since K is compact), and take U to be a tubular neighborhood of $\mathcal{O}$ in X. □

Now let G be a linear algebraic group acting on a smooth variety X. In algebraic geometry, the topology is much coarser, so we will need additional hypotheses.

Luna's slice theorem. *Let $X = \operatorname{Spec} A$ be a smooth affine variety, G a reductive group acting on X, and $\mathcal{O} = Gx$ a closed G-orbit. Then there exists an étale neighborhood U of $\mathcal{O}$ in X, which is isomorphic to an étale neighborhood V of $\mathcal{O}$ in $N_{\mathcal{O}}X$.*

Sketch of proof. Let us denote by G_x the stabilizer of $x \in \mathcal{O}$. Since $\mathcal{O}$ is closed and X is affine, we immediately see that G/G_x is affine. By Matsushima's criterion, this implies that G_x is reductive.

Let $\mathfrak{m} \subset A$ be the maximal ideal of x, and consider the projection $\mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$. Since G_x is reductive, it admits a G_x-equivariant section. In particular, this provides us with a map $\operatorname{Spec}\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m} \subset A$, and so with a map $\zeta : X \to T_xX$, which is by construction étale at x. Using reductiveness of G_x again, we can split $T_xX = T_x\mathcal{O} \oplus N$ in a G_x-equivariant way. Denoting $W = \zeta^{-1}(N)$, we can extend the inclusion $W \subset X$ to a G-equivariant map $G \times^{G_x} W \to X$ via G-action. At the same time, ζ gives rise to a map $G \times^{G_x} W \to G \times^{G_x} N$. Both of these maps are étale at $x \in \mathcal{O}$.

We omit the proof of the following lemma of Luna:

Lemma 3.2. *Let $\varphi : X \to Y$ be a G-equivariant map of smooth affine G-varieties, and $\mathcal{O} \subset X$ a closed G-orbit. Assume that φ is étale at a point $x \in \mathcal{O}$, $\varphi(\mathcal{O})$ is closed in Y, and φ is injective on $\mathcal{O}$. Then there exists opens $\mathcal{O} \subset U \subset X$, $\varphi(\mathcal{O}) \subset V \subset Y$, such that $\varphi : U \to V$ is étale.* □

This lemma applies to both maps $G \times^{G_x} W \to X$, $G \times^{G_x} W \to G \times^{G_x} N$, and so we may conclude. □

What about the case when the variety is not affine? Luckily, we can get around this requirement in the cases we need.

Sumihiro's linearization. *Let X be a normal quasi-projective G-variety, where G is a connected linear algebraic group. There exists a locally closed embedding $X \hookrightarrow \mathbb{P}^N$ together* *with a group homomorphism $\varphi : G \to PGL_{N+1}$, such that the G-action on X is induced from the natural PGL_{N+1}-action on $\mathbb{P}^N$. Moreover, when X is only normal, each point $x \in X$ admits a G-invariant quasi-projective neighborhood.*

We will not prove this theorem, and instead deduce the result we need.

Corollary 3.3. *Let G be a reductive group, X a smooth G-variety, $x \in X^G$ a fixed point. Then x has a G-invariant affine neighborhood.*

Proof. By Sumihiro's result, we can assume that X is a locally closed subvariety of $\mathbb{P}(V)$, where V is a G-representation. Since x is a fixed point and G is reductive, we can further assume that $V = L \oplus V'$ as a G-representation, where $x = [L]$.

If X is closed in $\mathbb{P}(V)$, we obtain the desired affine neighborhood as the complement of $\mathbb{P}(V') \cap X$. Otherwise, let $Y = \overline{X} \setminus X$, and it suffices to find a homogeneous G-invariant function f on $\mathbb{P}(V)$, such that $f|_Y = 0$ and $f(x) \neq 0$. Let $x_0, \dots, x_N$ be the homogeneous coordinates on $\mathbb{P}(V)$, where x_0 corresponds to the line L. Let I be the homogeneous ideal of Y, and $J = (x_1, \dots, x_N)$ the homogeneous ideal of $x = [L]$. Since $x \notin Y$, for any homogeneous function h on $\mathbb{P}(V)$ of positive degree we can find $k \in \mathbb{N}$ and $h_1 \in I$, $h_2 \in J$ such that $f^k = h_1 + h_2$. Furthermore, since G is reductive we have a G-equivariant projector map $\pi : \mathbb{C}[x_0, \dots, x_N] \to \mathbb{C}[x_0, \dots, x_N]^G$. Let $h = x_0$. By the above, we have

$$x_0^k = \pi(x_0^k) = \pi(h_1) + \pi(h_2),$$

and we can set $f = \pi(h_1)$. □

Remark 3.4. A similar proof shows that any affine G-orbit in X has a G-invariant affine neighborhood.

3.2. **Localization theorem for smooth varieties.** Let us begin with a local characterization of fixed points.

Lemma 3.5. *Let G be a connected reductive group acting on a smooth variety G, and $p \in X^G$. Then p is an isolated fixed point iff the G-representation T_pX does not contain the trivial representation.*

Proof. By Luna's slice theorem and Corollary 3.3, we can assume that p is the origin in a G-representation X. The lemma easily follows. □

In other words, when $G = T$ is torus the lemma tells us that p is isolated iff $c_{\mathrm{top}}(T_pX) = 0$.

Remark 3.6. Note that the condition of G being reductive is crucial for the lemma to hold. Indeed, let the additive group $\mathbb{C}$ act on $\mathbb{P}^1$ via $t[a:b] = [a:b+ta]$. Then the only fixed point is $[0:1]$, but $\mathbb{C}$ does not admit non-trivial one-dimensional algebraic representations.

Theorem 3.7. *Let X be smooth, and T an algebraic torus acting on X. Assume that X^T is finite, and denote $e = \prod_{p \in X^T} c_{\mathrm{top}}(T_pX)$. Suppose that we have a collection of $m \leq \#X^T$ classes in $H_T^*(X)$, which restrict to a basis of $H^*(X)$. Then $m = \#X^T$, and both maps*

$$i^* : H_T^*(X) \to H_T^*(X^T), \qquad i_* : H_T^*(X^T) \to H_T^*(X)$$

become isomorphisms upon inverting e. Furthermore, i^* is injective.

Proof. Let $n = \#X^T$, and write

$$\Lambda_T^{\oplus n} = H_T^*(X^T) \xrightarrow{i_*} H_T^*(X) \xrightarrow{i^*} H_T^*(X^T) = \Lambda_T^{\oplus n}.$$

By the self-intersection property of Gysin map, the composition $i^*i_* \in \mathrm{End}_{\Lambda_T}(\Lambda_T^{\oplus n})$ can be represented as a diagonal matrix with entries $\{c_{\mathrm{top}}(T_pX) : p \in X^T\}$. Thus $\det(i^*i_*) = e$, which implies that i_* is injective, and i^* becomes surjective upon inverting e. The condition on a collection of classes in $H_*^T(X)$ shows (say, by graded Nakayama lemma) that $H_*^T(X)$ is generated by m elements as a Λ_T-module. From the injectivity of i_* we conclude that $m = n$, and $H_*^T(X)$ is a free Λ_T-module of rank n. Thus i^* becomes an isomorphism after inverting e, and so does i_*.

Finally, for the injectivity of i^* look at the following commutative square:

$$\begin{tikzcd}
H_T^*(X) \arrow[r, "i^*"] \arrow[d, hook] & H_T^*(X^T) \arrow[d, hook] \\
H_T^*(X)[e^{-1}] \arrow[r, equal] & H_T^*(X^T)[e^{-1}]
\end{tikzcd}$$

The vertical maps are injective, therefore the horizontal one is as well. □

Example 3.8. Let $X = \mathrm{Gr}(d, \mathbb{C}^n)$, together with an action of $T = (\mathbb{C}^*)^n$, induced from its action on $\mathbb{C}^n$ with n distinct characters $\chi_1, \ldots, \chi_n$. We write $\mathbb{C}^n = \bigoplus_{i=1}^n L_i$, where T acts on the line L_i via the character χ_i. In this case the fixed points are just the coordinate subspaces:

$$X^T = \{V_I : I \subset [1, n],\ \#I = d\}, \qquad V_I = \bigoplus_{i \in I} L_i.$$

Furthermore, the tangent spaces are given by

$$T_{V_I} = \mathrm{Hom}(V_I, \mathbb{C}^n / V_i) = \bigoplus_{\substack{i \in I \\ j \notin I}} L_i^\vee \otimes L_j,$$

and so

$$c_{\mathrm{top}}(T_{V_I}) = \prod_{\substack{i \in I \\ j \notin I}} (\chi_j - \chi_i).$$

Thus in localization theorem, we need to invert Van der Monde determinant $e = \Delta_n = \prod_{i<j}(\chi_i - \chi_j)$. Finally, in order to check the basis condition, recall that $H^*(\mathrm{Gr}(d, \mathbb{C}^n))$ is generated by the Chern classes of the tautological vector bundle S. We can thus pick a basis of $H^*(X)$ monomial in $c_i(S)$'s, and lift it to equivariant Chern classes in $H_T^*(X)$. Thus Theorem 3.7 applies.

In fact, we can rather easily get rid of the condition that X^T is finite.

Lemma 3.9 (Iversen). *Let G be reductive. Then for any smooth G-variety X, the fixed locus X^G is smooth.*

Proof. Omitted; the idea is to use splitting as in the proof of Luna's slice theorem, but for all powers of $\mathfrak{m}$. □

Remark 3.10. It is not hard to show that the property above *characterizes* reductive groups. More precisely, whenever G is not reductive, one can find a smooth G-variety X such that X^G is not reduced.

Lemma 3.11. *Let T be an algebraic torus, X a smooth T-variety, $Z \subset X^T$ a connected component of fixed locus, and denote $d = \operatorname{codim} Z$. Then there exists a collection of non-zero characters $\chi_1, \dots, \chi_d$ of T, such that for any $p \in Z$ the T-action on the fiber of the normal bundle $N_Z(X)|_p$ is by these weights. Moreover, T acts trivially on T_pX.*

Proof. Use Luna's slice theorem as in Lemma 3.5. ☐

As before, self-intersection formula implies that $i^* i_*(-) = c_d(N_Z X) \cup -$, where $i : Z \hookrightarrow X$ is the inclusion of a fixed locus component. By e.g. splitting principle for Chern classes, we have

$$c_d(N_Z X) = \chi_1 \dots \chi_d + \sum_{i=1}^{d} a_{d-i} \gamma_i, \qquad a_i \in \Lambda_T^{2i}, \gamma_i \in H^{2i}(Z).$$

Since the classes γ_i are all nilpotent, the class $c_d(N_Z X)$ is a non-zero divisor, and becomes invertible after inverting $e_Z := \chi_1 \dots \chi_d$. Thus the same proof as before yields

Theorem 3.12. *Let X be smooth, and T an algebraic torus acting on X. Let $S \subset \Lambda_T$ be a multiplicative subset, which contains e_Z for each $Z \subset X^T$. Assume that we have a collection of m classes in $H_T^*(X)$ restricting to a basis of $H^*(X)$, with $m \leq \sum_{Z \subset X^T} \operatorname{rk} H^*(Z)$. Then $m = \sum_{Z \subset X^T} \operatorname{rk} H^*(Z)$, the maps*

$$i^* : S^{-1} H_T^*(X) \to S^{-1} H_T^*(X^T), \qquad i_* : S^{-1} H_T^*(X^T) \to S^{-1} H_T^*(X)$$

are isomorphisms, and $i^* : H_T^*(X) \to H_T^*(X^T)$ is injective. ☐

Exercise 3.13. Let $T = (\mathbb{C}^*)^n$ act on $\mathbb{C}^n$ via k distinct characters $\chi_1, \dots, \chi_k$, and write $\mathbb{C}^n = \bigoplus_{i=1}^n V_i$, where V_i is the isotypic component for the character χ_i. Show that

$$X^T = \bigsqcup_{\substack{\sum_i d_i = d \\ d_i \leq \dim V_i}} \mathrm{Gr}(d_1, V_1) \times \dots \times \mathrm{Gr}(d_k, V_k).$$

Compute the classes e_Z for every fixed component of X^T.

3.3. **Equivariant formality.** The condition on lifting a basis of $H^*(X)$ to $H_T^*(X)$ is annoying to check. Fortunately, it holds in many situations of interest for general reasons. Let us begin by encapsulating this condition into a definition, where we for once emphasize the dependence on the ring of coefficients.

Definition 3.14. Let X be a G-variety. X is called *equivariantly formal* over a ring R if for all $i > 0$

- $H^i(X, R)$ is finitely generated and free over R;
- There exists a finite collection of classes $x_{ij} \in H_G^i(X, R)$ restricting to a basis of $H^i(X, R)$.

Proposition 3.15. *Let X be equivariantly formal. Then $H_G^*(X, R)$ is a free Λ_G-module with basis $\{x_{ij}\}$, and the forgetful map $H_G^*(X, R) \otimes_{\Lambda_G} R \to H^*(X, R)$ is an isomorphism. Moreover, for any group homomorphism $G' \to G$ the map $H_G^*(X, R) \otimes_{\Lambda_G} \Lambda_{G'} \to H_{G'}^*(X, R)$ is an isomorphism as well.*

Proof. Recall that equivariant cohomology can be computed from the Leray spectral sequence associated to fibration $\mathbb{E} \times^G X \to \mathbb{B}$ with fiber X. The formality condition is equivalent to asking that this spectral sequence degenerates at the E_2-page. ☐

We have a very powerful theorem which describes the topology of X in terms of its torus fixed points.

Theorem 3.16 (Białynicki-Birula)**.** *Let T be an algebraic torus acting on X, where X is a smooth* proper *variety. Then there exists a filtration of X by T-invariant closed subvarieties*

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_1 \supset \emptyset,$$

where n is the number of connected components of X^T, and an ordering of said connected components Z_i, such that $X_i \setminus X_{i-1}$ is an affine fibration over Z_i for all i. □

In particular, assume that X is smooth, proper, and has finitely many fixed points. Then Białynicki-Birula theorem tells us that X stratifies into a union of affine cells:

$$X = \bigsqcup U_i, \qquad U_i \simeq \mathbb{A}^{n_i}.$$

Moreover, by the long exact sequence in cohomology the classes $[\overline{U_i}]$ provide a basis of both $H^*(X)$ and $H^*_T(X)$, and both of these cohomology groups are even. Thus X is equivariantly formal over $\mathbb{Z}$, and so localization theorem applies.

Remark 3.17. If we don't care about torsion in cohomology, we can further lift the restriction of X having finitely many fixed points. Namely, we implicitly used the fact the cohomology of each connected component of X^T is even, in order to split the long exact sequences. If we work with $\mathbb{Q}$-coefficients, there is a additional piece of structure one can introduce to cohomology groups, namely "mixed Hodge structure". Working in this richer framework, one can replace our parity argument with a "purity" argument, and show that the long exact sequences split. This implies equivariant formality *over* $\mathbb{Q}$ for any smooth proper T-variety.

3.4. **Poincaré duality.** Before continuing with localization theorems, let's pause for a quick interlude about equivariant Poincaré duality.

Let us begin with the non-equivariant setting. Let $f : X \to Y$ be a fiber bundle with a proper fiber F. When do we have relative Poincaré duality on $H^*(X)$, that is linear over $H^*(Y)$ and of degree $2 \dim F$?

Example 3.18. Consider the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ with fiber $\mathbb{S}^1$; here we have no relative Poincaré for degree reasons.

The example above suggests that we might have better luck when at the very least $H^*(X)$ is a free module over $H^*(Y)$.

Proposition 3.19. *Assume that X is a smooth proper G-variety, and X is equivariantly formal with the basis $\{\beta_i\}$ of $H^*_G(X)$ as a Λ_G-module. Denote by $p : X \to \mathrm{pt}$ the projection map. Then we have a (unique) basis $\{\gamma_i\}$ of $H^*_G(X)$ over Λ_G with $p_*(\beta_i \cup \gamma_j) = \delta_{ij}$.*

Proof. Order β_i's so that the cohomological degree non-strictly decreases. Denote by $\overline{\gamma}_j$ the Poincaré dual basis to the images of β_i's in $H^*(X)$. We will look for γ_k's by induction.

For $k = 1$, this is the unique lift of $\overline{\gamma}_1$ by $H^0_G(X) \xrightarrow{\sim} H^0(X)$. Now assume we constructed $\gamma_1, \ldots, \gamma_{k-1}$; then take any lift γ'_k of $\overline{\gamma}_k$, and set

$$\gamma_k = \gamma'_k - \sum_{j=1}^{k-1} \langle \gamma'_k, \beta_j \rangle \gamma_j.$$

It is easy to check this choice works.

It remains to show that γ_j's form a basis. Consider the inclusion of Λ_G-modules $\Lambda_G\gamma_1 \oplus \ldots \oplus \Lambda_G\gamma_n \hookrightarrow H^*_G(X)$. By construction, its reduction modulo the augmentation ideal of Λ_G is onto, therefore it is itself onto by graded Nakayama lemma. ☐

Example 3.20. Let us return to the setup of Example 2.31, and denote by p the projection of $\mathbb{P}^1$ to a point. For any $a \in T^\vee$, the basis $(1, \zeta + a)$ satisfies the conditions of equivariant formality, since it restricts to the basis $(1, [\mathrm{pt}])$ of $H^*(\mathbb{P}^1)$. Note that the Poincaré dual basis is $(\zeta + \chi_1 + \chi_2 - a, 1)$, because, using image of subvarieties property,

$$\begin{aligned} p_*((\zeta + a)(\zeta + b)) &= p_*((a + b - \chi_1 - \chi_2)\zeta + ab) \\ &= p_*((a + b - \chi_1 - \chi_2)([0] - \chi_2)) \\ &= a + b - \chi_1 - \chi_2. \end{aligned}$$

In particular, we do not get a self-dual basis unless $\chi_1 + \chi_2$ is divisible by 2, unlike the non-equivariant case.

3.5. **Integration formula.** Let X, Y, smooth T-varieties, for which the localization theorem applies. Consider a proper T-equivariant map $f : X \to Y$. Each connected component $P \subset X^T$ gets sent to some component $Q \subset Y^T$; let us denote by $f_P : P \to Q$ the restriction of f. We further consider the closed embeddings $i_P : P \to X$, $i_Q : Q \to Y$; we will sometimes denote the pullback along these maps as restrictions: $i_P^*(-) = (-)|_P$. Since both f and f_P are proper, we have the following (non-commutative!) square:

$$\begin{tikzcd} H^*_T(X) \arrow[r, "i_P^*"] \arrow[d, "f_*"] & H^*_T(P) \arrow[d, "(f_P)_*"] \\ H^*_T(Y) \arrow[r, "i_Q^*"] & H^*_T(Q) \end{tikzcd}$$

Proposition 3.21. *For any $u \in H^*_T(X)$, we have*

$$f_*(u)|_Q = c_{\mathrm{top}}(N_Q Y) \sum_{f(P)\subset Q} (f_P)_* \left(\frac{u|_P}{c_{\mathrm{top}}(N_P X)} \right).$$

Remark 3.22. Note that a priori, this equation lives in the localization $S^{-1}H^*_T(Q)$ for some multiplicative set $S \subset \Lambda_T$. However, $H^*_T(Q)$ obviously embeds into $S^{-1}H^*_T(Q)$. In particular, a part of the theorem above is that the right-hand side takes values in $H^*_T(Q)$, which implies various divisibility properties!

Proof. By localization theorem, we can replace $H^*_T(X)$ by $S^{-1}H^*_T(X^T)$, and so by linearity it is enough to check the formula for $u = (i_P)_*(z)$, $z \in H^*_T(P)$. On the left-hand side, we have

$$f_*(u)|_Q = i_Q^* f_*(i_P)_*(z) = i_Q^*(i_Q)_*(f_P)_*(z) = \begin{cases} c_{\mathrm{top}}(N_Q Y) \cup (f_P)_*(z) & \text{if } f(P) \subset Q, \\ 0 & \text{otherwise.} \end{cases}$$

By self-intersection formula, we have

$$u|_P = i_P^*(i_P)_*(z) = c_{\mathrm{top}}(N_P X) \cup z,$$

and so

$$c_{\mathrm{top}}(N_Q Y) \sum_{f(P)\subset Q} (f_P)_* \left(\frac{u|_P}{c_{\mathrm{top}}(N_P X)} \right) = c_{\mathrm{top}}(N_Q Y)(f_P)_* \left(\frac{i_P^*(i_P)_*(z)}{c_{\mathrm{top}}(N_P X)} \right)$$
$$= c_{\mathrm{top}}(N_Q Y) \cup (f_P)_*(z),$$

which allows us to conclude. □

In particular, if X is proper and $\pi : X \to \mathrm{pt}$ is the projection to a point, we get the *integration formula*:

$$\int_X u := \pi_*(u) = \sum_{P\subset X^T} \int_P \left(\frac{u|_P}{c_{\mathrm{top}}(N_P X)} \right).$$

If X has finitely many fixed points, this assumes a particularly simple form:

$$\int_X u = \sum_{p\in X^T} \frac{u|_P}{c_{\mathrm{top}}(T_p X)}.$$

One can write a similar simplified form for a more general proper map between two varieties with finitely many fixed points.

Example 3.23. Let T act on $\mathbb{P}^{n-1}$ via distinct characters $\chi_1, \dots, \chi_n$, and let $\zeta = c_1(\mathcal{O}(1))$ as usual. From basic geometry, we know that

$$\int_{\mathbb{P}^{n-1}} \zeta^k = \begin{cases} 1 & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the integration formula tells us that

$$\int_{\mathbb{P}^{n-1}} \zeta^k = \sum_{i=1}^{n} \frac{-\chi_i^k}{\prod_{j\neq i}(\chi_j - \chi_i)}.$$

In particular, for $k = n-1$ one essentially recovers the inductive formula for Van der Monde determinant.

Example 3.24. Consider $T = \mathbb{C}^*$ acting on $\mathbb{P}^2$ by $t[a:b:c] = [a:tb:t^2c]$. The fixed points are coordinate lines L_1, L_2, L_3. Given a class $u \in H_T^*(\mathbb{P}^2)$, denote $u_i = u|_{L_i}$. Then the integration formula says

$$\int_{\mathbb{P}^2} u = \frac{u_1}{t\cdot 2t} + \frac{u_2}{(-t)\cdot t} + \frac{u_3}{2t\cdot t} = \frac{u_1 - 2u_2 + u_3}{2t^2}.$$

Thus $u_1 - 2u_2 + u_3$ must be divisible by $2t^2$!

Remark 3.25. Integration formula can also be used to formally define integration map for non-proper maps. Namely, assume that $f : X \to Y$ is not proper, but its restriction $X^T \to Y^T$ is. Then we can *define*

$$f_*(u) := \sum_{\substack{Q\subset Y^T \\ f(P)\subset Q}} (i_Q \circ f_P)_* \left(\frac{u|_P}{c_{\mathrm{top}}(N_P X)} \right) \in S^{-1}H_T^*(Y).$$

Note that in this case we cannot expect any divisibility properties to hold. For example, for $T = \mathbb{C}^*$ acting on $\mathbb{C}$ with character ξ we get $f_*(1) = 1/\xi$.

Example 3.26. Let us compute the number of lines in $\mathbb{P}^3$ meeting 4 general lines. Lines in $\mathbb{P}^3$ are parameterized by points in $X = \mathrm{Gr}(2,4)$. Let a torus T act on it via the action on $\mathbb{C}^4$ by distinct characters $\chi_1, \dots, \chi_4$. We denote $E_{ij} = \mathrm{span}(v_i, v_j)$, where v_i is the χ_1-eigenvector. Fix a line $L = E_{12}$; then the subvariety of lines meeting L is

$$\Omega = \{E \in \mathrm{Gr}(2,4) \,:\, \dim(E \cap E_{12}) \geq 1\}.$$

Alternatively, Ω is given by the condition $\mathrm{rk}(S \to \mathbb{C}^4/E_{12}) \leq 1$, or $\Lambda^2 S \to \Lambda^2(\mathbb{C}^4/E_{12})$ is a zero map; here S is the tautological vector bundle on $\mathrm{Gr}(2,4)$. This tells us that Ω is the zero set of a section of

$$\mathrm{Hom}(\Lambda^2 S, \Lambda^2(\mathbb{C}^4/E_{12})) = \Lambda^2 S^\vee \otimes \mathbb{C}_{\chi_3+\chi_4},$$

and so $[\Omega] = c_1(\Lambda^2 S^\vee \otimes \mathbb{C}_{\chi_3+\chi_4})$. We want to compute $\int_X [\Omega]^4$ using integration formula. Recall that the fixed points of X are precisely E_{ij}, and

$$c_1(\Lambda^2 S^\vee \otimes \mathbb{C}_{\chi_3+\chi_4})|_{E_{ij}} = \chi_3 + \chi_4 - \chi_i - \chi_j.$$

Putting everything together, integration formula yields

$$\int_X [\Omega]^4 = \sum_{1 \leq i < j \leq 4} \frac{(\chi_3 + \chi_4 - \chi_1 - \chi_2)^4}{(\chi_k - \chi_i)(\chi_k - \chi_j)(\chi_l - \chi_i)(\chi_l - \chi_j)}.$$

Since the answer is a number, we can substitute χ_i with any numbers such that the denominators do not vanish, for example $\chi_i = i$. Doing this, we obtain:

$$\frac{4^4}{12} - \frac{3^4}{3} + \frac{2^4}{4} + \frac{2^4}{4} - \frac{1^4}{3} - \frac{0^4}{12} = \frac{64}{3} - 27 + 4 + 4 - \frac{1}{3} = 21 - 19 = 2.$$

The answer is two lines!

Exercise 3.27. Compute the number of lines in $\mathbb{P}^4$ meeting 6 generic planes.

3.6. **General localization theorem.** Now let us assume X is any T-variety, not necessarily smooth or proper. Let L be a subgroup of the character lattice $T^\vee$, and define

$$T(L) = \bigcap_{\chi \in L} \ker \chi.$$

Furthermore, let $S(L) \subset \Lambda_T$ be the multiplicative set generated by the complement $T^\vee \setminus L$. We will prove a general version of localization theorem:

Theorem 3.28. *Let $i \,:\, X^{T(L)} \hookrightarrow X$ be the inclusion of $T(L)$-fixed points. Then the restriction map*

$$i^* \,:\, S(L)^{-1} H_T^*(X) \to S(L)^{-1} H_T^*(X^{T(L)})$$

is an isomorphism.

Let us begin with a couple of useful lemmas. The first lemma is a straightforward corollary of Λ_T-linearity of the pullback.

Lemma 3.29. *Let $Y' \to Y$ be a T-equivariant map. If $c \in \Lambda_T$ annihilates $H_T^*(Y)$, then it also annihilates $H_T^*(Y')$.* □

Lemma 3.30. *There exists a T-invariant open $U \subset X$ such that all points have the same stabilizer $T' \subset T$. Moreover, one can assume that $U \simeq U' \times T/T'$.*

Proof. Without loss of generality, we can assume that X smooth and affine. By Sumihiro's linearization, X can be T-equivariantly embedded into a linear representation V of

T. The latter only has finitely many classes of stabilizers (namely, various intersections of $\ker \chi_i$'s, where χ_i's are the characters appearing in V), so the first claim follows.

For the second claim, use Sumihiro's linearization again to embed U into a T/T'-representation V'. Decompose $V' = \bigoplus_i V'_i$ into the isotypic components. The complement of coordinate planes is clearly a T/T'-torsor over $\prod_i \mathbb{P}(V'_i)$, and so we can pick an open U' inside the image of U over which this torsor is trivial. $\square$

Proof of Theorem 3.28. Using Lemma 3.30 recursively, we obtain the following filtration of X:

$$X^{T(L)} = X_0 \subset X_1 \subset \ldots \subset X_k = X, \qquad X_i \setminus X_{i-1} = U_i \times T/T_i,$$

where U_i has trivial $T(L)$-action, and $T_i \subset T(L)$ is a torus. Reasoning by Noetherian induction, it suffices to show that the pullback

$$S(L)^{-1}H^*_T(X_i) \to S(L)^{-1}H^*_T(X_{i-1})$$

is an isomorphism, which by the long exact sequence in cohomology is equivalent to the vanishing of relative cohomology $H^*_T(X_i, X_{i-1})$. Let us pick an approximating space $\mathbb{E} \to \mathbb{B}$ for T. Then by tautness,

$$\begin{aligned} H^k_T(X_i, X_{i-1}) &= H^k(\mathbb{E} \times^T X_i, \mathbb{E} \times^T X_{i-1}) \\ &= \lim_{V \supset \mathbb{E} \times^T X_{i-1}} H^k(\mathbb{E} \times^T X_i, V) \\ &= \lim_{V \supset \mathbb{E} \times^T X_{i-1}} H^k(\mathbb{E} \times^T (U_i \times T/T_i), V \setminus (\mathbb{E} \times^T X_{i-1})). \end{aligned}$$

Let $\chi \in T^\vee$ be a character with $\chi|_{T_i} = 0$, and $\chi|_{T(L)} \neq 0$. It is clear that such $\chi \in S(L)$ annihilates $H^*_T(T/T_i)$, and so we have

$$S(L)^{-1}H^*(\mathbb{E} \times^T (U_i \times T/T_i)) = H^*(U_i) \otimes S(L)^{-1}H^*_T(T(L)/T') = 0.$$

By Lemma 3.29, this also means that $H^k(V \setminus (\mathbb{E} \times^T X_{i-1})) = 0$, and so in particular all the relative cohomology groups above vanish. $\square$

Remark 3.31. Note that during the proof we only needed to invert a finite set of characters $\{\chi_i\}$, such that stabilizers in the complement $X \setminus X^{T(L)}$ all lie in $\bigcup_i \ker \chi_i$. We can therefore replace $S(L)$ by a smaller, finitely generated multiplicative set.

3.7. **Description of the image.** Let us begin with a very general, and not very explicit, description of the image of localization map.

We say that and element $f \in \Lambda_T$ is an *irreducible factor* if it is either a prime in $\mathbb{Z} = \Lambda^0_T$, or a primitive character $\chi \in \Lambda^2_T$. Given an irreducible factor f, we will write $L_f \subset T^\vee$ for the sublattice of characters divisible by f, and $T(f) = T(L_f)$. When $f \in \Lambda^2$, then $T(f) \subset T$ is a codimension 1 subtorus; when $f \in \mathbb{Z}$, $T(f)$ is the finite subgroup of order p elements.

Theorem 3.32 (Chang-Skjelbred)**.** *Assume that $H^*(X^T, R)$ is a free R-module, and that $H^*_T(X, R)$ is a free Λ_T-module. Then $\alpha \in H^*_T(X^T, R)$ belongs to the image of $i^* : H^*_T(X, R) \to H^*_T(X^T, R)$ iff it lies in the image of $H^*_T(X^{T(f)}, R) \to H^*_T(X^T, R)$ for all irreducible factors f.*

Proof. Since $X^{T(f)} \supset X^T$, the "only if" direction is clear. For the other direction, assume that $\alpha \in H^*_T(X^T)$ lies in the image of all $H^*_T(X^{T(f)})$. We can find an element $g \in S$ such that $g\alpha \in H_T^*(X)$ by localization theorem. Suppose it's minimal, that is for any g' proper divisor of g, we have $g'\alpha \notin H_T^*(X)$.

Write $g\alpha = \sum_i a_i e_i$, where $\{e_i\}$ is an R-basis of $H_T^*(X)$ over Λ_T. Suppose α does not belong to the image of $H_T^*(X)$; then $g \in \Lambda_T$ is not a unit. Let f be an irreducible factor of g, then by minimality of g we can assume that a_1 is relatively prime to f.

By the general localization theorem, $S(f)^{-1}H_T^*(X) = S(f)^{-1}H_T^*(X^{T(f)})$, where $S(f)$ is the multiplicative set generated by characters not divisible by f. Then $\psi_f\alpha \in H_T^*(X)$ for some $\psi_f \in S(f)$. Write $\psi_f\alpha = \sum_i b_i e_i$. Then on one hand, the coefficient of e_1 in $(g\psi_f)\alpha$ is $\psi_f a_1$ relatively prime to f, and on the other gb_1 divisible by f. We arrive at a contradiction. □

Let us now assume that X is smooth. Denote by $\mathscr{S} \subset T^\vee$ the finite set of all non-trivial characters occuring as weights of T_pX, $p \in X^T$. For any irreducible factor f, we can divide $\mathscr{S} = \mathscr{S}_f^+ \sqcup \mathscr{S}_f^-$, where $\mathscr{S}_f^+$ are the characters divisible by f. Define further

$$T_f := \bigcap_{\chi \in \mathscr{S}_f^+} \ker\chi, \qquad X^f := X^{T_f}.$$

Then carefully looking at the proof of both Theorem 3.28 and Theorem 3.32, it suffices to take only $f \in \mathscr{S}$, and replace $X^{T(f)}$ by X^f.

Corollary 3.33. *Let X be smooth, and $H_T^*(X)$ a free Λ_T-module. Assume that for any irreducible factor f, the fixed locus X^f is compact and $H_T^*(X^f)$ is free over Λ_T. Then an element $(\alpha_Z)_{Z\subset X^T}$ belongs to $H_T^*(X)$ if and only if we have*

$$\sum_{Z\subset X^T} \frac{\alpha_Z \beta|_Z}{c_{\mathrm{top}}(N_Z X^f)} \in \Lambda_T$$

for all irreducible factors f and $\beta \in H_T^*(X^f)$.

Proof. The condition is necessary by the integration formula. For sufficiency, we only need to prove that $\alpha = (\alpha_Z)$ lifts to $H_T^*(X^f)$ for all f. Let $\{x_i\}$ be a Λ_T-basis of $H_T^*(X^f)$, and $\{y_i\}$ the Poincaré dual basis. Write $\alpha = \sum a_i x_i$, for $a_i \in S^{-1}\Lambda_T$. By Poincaré duality and integration formula, we have

$$a_i = \int_{X^f} \alpha y_i = \sum_{Z\subset X^T} \frac{\alpha_Z y_i|_Z}{c_{\mathrm{top}}(N_Z X^f)},$$

and so all a_i's belong to Λ_T. □

Thus, the motto is "image of localization map is determined by divisibilities from integration formula".

3.8. **GKM varieties.** When X is smooth with isolated fixed points, the description of image drastically simplifies.

Definition 3.34. A *T-curve* $C \subset X$ is the closure $\overline{T\cdot x}$ of a one-dimensional T-orbit in X. Each T-curve has the associated non-zero character $\pm\chi$, obtained from the orbit $T\cdot x$ itself; sign comes from the choice of an identification of $T\cdot x$ with $\mathbb{C}^*$.

Note that by definition every T-curve is rational. The only normal rational curve with torus action having 2 fixed points is $\mathbb{P}^1$, so every T-curve connecting two fixed points gives a T-equivariant embedding $\mathbb{P}^1 \to X$. By Example 2.31, the character χ can be expressed as $\chi = \chi_2 - \chi_1$, where T acts on $\mathbb{P}^1$ with the characters χ_1, χ_2.

We can have infinitely many T-curves, even when the fixed locus is finite.

Exercise 3.35. List all T-curves in Example 3.24. In particular, show that there are infinitely many T-curves between L_1 and L_3.

Exercise 3.36. We say that two characters χ_1, χ_2 are *parallel* if $\chi_1 = c_1\eta$, $\chi_2 = c_2\eta$, with η primitive, and $c_1, c_2 \in \mathbb{Z}$. Let $p \in X^T$ be an isolated fixed point. Using Sumihiro's linearization, show that there are finitely many T-curves through p if and only if no two weights of T_pX are parallel.

Lemma 3.37. *Suppose X is smooth, X^T is finite, and $H_T^*(X)$ is free over Λ_T. Assume that at each $p \in X^T$ the weights of T_pX are relatively prime. If f is a character, X^f is the union of all T-curves whose character is divisible by f, together with all fixed points $p \in X^T$ where no weight is divisible by f.*

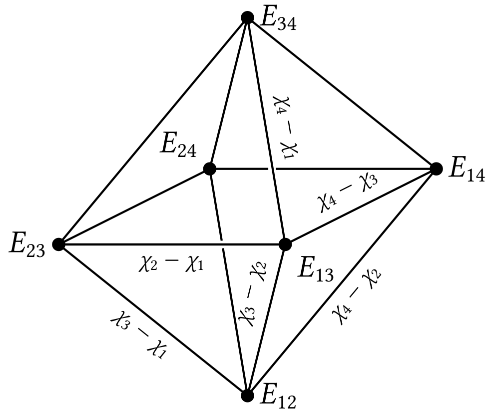
Proof. First of all, note that every connected component of X^f must contain a T-fixed point, otherwise the localization map cannot be injective. Furthermore, by Lemma 3.9 the fixed locus X^f is a smooth variety. By the condition on tangent weights, at each fixed point we can have at most one line which can be tangent to X^f. Thus X^f must consist of T-curves and T-fixed points.

It is clear that all T-fixed points belong to X^f. For the curves, a T-curve with character χ belongs to X^f iff $T^f \subset \ker \chi$ iff χ belongs to the lattice $K = \langle \mathscr{S}_f^+ \rangle \subset T^\vee$. If χ is divisible by f it belongs to $\mathscr{S}_f^+$ by definition; if it is not, it does not belong to K because all elements of K are divisible by f. $\square$

Corollary 3.38. *Let X be a smooth variety with X^T finite, and assume $H_T^*(X)$ is free over Λ_T. Suppose that for each $p \in X^T$, the weights on T_pX are relatively prime. Then $(u_p)_p \in H_T^*(X^T)$ lies in the image of $H_T^*(X)$ iff for each T-curve $C_{pq} \simeq \mathbb{P}^1$ connecting $p \neq q \in X^T$, the difference $u_p - u_q$ is divisible by the character $\pm\chi_{pq}$ of C_{pq}.*

Proof. The condition is necessary, because for any T-curve $\{p, q\} \to \mathbb{P}^1 \to X$ we can factor the restriction map through $\mathbb{P}^1$ and apply Example 2.31. For sufficiency, by our refinement of Theorem 3.32 we only need to prove that (u_p) lifts to $H_T^*(X^f)$, for each character f. By Lemma 3.37, X^f is a disjoint union of T-curves whose character is divisible by f, together with some isolated fixed points. Classes of isolated fixed points clearly lift, so we are only concerned with T_f-curves containing a fixed point. Each such T-curve contains a fixed point, so must be either $\mathbb{P}^1$ or $\mathbb{C}$. The first case follows from Example 2.31, and in the second case the pullback map is an isomorphism by homotopy invariance. $\square$

Example 3.39. Let $X = \mathrm{Gr}(2,4)$ again. We have already computed the weights at fixed points in Example 3.26, and this is enough to determine the characters of T-curves:



Let us again consider the subvariety

$$\Omega = \{E \in \mathrm{Gr}(2,4) \,:\, \dim(E \cap E_{12}) \geq 1\}.$$

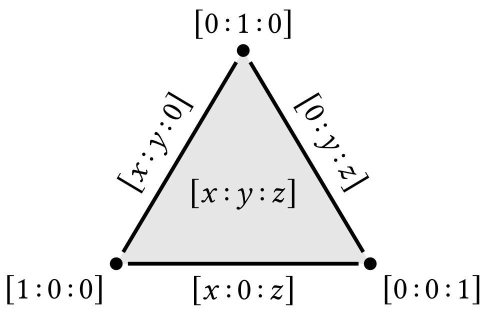
We know that $E_{34} \notin \Omega$, and E_{24} is a non-singular point with normal character $t_3 - t_2$. This tells us that $[\Omega]|_{E_{34}} = 0$, $[\Omega]|_{E_{24}} = t_3 - t_2$. The rest of the restrictions we can recover from divisibility conditions!

Exercise 3.40. Let T act on $\mathbb{P}^1$ by characters χ_1, χ_2, and consider the diagonal action of T on $\mathbb{P}^1 \times \mathbb{P}^1$. Check that this action has 4 fixed points, but infinitely many T-curves. What is the image of the localization map?

4. Cohomology of toric varieties

In this section we will study the situation when a variety X has a dense T-orbit, where T is a torus. This case allows us to reduce everything there is to know about $H^*_T(X)$ to the combinatorics of convex polyhedra. These varieties are quite special, but one cares about them much in the same way human doctors care about lab rat biology.

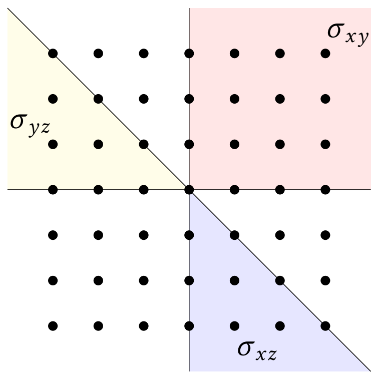
4.1. **Motivating example.** Let us begin by considering a familiar example. Take $X = \mathbb{P}^2$, together with an action of a two-dimensional torus T. We have one dense T-orbit, three T-curves, and three T-fixed points:



Let $[x:y:z]$ be homogeneous coordinates on $\mathbb{P}^2$. The projective plane is covered by three charts isomorphic to $\mathbb{C}^2$, which we obtain by throwing out each T-curve, i.e. by declaring each coordinate is non-zero:

$$\begin{aligned} U_{xy} &= \{z \neq 0\} = \operatorname{Spec} \mathbb{C}[X, Y], \quad X = x/z, Y = y/z, \\ U_{yz} &= \{x \neq 0\} = \operatorname{Spec} \mathbb{C}[y/x, z/x] = \operatorname{Spec} \mathbb{C}[YX^{-1}, X^{-1}], \\ U_{xz} &= \{y \neq 0\} = \operatorname{Spec} \mathbb{C}[XY^{-1}, Y^{-1}]. \end{aligned}$$

Note that each chart is a monoid ring of some monoid of monomials in $X^{\pm 1}$, $Y^{\pm 1}$. Let us identify $X^m Y^n$ with an integral point $(m, n) \in \mathbb{Z}^2$ on a plane, and draw these rings:

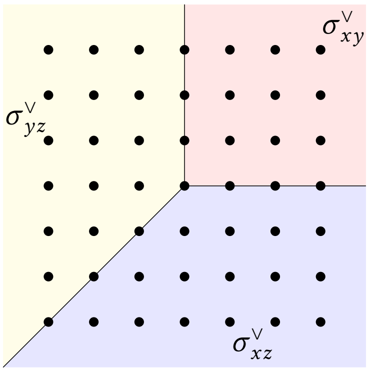


These are all polyhedral cones, which contain the origin, and the slopes of all faces are rational. In order to get an even nicer picture, let us take a dual of each cone:

$$\sigma^\vee = \{u \in \mathbb{Z}^2 \,:\, \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

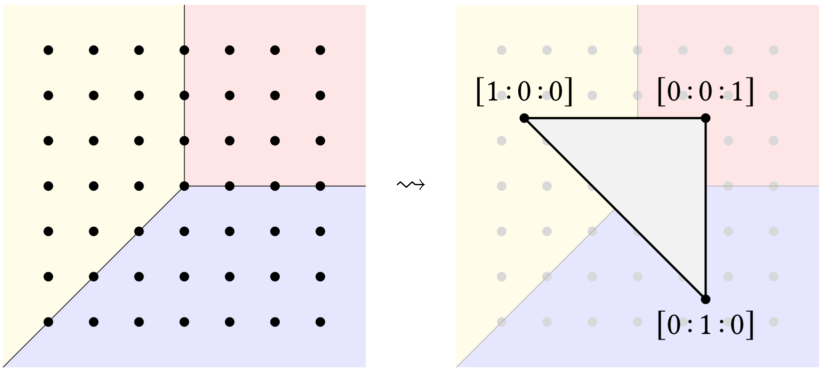
Let us denote $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then it is clear that the duals are as follows:

$$\sigma^\vee_{xy} = \sigma_{xy}, \quad \sigma^\vee_{yz} = \mathbb{Z}[e_2, -e_1 - e_2], \quad \sigma^\vee_{xz} = \mathbb{Z}[e_1, -e_1 - e_2].$$



$$\sigma_{xy}^\vee = \sigma_{xy}$$
$$\sigma_{yz}^\vee = \mathbb{Z}[e_2, -(e_1+e_2)]$$
$$\sigma_{xz}^\vee = \mathbb{Z}[e_1, -(e_1+e_2)]$$

In the picture above, the intersections of closed cones $\sigma_1^\vee \cap \sigma_2^\vee$ correspond to cones generated by unions $\langle \sigma_1^\vee, \sigma_2^\vee \rangle$, so we can easily read off how these charts glue. For instance, the origin is the open dense T-orbit, rays are $\mathbb{C}^* \times \mathbb{C}$ and so on. We can further identify every stratum with one particular T-orbit. This picture is somewhat confusing though, because the "biggest" orbit corresponds to the "smallest" stratum. We remedy this by taking the intersection of normal cones, which swaps dimension with codimension:



4.2. **Crash course into toric geometry.** It turns out we can play the same game with every (normal) variety with a dense T-orbit. The easiest place to make the definitions from is from the middle of our example, namely the partitions of the plane. We begin with $N = \mathbb{Z}^n$ a lattice.

Definition 4.1. A *fan* is a collection Δ of strongly convex rational polyhedral cones $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ with apex at the origin, such that every face of a cone in Δ lies in Δ, and the intersection of two cones $\sigma_1 \cap \sigma_2$ is a face of both of them.

Let us unwind this definition slightly. Polyhedral is just piecewise linear; *rational* means that the cone is generated by some vectors in N (so, of rational slope). Finally, *strongly convex* means that we disallow cones containing a whole line (because in this case, the dual cone would not have full dimension).

Definition 4.2. Let Δ be a fan, and $\sigma \in \Delta$ a cone. Consider the sub-monoid $S_\sigma \subset N^\vee =: M$, given by

$$S_\sigma = \sigma^\vee \cap M = \{u \in M \,:\, \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

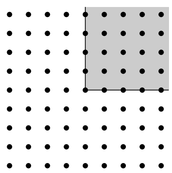
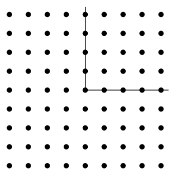
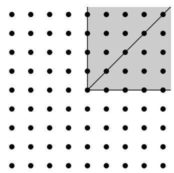
We call the affine variety $U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$ the *toric chart* associated to σ.

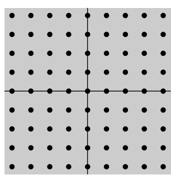
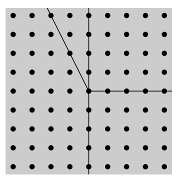
Note that by definition, for every inclusion of cones $\tau \subset \sigma$ we have a canonical open embedding of charts $U_\tau \subset U_\sigma$. Therefore all toric charts glue together, and we obtain the *toric variety* $X(\Delta)$ associated to Δ. One can show that this is a normal variety. Furthermore, we can identify the lattice M with the character lattice $T^\vee$ of a torus $T = (\mathbb{C}^*)^n$. The natural inclusions $\mathbb{C}[S_\sigma] \subset \mathbb{C}[M]$ give rise to $T^\vee$-grading on $\mathbb{C}[S_\sigma]$, and so a T-action on U_σ, together with an open dense free T-orbit $\operatorname{Spec} \mathbb{C}[M] \subset U_\sigma$. This glues together to a T-action on $X(\Delta)$.

Definition 4.3. A *toric variety* is a normal T-variety which contains an open dense free T-orbit.

Sumihiro linearization implies that every toric variety X is covered by (affine) toric charts. With some effort, one can show that there exists a finite open cover of X by affine toric charts such that all intersections of charts are affine. This implies that every toric X is of the form $X(\Delta)$, where Δ is a fan.

Exercise 4.4. Verify that the following fans give rise to the indicated varieties:


$\rightsquigarrow \mathbb{C}^2$,

$\rightsquigarrow \mathbb{C}^2 \setminus \{0\}$,

$\rightsquigarrow \mathrm{Bl}_0\mathbb{C}^2$,


$\rightsquigarrow \mathbb{P}^1 \times \mathbb{P}^1$,

$\rightsquigarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})(-a)$,

where the slope of diagonal line is $(1, 1)$ in the first row, and $(-1, a)$ in the second row.

Example 4.5. Let $\sigma = \mathbb{Z}[e_2, 2e_1 - e_2]$. Then $\sigma^\vee = \mathbb{Z}[e_1^\vee, e_1^\vee + 2e_2^\vee]$, and so the corresponding toric chart is

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X, XY, XY^2] = \mathbb{C}[x, y, z]/(y^2 - xz).$$

Observe that this variety is singular.

Exercise 4.6. How can one interpret the blow-up of the quadratic cone above in terms of toric varieties?

We can read off the fan when $X(\Delta)$ is smooth or projective.

Proposition 4.7. *$X(\Delta)$ is projective iff the fan Δ is complete, that is $\bigcup \sigma = N \otimes_{\mathbb{Z}} \mathbb{R}$. $X(\Delta)$ is smooth iff the fan Δ is non-singular, that is each cone $\sigma \in \Delta$ is generated by a part of basis of N.* $\square$

What is the T-invariant topology of $X(\Delta)$? In each toric chart, we only have finitely many T-orbits, parameterized by faces of the corresponding cone. The same observation holds after gluing.

Definition 4.8. Let $\tau \in \Delta$ a cone. On each $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$, consider the map

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M],$$
$$u \in \sigma^\vee \cap M \mapsto \begin{cases} u & \text{if } u \in \tau^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Define $V(\tau) \cap U_\sigma := \operatorname{Spec} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M]$. This glues to a closed subvariety $V(\tau) \subset X(\Delta)$.

The varieties $V(\tau)$, $\tau \in \Delta$ are the only closed T-invariant irreducible subvarieties of $X(\Delta)$. Note that $\dim_{\mathbb{R}} \tau = \operatorname{codim} V(\tau)$. In particular:

- n-dimensional cones in Δ correspond to T-fixed points in $X(\Delta)$;
- $(n-1)$-dimensional cones correspond to T-curves;
- rays correspond to divisors.

Moreover, we have

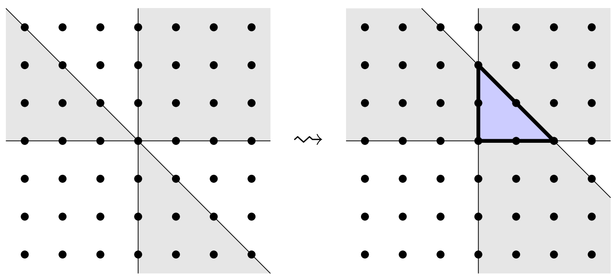
$$V(\sigma) \cap V(\tau) = \begin{cases} V(\gamma) & \text{if } \sigma, \tau \text{ together span a cone } \gamma \in \Delta, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.1}$$

4.3. **Polytopes and cohomology basis.** Let us now recover the "normal" picture.

Definition 4.9. A *lattice polytope* in $M_{\mathbb{R}}$ is the convex hull of a finite set of points in M. Given a lattice polytope P, we define the fan Δ_P via

$$\Delta_P = \{\sigma_F,\ F \text{ a face of } P\}, \qquad \sigma_F = \{v \,:\, \langle u', v\rangle \geq \langle u, v\rangle \text{ for all } u' \in P, u \in F\}.$$

Given a complete fan Δ, we can always find a lattice polytope P such that $\Delta = \Delta_P$; roughly speaking, we can recover P by intersecting the "translations" of duals of cones of maximal dimension:



Exercise 4.10. The fan Δ_P is non-singular if and only if at each vertex of P, the primitive vectors along the one-dimensional edges form a basis of M.

Let us fix a lattice polytope $P \subset M_{\mathbb{R}}$, and denote its vertices by $p_1, \ldots, p_k$. Pick a generic vector $v \in N$, such that all pairings $\langle p_i, v\rangle$ are distinct. Renumbering the vertices, let us assume that $\langle p_1, v\rangle < \ldots < \langle p_k, v\rangle$. Recall that each vertex p_i of the polytope corresponds to an n-dimensional cone $\sigma_i \in \Delta_P$. Define

$$\tau_i = \bigcap_{\substack{j>i \\ \dim(\sigma_i \cap \sigma_j) = n-1}} \sigma_i \cap \sigma_j.$$

Such collection of cones is called a *shelling* of Δ. Each τ_i gives rise to a cohomology class $[V(\tau_i)]$, and these classes form a basis of both $H^*(X(\Delta))$ and $H_T^*(X(\Delta))$.

Example 4.11. Consider $X = \mathbb{P}^2$, and pick the vector $v = (0, 1, 2)$. Then $V(\tau_2) = [0:0:1]$, $V(\tau_1) = [0:1:z]$, and $V(\tau_0) = [1:y:z]$.

Remark 4.12. Let $\mathbb{C}^*$ act on $\mathbb{P}^2$ with characters $(0, t, 2t)$. For every $L \in (\mathbb{P}^2)^{\mathbb{C}^*}$, consider the set $A(L)$ of all points $L' \in \mathbb{P}^2$ such that $\lim_{t\to 0} e^t(L') = L$. It is easy to check that $A([\mathbb{C}e_i]) = \tau_i$.

More generally, our identification $M = T^\vee$ induces the equality $N = \mathsf{Hom}(\mathbb{C}^*, T)$. Then the shelling of X for $v \in N$ is precisely the cellular decomposition obtained from Białynicki-Birula theorem for the action of the one-dimensional torus $v : \mathbb{C}^* \hookrightarrow T$.

4.4. **Stanley-Reisner ring.** Recall that every ray ρ in Δ corresponds to a T-invariant divisor $D_\rho := V(\rho)$ in $X(\Delta)$. Let us denote by $v_\rho \in N$ the indivisible integral generator of ρ.

By our construction, every $u \in M$ is a rational function on $X(\Delta)$. Another way to phrase this is that u defines a section s_u of a (topologically trivial) line bundle L_u of character u on $X(\Delta)$. In particular, by the properties of Chern classes

$$u = c_1(L_u) = [Z(s_u)] = \sum_{\text{rays } \rho\in\Delta} \langle u, v_\rho\rangle D_\rho.$$

This equality, together with (4.1), motivates the following definition. Let us consider a polynomial ring $\mathbb{Z}[\underline{x}] = \mathbb{Z}[x_\rho]$, where ρ runs through the set of all rays in Δ. We have the following ideals in $\mathbb{Z}[\underline{x}]$:

- I is generated by all monomials $x_{\rho_1} \dots x_{\rho_r}$ such that the rays $\rho_1, \dots, \rho_r$ do not span a cone;
- J is generated by all elements $\sum_\rho \langle u, \rho\rangle x_\rho$ with $u \in M$.

Definition 4.13. The ring $\mathbb{Z}[\underline{x}]/I$ is called the *Stanley-Reisner ring* of Δ.

Similarly, let $\Lambda = \Lambda_T$, and $\Lambda[\underline{x}] := \mathbb{Z}[\underline{x}] \otimes_{\mathbb{Z}} \Lambda$. Here, we have two similar ideals:

- I' has the same generators as I above;
- J' is generated by all elements $u - \sum_\rho \langle u, \rho\rangle x_\rho$ with $u \in M$.

Lemma 4.14. *The map $\mathbb{Z}[\underline{x}]/I \to \Lambda[\underline{x}]/(I' + J')$ is an isomorphism.*

Proof. We can identify Λ with $\mathbb{Z}[M]$. Given a basis $u_1, \dots, u_n$ of M, the elements

$$u_1 - \sum_i \langle u_1, \rho\rangle x_\rho, \dots, u_n - \sum_i \langle u_n, \rho\rangle x_\rho$$

form a regular sequence in $\Lambda[\underline{x}]$ and generate J', with quotient $\Lambda[\underline{x}]/J' = \mathbb{Z}[\underline{x}]$. Since $I' = I \cdot \Lambda[\underline{x}]$, we may conclude. □

By the considerations above, we have homomorphisms

$$\mathbb{Z}[\underline{x}]/(I + J) \to H^*(X), \qquad \Lambda[\underline{x}]/(I' + J') \to H^*_T(X),$$

which send each x_ρ to the class of the divisor $D_\rho \subset X(\Delta)$.

Theorem 4.15. *The maps above induce ring isomorphisms*

$$H^*(X) \simeq \mathbb{Z}[\underline{x}]/(I + J), \qquad H^*_T(X) \simeq \Lambda[\underline{x}]/(I' + J').$$

We will only give the proof for X smooth and projective (even though the statement holds for all toric varieties). Let us begin by introducing a certain intermediate ring.

Given a cone $\sigma \in \Delta$, its intersection with N generates a sublattice $N_\sigma \subset N$, and so dually a quotient $M \twoheadrightarrow M_\sigma$. If σ is spanned by the rays $\rho_1, \dots, \rho_k$, we can identify $\mathbb{Z}[M_\sigma]$ with Laurent polynomials in $x_{\rho_1}, \dots, x_{\rho_k}$. Let us denote by $\operatorname{Sym} M_\sigma \subset \mathbb{Z}[M_\sigma]$ the subring of polynomials in $\{x_{\rho_i} : i \in [1,k]\}$. With all these notations, we define the ring of *piecewise polynomial functions* on Δ:

$$PP(\Delta) = \{(f_\sigma)_{\sigma\in\Delta} \ : \ f_\sigma \in \operatorname{Sym} M_\sigma, f_\sigma|_\tau = f_\tau \text{ for all } \tau \subset \sigma\}.$$

Lemma 4.16. *We have an isomorphism $\mathbb{Z}[\underline{x}]/I \simeq PP(\Delta)$.*

Proof. With our identifications, we can write $PP(\Delta)$ as an inverse limit:

$$PP(\Delta) = \varprojlim_{\sigma\in\Delta, \sigma=\langle\rho_1,\dots,\rho_k\rangle} \mathbb{Z}[x_{\rho_1}, \dots, x_{\rho_k}],$$

where restriction to a face of a cone kills the variable corresponding to the missing ray. In particular, we obtain a natural map $\mathbb{Z}[\underline{x}] \twoheadrightarrow PP(\Delta)$. For each monomial belonging to I, its image in each $\operatorname{Sym} M_\sigma$ vanishes, therefore this map factors through $\mathbb{Z}[\underline{x}]/I \twoheadrightarrow PP(\Delta)$. On the other hand, take $f \notin I$. It contains a monomial $x_{\rho_1}^{i_1} \dots x_{\rho_k}^{i_k}$ such that $\sigma = \langle \rho_1, \dots, \rho_k\rangle$ is a cone in Δ. In particular, the image of f in $\mathbb{Z}[x_{\rho_1}, \dots, x_{\rho_k}]$ is non-zero, and so the map $\mathbb{Z}[\underline{x}]/I \to PP(\Delta)$ is an isomorphism.thm:SR-iso ☐

The following lemma is obvious from definitions.

Lemma 4.17. *The ring $PP(\Delta)$ can be rewritten in the following way:*

$$PP(\Delta) = \left\{ (f_\sigma)_{\substack{\sigma\in\Delta,\\ \dim\sigma=n}} \ : \ f_\sigma \in \operatorname{Sym} M, f_\sigma|_\tau = f_{\sigma'}|_\tau \textit{ for } \tau \subset \sigma\cap\sigma', \dim\tau = n-1 \right\}.$$

Proof of Theorem 4.15. First of all, note that we only need to prove the second isomorphism. Indeed, the first one follows immediately:

$$H^*(X(\Delta)) \simeq H_T^*(X(\Delta)) \otimes_\Lambda \mathbb{Z} \simeq \Lambda[\underline{x}]/(I' + J') \otimes_\Lambda \mathbb{Z} \simeq \mathbb{Z}[\underline{x}]/(I+J).$$

By Corollary 3.38, $H_T^*(X(\Delta))$ is a subring of $H_T^*(X(\Delta)^T) = \sum_{\sigma\in\Delta,\dim\sigma=n} \operatorname{Sym} M_\sigma$, consisting of tuples $(f_\sigma)_\sigma$ where $f_\sigma - f_{\sigma'}$ is divisible by u whenever there is a T-curve of character u between σ and σ'. The condition of having T-curve is equivalent to cones σ and σ' having a common face of dimension $n-1$. Let $\tau = \langle \rho_1, \dots, \rho_{n-1}\rangle$, $\sigma = \langle \tau, \rho_n\rangle$, $\sigma' = \langle\tau, \rho_n'\rangle$. Since $X(\Delta)$ is smooth, and so Δ is non-singular, we can identify $\operatorname{Sym} M_\sigma$ with $\mathbb{Z}[x_{\rho_1}, \dots, x_{\rho_{n-1}}, x_{\rho_n}]$, and $\operatorname{Sym} M_{\sigma'}$ with $\mathbb{Z}[x_{\rho_1}, \dots, x_{\rho_{n-1}}, x_{\rho_n'}]$. Under these notations the character u of the T-curve τ gets identified with x_{ρ_n} and $x_{\rho_n'}$ respectively. In particular, $f_\sigma - f_{\sigma'}$ is divisible by u if and only if $f_\sigma|_{x_{\rho_n}=0} = f_{\sigma'}|_{x_{\rho_n'}=0}$ in $\mathbb{Z}[x_{\rho_1}, \dots, x_{\rho_{n-1}}]$. This is precisely the defining condition of $PP(\Delta)$ by Lemma 4.17, and so

$$H_T^*(X(\Delta)) = PP(\Delta) = \mathbb{Z}[\underline{x}]/I$$

by Lemma 4.16. ☐

This description of the equivariant cohomology almost instantly yields the dimensions of $H_T^k(X(\Delta))$.

Proposition 4.18. *For $k \in [0,n]$, define*

$$C_k = \bigoplus_{\substack{\sigma\in\Delta,\\ \dim\sigma=k}} \mathbb{Z}[\underline{X}]/\langle x_\rho \ : \ \rho \not\subset \sigma\rangle.$$

Let $d_k : C_k \to C_{k-1}$ be the sum of restriction maps from cones to their codimension 1 faces, taken with the sign $(-1)^p$, where p is the index of the ray being thrown out. Then we have an exact sequence of $\mathbb{Z}[\underline{x}]/I$-modules

$$0 \to \mathbb{Z}[\underline{x}]/I \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \to 0.$$

Proof. Left as an exercise. □

Let us denote $P_t(X) = \sum_i \dim H^i(X)t^i$, $P_t^T(X) = \sum_i \dim H_T^i(X)t^i$ for any T-variety X.

Corollary 4.19. *Let a_k be the number of k-dimensional cones in $X = X(\Delta)$. Then*

$$P_t^T(X) = \sum_{i=0}^{n} \frac{(-1)^{n-i}a_i}{(1-t^2)^i} = \frac{P_t(X)}{(1-t^2)^n},$$

$$P_t(X) = \sum_{i=0}^{n} (-1)^{n-i}a_i(1-t^2)^i.$$

Assume $\Delta = \Delta_P$ for P a polytope satisfying the conditions of Exercise 4.10; note that the number f_{n-i} of i-dimensional faces of P is equal to a_i. Then the Poincaré duality for $X(\Delta)$ immediately yields the following combinatorial formulas for P, called *Dehn-Sommerville relations*:

$$h_k = h_{n-k}, \qquad h_k := \sum_{i=k}^{n} (-1)^{i-k}\binom{i}{k} f_{n-i-1}.$$

Remark 4.20. Theorem 4.15 imposes strong cohomological conditions on a variety to be toric. For example, we can easily see that $\mathrm{Gr}(2,4)$ cannot be toric. Indeed, cohomology of a toric variety is even and generated by degree 2 elements, whereas

$$H^*(\mathrm{Gr}(2,4)) \simeq \mathbb{Z}[c_1, c_2]/(c_1^3 - c_1c_2, c_2^2), \qquad \deg c_i = 2i$$

as per Exercise 2.16.

5. NilHecke algebra and abelianization

We have extensively studied equivariant cohomology for varieties X equipped with an action of an algebraic torus T. What about other algebraic groups G? In this section, we will mostly reduce the study of $H^*_G(X)$ to $H^*_T(X)$, where $T \subset G$ is a maximal torus; the upshot is that $H^*_G(X)$ is determined by $H^*_T(X)$, up to p-torsion for some small list of primes p, depending on the group G. This reduction is known under the name *abelianization.*

5.1. Linear algebraic groups.

Let us begin by summarizing some notions of the theory of linear algebraic groups. We will omit all of the proofs.

Definition 5.1. Let $G \subset GL(V)$ be a connected linear algebraic group. Denote by $G^{(1)}$ its commutator $[G,G] := \langle ghg^{-1}h^{-1} \,:\, g,h \in G\rangle$, and consider the *derived series* of G:

$$G \supset G^{(1)} \supset G^{(2)} \supset \dots, \qquad G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

If $G^{(i)}$ is the trivial group for $i \gg 0$, we say that G is *solvable.* Furthermore, if each quotient $G^{(i+1)}/G^{(i)}$ is isomorphic to an additive group $\mathbb{C}^{k_i}$, we say that G is *unipotent.*

Solvable groups have very few irreducible representations, as evidenced by the following theorem:

Theorem 5.2 (Lie). *Let G be a solvable group, and $G \to GL(V)$ a representation. Then there exists a G-stable line $L \subset V$. If G is unipotent, then the action of G on L is trivial.* $\square$

Corollary 5.3. *Let $G \subset GL(V)$ be solvable. Then there exists a basis of V, such that any $g \in G$ is represented by an upper-triangular matrix. If G is unipotent, all these matrices have 1's on the diagonal.*

Proposition 5.4. *Every linear algebraic group G has the unique maximal normal unipotent (resp. solvable) subgroup, which we denote by $R_u(G)$ (resp. $R(G)$).* $\square$

We say that G is *reductive* if $R_u(G) = \{e\}$, and G is *semisimple* if $R(G) = \{e\}$. By definition, any G linear algebraic contracts to $G/R_u(G)$; in particular, $H^*_G(X) = H^*_{G/R_u(G)}(X)$. Thus for our purposes it suffices to only consider reductive algebraic groups. Furthermore, any reductive group G can be written as $G = (G' \times T')/\Gamma$, where G' is semisimple, T' an algebraic torus, and Γ a finite group. In order to streamline the presentation, we will only work with semisimple groups in this subsection.

Let G be a connected semisimple group, and pick a maximal connected solvable subgroup $B \subset G$; we will call B a *Borel* subgroup. Further, let $T \subset B$ be the maximal torus, and denote by $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ the corresponding Lie algebras. We write $M = T^\vee$ for the character lattice of T.

Definition 5.5. A *root* of G is a non-zero character appearing in the adjoint action $T \curvearrowright \mathfrak{g}$. We denote the (finite) set of roots by Δ.

By definition, we have the following decomposition of vector spaces:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the subspace where T acts via α.

Exercise 5.6. Check that for any $\alpha, \beta \in \Delta$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Proposition 5.7. *Each $\mathfrak{g}_\alpha$ is 1-dimensional.* □

Let us write $\Delta^+ \subset \Delta$ for the roots occurring in $\mathfrak{b}$. One can show that $\Delta^- := \Delta \setminus \Delta^+ = -\Delta^+$; because of this we call Δ^+ the set of *positive* roots. By definition, $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$; in fact, there exists the *opposite* Borel subgroup $B^- \subset G$ such that $\mathfrak{b}^- = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$.

We denote $M_{\mathrm{rt}} := \mathbb{Z}\langle \Delta \rangle \subset M$, and call it the *root lattice*. There exists a $\mathbb{Z}$-basis $\mathcal{S}$ of M_{rt} consisting of positive roots; we call elements of $\mathcal{S}$ *simple roots*.

Dually, let us write $M^\vee = \mathrm{Hom}(M, \mathbb{Z})$ for the set of 1-dimensional subgroups of T, and $M^\vee_{\mathbb{R}} := M^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. For each $\alpha \in \Delta$, there exists the unique *coroot* $\alpha^\vee \in M^\vee_{\mathbb{R}}$, defined by the following conditions:

$$\langle \alpha, \alpha^\vee \rangle = 2, \qquad \langle \beta, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall \beta \in \Delta.$$

We denote by $M_{\mathrm{wt}} \subset M_{\mathbb{R}}$ the dual lattice of $\mathbb{Z}\langle \alpha^\vee : \alpha \in \Delta \rangle$, and call it the *weight lattice*. We have inclusions $M_{\mathrm{rt}} \subset M \subset M_{\mathrm{wt}}$. We have isomorphism of groups $M_{\mathrm{wt}}/M = \pi_1(G)$ and $M/M_{\mathrm{rt}} = Z(G)$. In general, both M_{rt} and M_{wt} only depend on $\mathfrak{g}$.

Example 5.8. Let $G = SL_2$. In this case, $B = \begin{pmatrix} t & * \\ 0 & t^{-1} \end{pmatrix}$, $T = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $\mathfrak{g} = \mathfrak{sl}_2 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. We have $M = \mathbb{Z}\langle \omega \rangle$, where ω is the character of weight 1. Note that

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & -a \end{pmatrix},$$

so that $\mathfrak{g} = \mathfrak{g}_{-2\omega} \oplus \mathfrak{t} \oplus \mathfrak{g}_{2\omega}$. In particular, the root lattice is $M_{\mathrm{rt}} = \mathbb{Z}\langle \alpha \rangle$, $\alpha = 2\omega$. On the other hand, the coroot is defined by $\langle \alpha^\vee, \alpha \rangle = 2$, which means that the weight lattice is $M_{\mathrm{wt}} = \mathbb{Z}\langle \omega \rangle$. We have $M_{\mathrm{rt}} \subset M = M_{\mathrm{wt}}$.

Now, let us look at $G = PSL_2 = SL_2/\{\pm 1\}$. The maximal torus here is $T' = T/\{\pm 1\}$, which implies that $M = \mathbb{Z}\langle 2\omega \rangle$. In particular, in this case we have $M_{\mathrm{rt}} = M \subset M_{\mathrm{wt}}$.

In fact, the whole theory of reductive groups is built out of the fundamental example above. The critical result is

Theorem 5.9 (Jacobson-Morozov)**.** *For any nilpotent $x \in \mathfrak{g}$, there exists an inclusion of Lie algebras $\mathfrak{sl}_2 \subset \mathfrak{g}$, such that $x \in \mathfrak{sl}_2$.*

This produces an $\mathfrak{sl}_2$ for each positive root α, and the rest of the theory is roughly about how these $\mathfrak{sl}_2$'s interact with each other.

Example 5.10. Let $G = SL_n$, B upper-triangular matrices, T diagonal matrices. Denote by t_i the character that picks out i-th entry on the diagonal, then $M = (\mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n)/(t_1 + \cdots + t_n)$. We have:

$$\begin{aligned} \Delta &= \{(t_i - t_j) : 1 \le i \ne j \le n\}, \\ \Delta^+ &= \{(t_i - t_j) : 1 \le i < j \le n\}, \\ \mathcal{S} &= \{(t_i - t_{i+1}) : 1 \le i < n\}. \end{aligned}$$

We have $M = M_{\mathrm{wt}}$, and $M/M_{\mathrm{rt}} \simeq \mathbb{Z}/n\mathbb{Z}$, with the generator t_1.

5.2. **Weyl group.**

Definition 5.11. The *Weyl group* of G is $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T inside G.

The Weyl group W acts on M by conjugating T. This action is faithful; moreover, W preserves Δ and is generated by *simple reflections*:

$$\forall \alpha \in \Delta^+ : \qquad s_\alpha : \beta \mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha.$$

Denoting $\mathcal{S} = \{\alpha_1, \dots, \alpha_r\}$, one can also show that W is generated by s_α, $\alpha \in \mathcal{S}$. Since W acts faithfully on Δ, it is itself finite.

Example 5.12. Let $G = SL_2$. In this case, T is normalized by $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix} \right\}$, therefore $W = \mathfrak{S}_2$. Similarly, for $G = GL_n$ we have $W = \mathfrak{S}_n$.

Remark 5.13. While it so happens that $\mathfrak{S}_n \subset GL_n$, in general $W \not\subset G$!

In a certain sense, Weyl group W is the combinatorial core of G.

Theorem 5.14 (Bruhat decomposition)**.** *We have a decomposition $G = \bigsqcup_{w \in W} BwB$ of G into locally closed subvarieties.*

It is known that all Borels in G are conjugate; moreover, $N_G(B) = B$. This means that the set of all Borels in G acquires a structure of a homogeneous space $\mathcal{F} = G/B$. This is a projective variety, and we call it the *flag variety* of G. We have an action $T \curvearrowright \mathcal{F}$ by left multiplication; one can check that $\mathcal{F}^T \simeq W$. Furthermore, applying Białynicki-Birula theorem to $\mathcal{F}$ recovers precisely the Bruhat decomposition $\mathcal{F} = \bigsqcup_{w \in W} Bw$.

Remark 5.15. The flag varieties $\mathcal{F}$ are examples of *spherical varieties*, that is G-varieties with an open dense B-orbit. This notion is a rich generalization of toric varieties. Spherical varieties can also be classified in terms of some combinatorial data, which is more involved (coloured fans).

Any reductive group G over $\mathbb{C}$ contains a (real) maximal compact subgroup $K \subset G$, which is unique up to conjugation. It is known that G homotopically retracts to K.

Example 5.16. For $G = SL_n$, $K = SU_n$. Similarly, for $G = GL_n$, $K = U_n$.

Let us intersect all of our subgroups with K. We have $K \cap T = S = (\mathbb{S}^1)^r$ the maximal torus in K, and moreover $K \cap B = K \cap T$. In particular,

$$\mathcal{F} = G/B \simeq K/S.$$

Since $W = N_G(T)/T = N_K(S)/S$, we have a natural, albeit non-algebraic (see Remark 5.19), action of W on $K/S \simeq \mathcal{F}$ by multiplication on the *right*. This induces an action $W \curvearrowright H^*(\mathcal{F})$.

5.3. **Abelianization over $\mathbb{Q}$.** We finally have all pieces in place to state the first version of abelianization theorem. For now, let us restrict to equivariant cohomology with coefficients in $\mathbb{Q}$. Let X be a G-variety; then we have

$$H^*_T(X) \simeq H^*_B(X) \simeq H^*_G(G \times_B X) \simeq H^*_G(G/B \times X) \simeq H^*_K(K/S \times X).$$

Since W acts on K/S on the right, this action commutes with the K-action, and so we obtain an action $W \curvearrowright H^*_T(X)$.

Theorem 5.17 (Borel). *The map $H^*_G(X,\mathbb{Q}) \to H^*_T(X,\mathbb{Q})^W$ is a ring isomorphism.*

We begin with a topological lemma, which is the only place where we use the $\mathbb{Q}$-coefficients:

Lemma 5.18. *Let W be a finite group acting freely on a topological space Y. Then we have $H^*(Y/W,\mathbb{Q}) \simeq H^*(Y,\mathbb{Q})^W$.*

Proof. Interpreting H^* as simplicial cohomology, we can pick a chain complexes $C^\bullet(Y)$, $C^\bullet(Y/W)$ computing cohomology in such a way that $C^\bullet(Y/W) = C^\bullet(Y)^W$. Over $\mathbb{Q}$, any representation of a finite group W splits into isotypical components. Therefore we have

$$C^\bullet(Y) = \bigoplus_{\chi\,:\,\text{irrep of } W} \chi \otimes C^\bullet_\chi(Y),$$

where W acts trivially on all $C^\bullet_\chi(Y)$. In particular, $C^\bullet(Y)^W = \mathrm{triv} \otimes C^\bullet_{\mathrm{triv}}(Y)$, and applying cohomology we see that $H^*(Y/W,\mathbb{Q}) = H^*(C^\bullet(Y)^W) = H^*(C^\bullet(Y))^W = H^*(Y,\mathbb{Q})^W$. □

Proof of Theorem 5.17. Let us first consider the action of W on K/S. It is free with quotient $K/N_K(S)$; therefore applying Lemma 5.18 we have $H^*(K/N_K(S)) = H^*(K/S)^W$. By Bruhat decomposition, the cohomology of $\mathcal{F} = K/S$ is even and the Euler characteristic is $\chi(K/S) = \#W$. Thus

$$\chi(K/N_K(S)) = \frac{1}{\#W}\chi(K/S) = \frac{\#W}{\#W} = 1,$$

and so by evenness $H^*(K/N_K(S)) = \mathbb{Q}$, that is $K/N_K(S)$ is acyclic.

Now consider the fibration $\mathbb{E}K \times^{N_K(S)} X \to \mathbb{E}K \times^K X$ with fiber $K/N_K(S)$. Since $K/N_K(S)$ is acyclic, the pullback along this map induces an isomorphism on cohomology by Leray spectral sequence:

$$H^*_G(X) = H^*_K(X) = H^*(\mathbb{E}K \times^K X) = H^*(\mathbb{E}K \times^{N_K(S)} X).$$

Finally, the map $\mathbb{E}K \times^S X \to \mathbb{E}K \times^{N_K(S)} X$ is a finite cover with deck transformation group W, so by Lemma 5.18 we have

$$H^*(\mathbb{E}K \times^{N_K(S)} X) = H^*(\mathbb{E}K \times^S X)^W = H^*_S(X)^W = H^*_T(X)^W. \qquad \square$$

Remark 5.19. Let $G = SL_2$. Then $K = SU_2 \simeq \mathbb{S}^3$, $S = \mathbb{S}^1$, and $K \to K/S \simeq \mathbb{S}^2$ is the Hopf fibration. Embedding $\mathbb{S}^2$ into $\mathbb{R}^3$, the action of $W = \mathbb{Z}/2\mathbb{Z}$ is given by $(x,y,z) \mapsto (-x,-y,-z)$; in particular, $K/N_K(S) \simeq \mathbb{RP}^2$. Recall that $H^2(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$, and so $\mathbb{RP}^2$ only becomes acyclic after inverting 2.

The only place where we have used $\mathbb{Q}$-coefficients is Lemma 5.18, but there we had to invert $\#W$, which can be quite big. For example, for $G = GL_n$ we have to invert $n!$, but we know that at least for $X = \mathrm{pt}$ we have $\Lambda_G = (\Lambda_T)^W$ over $\mathbb{Z}$. Can we do better? The answer is yes, but for that we need to replace the non-algebraic action of W by algebraic action of a slightly bigger algebra.

5.4. **Convolution algebras.**

Let us begin with an abstract setup. Take a smooth projective G-variety F, and consider the space $\mathbf{H} = H^*_G(F \times F)$. We have the following correspondence:

$$\begin{tikzcd} & F \times F \times F \arrow[dl, "q"'] \arrow[dr, "p"] & \\ (F \times F) \times (F \times F) & & F \times F \end{tikzcd} \qquad \begin{aligned} q(f_1, f_2, f_3) &= (f_1, f_2, f_2, f_3), \\ p(f_1, f_2, f_3) &= (f_1, f_3). \end{aligned}$$

We treat the map q as being equivariant with respect to the diagonal group embedding $G \hookrightarrow G \times G$. Composing pullback along q with Gysin pushforward along p, we get a map $m : \mathbf{H} \otimes \mathbf{H} \to \mathbf{H}$.

Proposition 5.20. *The map m is an associative product.*

Proof. Consider the following diagram with obvious maps:

$$\begin{tikzcd} & & & F^4 \arrow[dl, "q"] \arrow[dr, "p"'] \arrow[dll, bend right, "q'"'] \arrow[drr, bend left, "p'"] & & \\ & & F^2 \times F^3 \arrow[dl, "q"] \arrow[dr, "p"'] & & F^3 \arrow[dl, "q"] \arrow[dr, "p"'] & \\ & F^2 \times F^2 \times F^2 & & F^2 \times F^2 & & F^2 \end{tikzcd}$$

Given $x, y, z \in H^*_G(X \times X)$, using base change we can write the product $x(yz)$ as

$$x(yz) = p_* q^* p_* q^*(x \otimes y \otimes z) = p'_*(q')^*(x \otimes y \otimes z).$$

Writing a similar diagram for $(xy)z$, we obtain that

$$(xy)z = p'_*(q')^*(x \otimes y \otimes z) = x(yz),$$

which proves associativity. □

Exercise 5.21. Check that that m is Λ_G-linear, and the class of diagonal $[\Delta] \in H^{2\dim F}_G(F \times F)$ is the unit in $\mathbf{H}$. In particular, m is different from cup product.

Similarly, let X be a G-variety. An analogous correspondence defines a left $\mathbf{H}$-module structure on $H^*_G(F \times X)$:

$$\begin{tikzcd} & F \times F \times X \arrow[dl, "q"'] \arrow[dr, "p"] & \\ (F \times F) \times (F \times X) & & F \times X \end{tikzcd}$$

5.5. **NilHecke algebras.** Let us apply our general setup to the case $F = \mathcal{F} = G/B$.

Definition 5.22. The *NilHecke algebra* of G is the associative algebra

$$\mathbf{NH}_G := H^*_G(G/B \times G/B).$$

Now let X be a G-variety. Then $\mathbf{NH}_G$ acts on $H^*_G(G/B \times X)$ by the above, and moreover

$$H^*_G(G/B \times X) = H^*_G(G \times_B X) = H^*_B(X) = H^*_T(X).$$

Thus we have obtained an algebraic replacement for the Weyl group action in Section 5.3.

Let us compute the NilHecke algebra for $G = GL_2$. In this particular case, we can easily reduce the computation in H^*_G to $(H^*_T)^W$.

We have $\mathcal{F} = \mathbb{P}^1$, $G = GL_2$ acts by $g(\mathbb{C}v) = \mathbb{C}g(v)$ for any $v \in \mathbb{C}^2 \setminus 0$, and $\mathbf{NH} := \mathbf{NH}_{GL_2} = H^*_G(\mathbb{P}^1 \times \mathbb{P}^1)$. Let us check that in this particular case, we have $H^*_G(\mathbb{P}^1 \times \mathbb{P}^1) = H^*_T(\mathbb{P}^1 \times \mathbb{P}^1)^W$. It is clear that $\mathbb{P}^1 \times \mathbb{P}^1$ has two G-orbits: the diagonal $\Delta \simeq \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, and the complement $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$. Since $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \xrightarrow{\mathrm{pr}_1} \mathbb{P}^1$ is an affine bundle, by homotopy invariance we have

$$H^*_G((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta) \simeq H^*_G(\mathbb{P}^1) \simeq H^*_B(\mathrm{pt}) = \Lambda_T.$$

Since Λ_T is even, the long exact sequence in cohomology splits into short exact sequences

$$0 \to H^{*-2}_G(\Delta) \to H^*_G(\mathbb{P}^1 \times \mathbb{P}^1) \to H^*_G((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta) \to 0,$$

and so it suffices to check that $H^*_G(\mathbb{P}^1) \simeq H^*_T(\mathbb{P}^1)^{\mathfrak{S}_2}$.

Let us recal the GKM description of the T-equivariant cohomology of $\mathbb{P}^1$:

$$\begin{aligned} H^*_T(\mathbb{P}^1) &= \{(f_0, f_\infty) \in \Lambda_T \oplus \Lambda_T \,:\, (f_\infty - f_0) \text{ is divisible by } (t_1 - t_2)\} \\ &= \{(f_0, f_0 - (t_1 - t_2)h) \,:\, f_0, h \in \Lambda_T\}. \end{aligned}$$

How do we see the $\mathfrak{S}_2$-invariant part? Note that on the T-fixed points, the natural action of $N_G(T)$ factors through the Weyl group $W = N_G(T)/T$. In our case, $W = \mathfrak{S}_2 = \langle s \rangle$; s swaps t_1 with t_2, and L_0 with L_∞. Therefore,

$$H^*_T(\mathbb{P}^1)^{\mathfrak{S}_2} = \{(f_O, f_0 - (t_1 - t_2)h) \,:\, f_0, h \in \Lambda_T, s(f_0) = f_0 - (t_1 - t_2)h\},$$

or equivalently

$$\frac{f_0 - s(f_0)}{t_1 - t_2} \in \Lambda_T.$$

Lemma 5.23. *For any $f \in \Lambda_T = \mathbb{Z}[t_1, t_2]$, the element $\partial(f) := \frac{f - s(f)}{t_1 - t_2}$ belongs to Λ_T.*

Proof. On monomials $t_1^m t_2^n$, $m \geq n$, we have

$$\partial(t_1^m t_2^n) = (t_1 t_2)^n \frac{t_1^{m-n} - t_2^{m-n}}{t_1 - t_2} = t_1^{m-1} t_2^n + t_1^{m-2} t_2^{n+1} + \ldots + t_1^n t_2^{m-1}.$$

For other polynomials, we conclude by linearity of ∂. □

We call $\partial : \Lambda_T \to \Lambda_T$ the *Demazure operator.*

Exercise 5.24. Check that $\partial(fg) = s(f)\partial(g) + g\partial(f) = f\partial(g) + s(g)\partial(f)$.

We conclude that

$$H^*_T(\mathbb{P}^1)^{\mathfrak{S}_2} \simeq H^*_T(\{L_0\}) = H^*_G(\mathbb{P}^1),$$

and so we may work with $H^*_T(\mathbb{P}^1 \times \mathbb{P}^1)$ instead of $H^*_G(\mathbb{P}^1 \times \mathbb{P}^1)$.

We compute this algebra via its natural action on $H^*_T(\mathbb{P}^1)$. Consider the following localization diagram:

$$\begin{tikzcd} H^*_T(\mathbb{P}^1 \times \mathbb{P}^1) \otimes H^*_T(\mathbb{P}^1) \arrow[r, "\Delta^*_{23}"] \arrow[d, "i^*"] & H^*_T(\mathbb{P}^1 \times \mathbb{P}^1) \arrow[r, "\mathrm{pr}_{1*}"] \arrow[d, "i^*"] & H^*_T(\mathbb{P}^1) \arrow[d, "i^*"] \\ \Lambda_T^{(\mathbb{P}^1 \times \mathbb{P}^1)^T} \otimes \Lambda_T^{(\mathbb{P}^1)^T} \arrow[r] & \Lambda_T^{(\mathbb{P}^1 \times \mathbb{P}^1)^T} \arrow[r, "p_*"] & \Lambda_T^{(\mathbb{P}^1)^T} \end{tikzcd}$$

The left square clearly commutes; for the right square integration formula yields

$$i^* p_{1*}(\alpha) = p_* i^* \left(\frac{\alpha}{t_1 - t_2} \right). \tag{5.1}$$

Thus in order to compute the composition in the top row, it is enough to localize to T-fixed and compute it on the bottom row instead. The convolution algebra there is nothing other than (2×2)-matrix algebra; however, we need to take into account the additional Euler class from (5.1). Let us only consider the $\mathfrak{S}_2$-invariant part. We know that $H^*_G(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by $[\Delta]$, $[\mathbb{P}^1 \times \mathbb{P}^1]$ over Λ_T; by equivariant localization to fixed points we obtain

$$[\mathbb{P}^1 \times \mathbb{P}^1] \begin{pmatrix} f \\ s(f) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f \\ s(f) \end{pmatrix} = \begin{pmatrix} \frac{f - s(f)}{t_1 - t_2} \\ \frac{s(f) - f}{t_2 - t_1} \end{pmatrix} = \begin{pmatrix} \partial f \\ \partial f \end{pmatrix},$$

$$[\Delta] \begin{pmatrix} f \\ s(f) \end{pmatrix} = \begin{pmatrix} t_1 - t_2 & 0 \\ 0 & t_2 - t_1 \end{pmatrix} \begin{pmatrix} f \\ s(f) \end{pmatrix} = \begin{pmatrix} \frac{f(t_1 - t_2)}{t_1 - t_2} \\ \frac{s(f)(t_2 - t_1)}{t_2 - t_1} \end{pmatrix} = \begin{pmatrix} f \\ s(f) \end{pmatrix}.$$

We can check that the action of $\mathbf{NH}$ on $H^*_G(\mathbb{P}^1)$ is faithful. To sum up, we obtained that

$$\mathbf{NH}_{GL_2} = \langle \Lambda_T, \partial \rangle \subset \mathsf{End}(\Lambda_T).$$

The action of $\mathbf{NH}_{GL_2}$ is clearly Λ_{GL_2}-linear, so in fact $\mathbf{NH}_{GL_2} \subset \mathsf{End}_{\Lambda_{GL_2}}(\Lambda_T)$. Note that Λ_T is a free Λ_{GL_2}-module of rank 2, with a basis $\{1, t_1\}$. On the other hand, $\mathbf{NH}_{GL_2}$ is a free Λ_{GL_2}-module of rank 4 with basis $\{1, t_1, \partial, t_1\partial\}$. Writing the action of these operators in matrix form, we obtain

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 & -t_1 t_2 \\ 1 & t_1 + t_2 \end{pmatrix}, \quad \partial = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_1 \partial = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows that the inclusion map is onto, and so

$$\mathbf{NH}_{GL_2} \simeq \mathrm{Mat}_{2\times 2}(\Lambda_{GL_2}).$$

Remark 5.25. Note that $\mathfrak{S}_2 \hookrightarrow \mathbf{NH}_{GL_2}$, $s \mapsto 1 - (t_1 - t_2)\partial$, and this inclusion is compatible with the action on Λ_T. In particular, if we invert $(t_1 - t_2)$, we obtain

$$\mathbf{NH}_{GL_2}[(t_1 - t_2)^{-1}] \simeq \mathfrak{S}_2 \ltimes \Lambda_T[(t_1 - t_2)^{-1}].$$

5.6. **NilHecke for** GL_n**.** Let us generalize the above example to $G = GL_n$. In this case, $\mathcal{F}$ is the variety of complete flags:

$$\mathcal{F} = GL_n/B = \{0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \,:\, \dim V_i = i\}.$$

Recall that we have Bruhat decomposition $G/B = \bigsqcup_{w \in \mathfrak{S}_n} Bw$. Equivalently, G-orbits in $G/B \times G/B$ are parameterized by elements of $\mathfrak{S}_n$. In order to describe what these orbits look like, let us assign some discrete data to each pair $(V_\bullet, W_\bullet)$ of complete flags in $\mathbb{C}^n$. For any $1 \leq i, j \leq n-1$, define

$$d_{ij} = \dim(V_i \cap W_j).$$

Note that these numbers are non-decreasing in i and j. Moreover, counting codimensions we see that

$$\max(i + j - n, 0) \leq d_{ij} \leq \min(i, j).$$

Let us define the set of $(n \times n)$-matrices with integer coefficients satisfying the two conditions above by $\mathcal{M}$.

Exercise 5.26. The map

$$\mathfrak{S}_n \to \mathcal{M}, \quad \sigma \mapsto (\{k \leq j \,:\, \sigma(k) \leq i\})_{1 \leq i,j \leq n-1}$$

is well-defined, and is a bijection.

Example 5.27. Let $n = 2$. Then the identity in $\mathfrak{S}_2$ goes to $\big(1\big)$, and s goes to $\big(0\big)$.

Let us consider the (closed) Bruhat cell $\overline{Z}_{s_i} \subset \mathcal{F} \times \mathcal{F}$, which corresponds to the transposition $s_i = (i\; i+1)$, $1 \le i \le n-1$. By the bijection above, we have

$$\overline{Z}_{s_i} = \left\{ (V_\bullet, W_\bullet) \,:\, V_j = W_j \text{ for } j \neq i \right\}.$$

In other words, $\overline{Z}_{s_i}$ can be seen as a fiber product:

$$\begin{tikzcd} \overline{Z}_{s_i} \arrow[r, equal] & \mathcal{F} \times_{\mathcal{F}_i} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_i \arrow[d, "\Delta"] \\ & \mathcal{F} \times \mathcal{F} \arrow[r] & \mathcal{F}_i \times \mathcal{F}_i \end{tikzcd} \tag{5.2}$$

where

$$\mathcal{F}_i := \{0 \subset V_1 \subset \cdots \subset V_{i-1} \subset V_{i+1} \subset \cdots \subset V_n \,:\, \dim V_k = k\}.$$

The fiber of $\overline{Z}_{s_i} \to \mathcal{F}_i$ at each point is $\mathbb{P}^1 \times \mathbb{P}^1$, which parameterizes pairs of lines in V_{i+1}/V_{i-1}. Moreover, we have a natural fiberwise GL_2-action on $\overline{Z}_{s_i}$. Therefore, we are in a "relative" version of the situation for GL_2. Essentially the same computation yields that $[\overline{Z}_{s_i}]$ acts on $H^*_G(\mathcal{F}) = \mathbb{Z}[t_1, \dots, t_n]$ by

$$\partial_i(f) = \frac{f - s_i(f)}{t_i - t_{i+1}}.$$

Exercise 5.28. Check the following relations between ∂_i, $1 \le i \le n-1$:

$$\partial_i^2 = 0, \quad \partial_i\partial_j = \partial_j\partial_i \text{ if } |i-j| > 1, \quad \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}.$$

These relations are almost the same as the defining relations between $s_i \in \mathfrak{S}_n$, save for $\partial_i^2 = 0$. In particular, for any reduced expression $w = s_{i_1} \dots s_{i_r} \in \mathfrak{S}_n$ we can define an operator $\partial_w := \partial_{i_1} \dots \partial_{i_r}$, which is independent of the presentation of w.

Proposition 5.29. *For any $w \in \mathfrak{S}_n$, we have $[\overline{Z}_w] = \partial_w$ as an operator on $H^*_G(\mathcal{F})$.*

Sketch of proof. Consider ∂_w as an operator on $H^*_G(\mathcal{F})$. By definition, it is obtained from the following chain of correspondences:

$$\begin{tikzcd} & \overline{Z}_{s_1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \overline{Z}_{s_r} \arrow[dl] \arrow[dr] & \\ \mathcal{F} & & \mathcal{F} & & \mathcal{F} & & \mathcal{F} \end{tikzcd}$$

All maps here are $\mathbb{P}^1$-bundles, so each fiber product has correct dimension. Taking all fiber products and applying base change, we see that ∂_w is also defined by the correspondence

$$\begin{tikzcd} & \overline{Z}_{s_1} \times_{\mathcal{F}} \dots \times_{\mathcal{F}} \overline{Z}_{s_r} \arrow[dl] \arrow[dr] & \\ \mathcal{F} & & \mathcal{F} \end{tikzcd}$$

Let us write $\widetilde{Z}_w := \overline{Z}_{s_1} \times_{\mathcal{F}} \dots \times_{\mathcal{F}} \overline{Z}_{s_r}$, or equivalently $\widetilde{Z}_w := \mathcal{F} \times_{\mathcal{F}_{i_1}} \dots \times_{\mathcal{F}_{i_r}} \mathcal{F}$. This is called the *Bott-Samelson variety* of w. It is known that $\widetilde{Z}_w$ surjects onto the closure $\overline{Z}_w$ of Z_w in $\mathcal{F} \times \mathcal{F}$, and this surjection is one-to-one over Z_w. In particular, we have the following diagram:

$$\begin{tikzcd}
 & \widetilde{Z}_w \arrow[dl, "q"'] \arrow[d, "\pi"] \arrow[dr, "p"] & \\
\mathcal{F} & \overline{Z}_w \arrow[l, "q'"'] \arrow[r, "p'"] & \mathcal{F}
\end{tikzcd}$$

By projection formula, we have

$$p_* q^*(\alpha) = p'_* \pi_* \pi^*(q')^*(\alpha) = p'_*(q')^*(\alpha),$$

and so we may conclude. □

Corollary 5.30. *The algebra* $\mathbf{NH}_{GL_n}$ *is a free* Λ_T*-module with a basis* ∂_w*,* $w \in \mathfrak{S}_n$*.*

Let $w_0 \in \mathfrak{S}_n$ be the longest element, that is the permutation sending 1 to n, 2 to $(n-1)$ and so on. Define

$$S_{w_0} := t_1^{n-1} t_2^{n-2} \dots t_{n-1}; \qquad S_w := \partial w^{-1} w_0 S_{w_0} \text{ for } w \in \mathfrak{S}_n.$$

These are called *Schubert polynomials.*

Proposition 5.31. *The Schubert polynomial* S_w*,* $w \in \mathfrak{S}_n$ *give a basis of* $\mathbb{Z}[t_1, \dots, t_n]$ *over* $\mathbb{Z}[t_1, \dots, t_n]^{\mathfrak{S}_n}$*.*

Proof. Order the monomials in each S_w lexicographically. The highest term of S_w is $t_1^{a_1} \dots t_{n-1}^{a_{n-1}}$, where $a_i = \#\{j > i \,:\, w(j) < w(i)\}$. Since the lowest terms form a basis, Schubert polynomials form a basis as well. □

Corollary 5.32. *We have* $\mathbf{NH}_{GL_n} \simeq \mathrm{Mat}_{n\times n}(\mathbb{Z}[t_1, \dots, t_n]^{\mathfrak{S}_n})$*.*

Proof. Omitted, see Proposition 5.46 for the general case. □

5.7. **General group.** Let us now sketch what happens for an arbitrary reductuve group G. In this case, we have

$$\mathbf{NH}_G = \Lambda_T\langle \partial_\alpha \,:\, \alpha \in \mathcal{S}\rangle \subset \mathsf{End}_{\Lambda_T^W}(\Lambda_T), \qquad \partial_\alpha = \frac{P - s_\alpha(P)}{\alpha},$$

where $\alpha \in \mathcal{S}$ is a simple reflection.

Remark 5.33. For any weight $\beta \in T^\vee$, we have $\partial_\alpha \beta = \langle \alpha^\vee, \beta\rangle$.

A similar argument gives rise to a basis ∂_w, $w \in W$ of $\mathbf{NH}_G$ over Λ_T. If we denote $\mathrm{d} := \prod_{\alpha\in\Delta^+} \alpha$, then $\mathbf{NH}_G[\mathrm{d}^{-1}] \simeq W \ltimes \Lambda_T[\mathrm{d}^{-1}]$ as in Remark 5.25. However, the inclusion $\mathbf{NH}_G \subset \mathsf{End}_{\Lambda_T^W}(\Lambda_T)$ is *not* an isomorphism in general.

Example 5.34. Let $G = PSL_2$. As in Example 5.8, we have $\Lambda_T = \mathbb{Z}[2t]$, and the (only) Demazure operator ∂_α sends $2t$ to 2. It is easy to conclude from this that for any element $A \in \mathbf{NH}_{PSL_2}$ the image $A(2t)$ is divisible by 2, and so e.g. the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ does not belong to $\mathbf{NH}_{PSL_2}$.

5.8. **Abelianization with modular coefficients.** Let us return to the question of computing $H_G^*(X)$ in terms of the action of $\mathbf{NH}_G$ on $H_T^*(X)$. We begin with a simple comparison claim. Let us consider the following left ideal in $\mathbf{NH}_G$:

$$I = \sum_{e\neq w\in W} \Lambda_T \partial_w = \langle \partial_w, w \neq e\rangle.$$

Given a left $\mathbf{NH}_G$-module M, we denote by $M^I \subset M$ the submodule of all elements annihilated by I.

Proposition 5.35. *Let X be a G-variety. Then we have natural maps*

$$H^*_G(X) \to H^*_T(X)^I \hookrightarrow H^*_T(X)^W.$$

Proof. Recall that the map $H^*_G(X) \to H^*_T(X)$ is defined as the pullback along $X \times G/B \to X$. For any simple root $\alpha \in \mathcal{S}$, we have the closed Bruhat cell $\overline{Z}_\alpha$, which fits into the fiber square

$$\begin{tikzcd} \overline{Z}_\alpha \arrow[r, "p"] \arrow[d, "p"] & \mathcal{F} \arrow[d, "\pi"] & \\ \mathcal{F} \arrow[r, "\pi"] & \mathcal{F}_\alpha \arrow[r, equal] & G/P_\alpha \end{tikzcd}$$

Here, $P_\alpha \subset G$ is the standard parabolic corresponding to α, that is the subgroup generated by B and the one-parameter group associated to $\mathfrak{g}_\alpha$. As in (5.2), base change yields $\partial_\alpha = p_*p^* = \pi^*\pi_*$. Therefore, using projection formula

$$\partial_\alpha(\pi^*\gamma) = \pi^*\pi_*\pi^*\gamma = \pi^*(\gamma \cup \pi_*(1)) = \pi^*\gamma \cup \partial_\alpha(1) = 0.$$

This shows that the image of $H^*_G(X) \to H^*_T(X)$ is contained in $H^*_T(X)^I$. For the second inclusion, note that $\partial_\alpha(\gamma) = 0$ implies, by definition of Demazure operator, that $\alpha\partial_\alpha(\gamma) = \gamma - s_\alpha(\gamma)$. The right-hand side obviously vanishes when γ is W-invariant. □

The map $H^*_T(X)^I \hookrightarrow H^*_T(X)^W$ is not bijective in general. Moreover, $H^*_G(X) \to H^*_T(X)^I$ may even fail to be injective! Thankfully, it turns out to be injective under fairly mild conditions. In order to formulate them, consider the following pullback along $G \to \mathrm{pt}$:

$$c : \Lambda_T = H^*_T(\mathrm{pt}) \simeq H^*_B(\mathrm{pt}) \to H^*_B(G) \simeq H^*(G/B).$$

Lemma 5.36. *The map c is surjective over $\mathbb{Q}$.*

Proof. Recall that $\Lambda_T = H^*_G(\mathcal{F})$ is a free $\Lambda_G = \Lambda_T^W$-module, when working over $\mathbb{Q}$. Therefore we can apply Künneth formula to obtain

$$H^*_T(\mathcal{F}) = H^*_B(\mathcal{F}) = H^*_G(\mathcal{F} \times \mathcal{F}) = \Lambda_T \otimes_{\Lambda_G} \Lambda_T.$$

Factoring off one copy of Λ_T, we see that the map $\Lambda_T \to \Lambda_T/\Lambda_G = H^*_T(\mathcal{F})/\Lambda_T = H^*(\mathcal{F})$ surjects. □

In particular, the map c has finite cokernel over $\mathbb{Z}$.

Definition 5.37. Let $t(G)$ be the order of the cokernel of the map $c : \Lambda^{2\dim\mathcal{F}} \to H^{2\dim\mathcal{F}}(\mathcal{F}) \simeq \mathbb{Z}$. We call this number the *torsion index* of G.

Remark 5.38. Let $\mathrm{d} = \prod_{\alpha\in\Delta^+} \alpha$ as before. One can show (e.g. by integration formula) that $c(\mathrm{d}) = |W|\theta$, where θ is the generator of $H^{2\dim\mathcal{F}}(\mathcal{F}) \simeq \mathbb{Z}$. This implies that the torsion index $t(G)$ divides the order of Weyl group $|W|$. In general, primes factors of $t(G)$ are the same as prime factors of $\pi_1(G)$, plus additionally

- 2 if G has a factor of type G_2 or B_n, $n \geq 3$;
- 2, 3 for factors of types E_6, E_7, F_4;
- 2, 3, 5 for factor of type E_8.

For instance, $t(GL_n) = 1$, and $t(SL_n)$ has the same prime factors as n.

Proposition 5.39. *The kernel of $H_G^*(X) \to H_T^*(X)^I$ is annihilated by $t(G)$. If $t(G)$ is invertible, this map is split injective.*

Proof. Pick a point $x \in X$, and consider the following diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook, "i_x"] \arrow[dr, "i"'] & \mathcal{F} \times X \arrow[r, "p"] \arrow[d, "\pi"] & X \\ & \mathrm{pt} & \end{tikzcd}$$

Recall that $c = i^*$. Choose a class $u \in \Lambda_T$ satisfying $i^*(u) = t(G)\theta$, and define $\overline{u} := \pi^*(u)$.

Exercise 5.40. Let $p : X \to Y$ be a G-equivariant map. Then the pullback $p^* : H_T^*(Y) \to H_T^*(X)$ commutes with $\mathbf{NH}_G$-action on both sides.

Using this exercise, we have

$$\begin{aligned} i_x^* p^* p_*(\overline{u}) &= i_x^* \partial_{w_0}(\overline{u}) = \partial_{w_0} i_x^*(\overline{u}) = \partial_{w_0} i^*(u) \\ &= t(G)\partial_{w_0}(\theta) = t(G) \in H^0(\mathcal{F}), \end{aligned}$$

which implies that $p_*(\overline{u}) = t(G)$. Define the maps

$$H_G^*(X) \xrightarrow{\Phi} H_T^*(X) \xrightarrow{\Psi} H_G^*(X)$$

by $\Phi = p^*$, $\Psi = p_*(\overline{u} \cup -)$. By projection formula, $\Psi\Phi(y) = p_*(\overline{u}) \cup y = t(G)y$. Therefore $p^*(y) = \Phi(y) = 0$ implies $t(G)y = 0$, which proves the first claim. Moreover, when $t(G)$ is invertible, $\frac{\Psi}{t(G)}$ provides a splitting for Φ. □

From now on, we will consider cohomology with coefficients in a field $\Bbbk$*, such that* $t(G)$ *is invertible in* $\Bbbk$. In this case, there exists an element $S \in \Lambda_T^{2\dim \mathcal{F}}$ such that $i^*(S) = c(S) = \theta$.

Definition 5.41. Let w_0 be the longest element in W. For each $w \in W$, define the *Schubert polynomial* S_w by $S_w := \partial_{w^{-1}w_0}(S)$.

Lemma 5.42. *We have $S_e = 1$. The set $\{c(S_w) : w \in W\}$ is a basis of $H^*(\mathcal{F})$.*

Proof. We have

$$i^* S_e = i^* \partial_{w_0}(S) = \partial_{w_0} i^* S = \partial_{w_0}(\theta) = 1,$$

which proves the first claim. An analogous computation shows that $c(S_w) = i^*(S_w) = \partial_{w^{-1}w_0}(\theta)$, and the latter is equal to the class of the closed Bruhat cell $[\overline{Z}_{w^{-1}w_0}]$. The second claim follows by Bruhat decomposition. □

Consider the $\Bbbk$-linear maps

$$\begin{aligned} s &: H^*(\mathcal{F}) \to \Lambda_T, & c(S_w) &\mapsto S_w, \\ \overline{s} &: H^*(\mathcal{F}) \to H_T^*(X), & c(S_w) &\mapsto \pi^*(S_w). \end{aligned}$$

Proposition 5.43. *The map*

$$H^*(\mathcal{F}) \otimes H_G^*(X) \to H_T^*(X), \qquad (x, b) \mapsto \overline{s}(x)p^*(b)$$

is a Λ_G-module isomorphism. The map

$$\Lambda_T \otimes_{\Lambda_G} H_G^*(X) \to H_T^*(X), \qquad (u, b) \mapsto \pi^*(u)p^*(b)$$

is an algebra isomorphism.

Proof. Note that $i_x^*(\pi^* S_w) = i^*(S_w) = c(S_w)$; in particular i_x^* is surjective, and $\overline{s}$ provides a section. This is exactly the setting where Leray-Hirsch theorem applies, and so the first isomorphism follows. The second map is manifestly multiplicative, and sends a basis $\{S_w \otimes 1\}$ to a basis $\{\pi^*(S_w)\}$. $\square$

Corollary 5.44. *We have* $H_G^*(\mathrm{pt}, \Bbbk) \simeq H_G^*(\mathrm{pt}, \mathbb{Z}) \otimes_{\mathbb{Z}} \Bbbk$ *and* $H_G^*(\mathrm{pt}, \Bbbk) \simeq (H_T^*(\mathrm{pt}, \mathbb{Z})^W) \otimes_{\mathbb{Z}} \Bbbk$. *Moreover,* $(\Lambda_T/\Lambda_T^W) \otimes_{\mathbb{Z}} \Bbbk \to H^*(\mathcal{F}, \Bbbk)$ *is an isomorphism.*

Proof. We have $\mathrm{Tor}^{\mathbb{Z}}(\Lambda_G, \Bbbk) = 0$, since the torsion submodule of Λ_G is killed by $t(G)$, and $t(G)$ is invertible in $\Bbbk$. The first isomorphism follows by universal coefficients. In particular,

$$p^* H_G^*(\mathrm{pt}, \Bbbk) = p^* H_G^*(\mathrm{pt}) \otimes \Bbbk \subset H_T^*(\mathrm{pt})^W \otimes \Bbbk.$$

For the opposite inclusion, note that $H_T^*(\mathrm{pt}, \Bbbk)$ is free over $p^* H_G^*(\mathrm{pt}, \Bbbk)$ with basis S_w, $w \in W$. Moreover, one can check that $\Lambda_T \otimes \Bbbk$ is free over $\Lambda_G \otimes \Bbbk$ with the same basis; this implies that $H_G^*(\mathrm{pt}, \Bbbk) \simeq (H_T^*(\mathrm{pt}, \mathbb{Z})^W) \otimes_{\mathbb{Z}} \Bbbk$.

Finally, for the last isomorphism we write

$$\begin{aligned}\left(\Lambda_T/\Lambda_{T,+}^W\right) \otimes \Bbbk &= (\Lambda_T \otimes \Bbbk) \otimes_{\Lambda_T^W \otimes_{\mathbb{Z}} \Bbbk} \Bbbk \\ &= (H^*(\mathcal{F}, \Bbbk) \otimes_{\Bbbk} (\Lambda_T^W \otimes_{\mathbb{Z}} \Bbbk)) \otimes_{\Lambda_T^W \otimes_{\mathbb{Z}} \Bbbk} \Bbbk = H^*(\mathcal{F}, \Bbbk),\end{aligned}$$

where we used Proposition 5.43 for the second equality. $\square$

Corollary 5.45. *The map* $\Lambda_T \otimes_{\Lambda_G} H_G^*(X) \to H_T^*(X)$ *is an isomorphism of* $\mathbf{NH}_G$*-modules, where the action on the left hand side is defined by* $\partial_w(u \otimes b) := \partial_w \otimes b$.

Proof. The action of $\mathbf{NH}_G$ on Λ_T is Λ_G-linear, therefore the action on the left hand side is well-defined. Furthermore, $\mathbf{NH}_G$-action on $H_T^*(X)$ is $H_G^*(X)$-linear by the proof of Proposition 5.35, therefore

$$\pi(\partial(u))p^*(b) = \partial(\pi^*(u))p^*(b) = \partial(\pi^*(u)p^*(b)),$$

and so we may conclude. $\square$

We are ready to prove Corollary 5.32 in a more general setting.

Proposition 5.46. *Assume that* $t(G)$ *is invertible in* $\Bbbk$. *Then we have an isomorphism* $\mathbf{NH}_G \simeq \mathrm{End}_{\Lambda_G}(\Lambda_T)$ *over* $\Bbbk$.

Proof. Let us consider the elements $\partial_w(S_{w'})$ for $w, w' \in W$. By definition of S_w and relations in $\mathbf{NH}_G$, we have $\partial_w(S_w) = 1$.

Lemma 5.47. *Denote* $l(w) := r$ *for a reduced expression* $w = s_{i_1} \dots s_{i_r}$. *If* $l(w) > l(w')$, *then* $\partial_w(S_{w'}) = 0$. *If* $l(w) = l(w')$, *then* $\partial_w(S_{w'}) = \delta_{w,w'}$.

Proof. For the first claim, consider the degree of resulting polynomial:

$$\deg \partial_w S_{w'} = \deg \partial_w \partial_{w^{-1} w_0} S = l(w_0) - (l(w_0) - l(w')) - l(w) = l(w') - l(w) < 0.$$

For the second claim, note that $\partial_w \partial_{(w')^{-1} w_0}$ unless $l(w) = l((w')^{-1} w_0)$, which is equivalent to $w = w'$. $\square$

As a consequence of the lemma above, the determinant of the $(|W| \times |W|)$-matrix $(\partial_w(S_{w'}))_{w,w' \in W}$ is equal to 1. In other words, the Λ_T-linear map

$$\bigoplus_w \Lambda_T \partial_w \to \bigoplus_w \Lambda_T S_w, \qquad A \mapsto (A(S_w) S_w)_w$$

is an isomorphism. We conclude by observing that $\mathbf{NH}_G \simeq \bigoplus_w \Lambda_T \partial_w$ as explained in Section 5.7, and $\mathrm{End}_{\Lambda_G}(\Lambda_T) \simeq \bigoplus_w \Lambda_T S_w$ by Proposition 5.43 applied to a point. □

An easy consequence of this structure result is the main statement of this chapter.

Theorem 5.48. *Assume that $t(G)$ is invertible in $\Bbbk$. Restriction from G to T induces an isomorphism $H^*_G(X, \Bbbk) \simeq H^*_T(X, \Bbbk)^I$.*

Proof. Let us consider the categories $\mathcal{A} = \mathbf{NH}_G$-mod, $\mathcal{B} = \Lambda_G$-mod. We have the following functors:

$$F : \mathcal{A} \to \mathcal{B}, \qquad A \mapsto \mathrm{Hom}_{\mathbf{NH}_G}(\Lambda_T, A);$$
$$G : \mathcal{B} \to \mathcal{A}, \qquad B \mapsto \Lambda_T \otimes_{\Lambda_G} B.$$

Recall that $\mathbf{NH}_G \simeq \mathrm{Mat}_{|W|\times|W|}(\Lambda_G)$. For such rings, we have the following useful result (which we will not prove).

Lemma 5.49 (Morita equivalence)**.** *Let R be a ring. Then for any $n > 0$, we have R-mod $\simeq$ $\mathrm{Mat}_{n\times n}(R)$, via the maps $M \mapsto M^{\oplus n}$, $\mathrm{Hom}_{\mathrm{Mat}}(M^{\oplus n}, N) \leftarrow N$.*

In particular, G is an equivalence of categories, and F is its inverse.

Lemma 5.50. *Let $J : \mathcal{A} \to \mathcal{B}$, $J(A) := A^I$. Then $F \simeq J$ via φ, sending $f \in F(A)$ to $f(1) \in J(A)$.*

Proof. The map φ is injective, since Λ_T is cyclic over $\mathbf{NH}_G$, and surjective because I is precisely the annihilator of $1 \in \Lambda_T$. □

Finally, let $A := H^*_T(X) \in \mathcal{A}$, and $B := H^*_G(X) \in \mathcal{B}$. We know that $A = G(B)$, therefore $B = F(A) = J(A) = A^I$. □

It only remains for us to compare $H^*_T(X, \Bbbk)^I$ with $H^*_T(X, \Bbbk)^W$.

Proposition 5.51. *Let A be an $\mathbf{NH}_G$-module. Then $A^I = A^W$ if either $|W|$ is invertible, or $\mathrm{d} = \prod_{\alpha\in\Delta^+} \alpha$ is not a zero divisor in A.*

Sketch of proof. One can check that $u \mapsto \partial_{w_0}(Su)$ defines a map $\rho : A \to A^I$. If $|W|$ is invertible, we can pick $S = \mathrm{d}|W|^{-1}$ and check that ρ is precisely the symmetrization. If d is not a zero divisor, then $\mathbf{NH}_G \simeq W \ltimes \Lambda_T$. □

Corollary 5.52. *If $t(G)$ is invertible and $H^*_T(X)$ is equivariantly formal (or just Λ_T-torsion free), then $H^*_G(X) \simeq H^*_T(X)^W$.* □

Remark 5.53. When $t(G)$ is not invertible in $\Bbbk$, not much is known about G-equivariant cohomology, or even about the ring structure of Λ_G.

6. Convolution algebras

In the previous section, we used the action of convolution algebra $\mathbf{NH}_G$ to study $H_G^*(X)$. Let us go the other way around, and ask:

- Given an algebra A, can we realize it as a convolution algebra?
- Given a convolution algebra $H_G^*(F \times F)$, do all of its modules arise from $H_G^*(F \times X)$ for various X? Maybe $X = \mathrm{pt}$ is enough?

A good testing ground is $A = \mathbb{Z}[\Gamma]$, where Γ is a finite group. Let us further restrict our attention to the symmetric group $\Gamma = \mathfrak{S}_n$.

We immediately run into a problem; namely, the product in $H_G^*(F \times F)$ changes cohomological degree by $-2 \dim F$, while $\mathbb{Z}[\Gamma]$ doesn't have any natural grading. This corresponds to the fact that while $\mathbb{Z}[\mathfrak{S}_n] \subset \mathbf{NH}_n$, the two algebras are meaningfully different. For the first attempt at repairing this, assume that $F \to B$ is a proper map, and consider $Z := F \times_B F$. We have the following correspondence:

$$\begin{tikzcd} & F \times_B F \times_B F \arrow[dl] \arrow[dr] & \\ Z \times Z & & Z \end{tikzcd}$$

If $\dim F \times_B F \times_B F = F \times_B F$, then the multiplication preserves cohomological degree. Moreover, if $\dim F^{\times_B 4} = \dim F \times_B F$, then the product on $H^*(Z)$ is associative by base change. However, we arrive at a new problem: $\mathbb{Z}[\Gamma]$ must belong to $H^0(Z)$ for degree reasons, and $H^0(Z) = \mathbb{Z}$ if Z is connected.

Example 6.1. Let $X = \{xy = 0\} \subset \mathbb{C}^2$. Then $H^*(X) = H^0(X) = \mathbb{Z}$ by homotopy equivalence. On the other hand, in compactly supported cohomology $H_c^0(X) = \mathbb{Z}$, $H_c^2(X) = \mathbb{Z}^2$ by an easy application of Mayer-Vietoris.

This suggest that we should replace cohomology with compactly supported cohomology. The remaining issue is that H_c^* is *contravariant* with respect to proper maps, so in order to define convolution algebras we need to dualize.

Let us recall the 4 flavours of singular (co)homology:

$$\begin{tikzcd} H_*(X) \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash, dashed] & H_*^{\mathrm{BM}}(X) \arrow[d, dash] \arrow[dl, dash, dashed] \\ H_c^*(X) \arrow[r, dash] & H^*(X) \end{tikzcd}$$

Here, the dashed line means duality (over $\mathbb{Q}$), blue line is Poincaré duality (when X is smooth), and red line is the trivial isomorphism when X is compact. $H_*^{\mathrm{BM}}(X)$ stands for *Borel-Moore* homology, and so this is what we need to work with. The main issue we will encounter is the definition of pullbacks; while pullback clearly makes sense for maps between smooth varieties by Poincaré duality, the fiber products $F \times_B F$ we want to consider are rarely smooth.

6.1. Borel-Moore homology.

From now on, we will work exclusively with the spaces admitting a closed embedding into $\mathbb{R}^n$; note that these covers all quasi-projective algebraic varieties over $\mathbb{C}$, since $\mathbb{CP}^n \subset \mathbb{R}^{4n}$.

Definition 6.2. Let $X \subset \mathbb{R}^N$ be a closed embedding. The *Borel-Moore homology* of X is defined to be the relative cohomology:

$$H_i^{\mathrm{BM}}(X) := H^{N-i}(\mathbb{R}^N, \mathbb{R}^n \setminus X).$$

Proposition 6.3. *This definition is independent of the closed embedding $X \subset \mathbb{R}^N$.*

Proof. Let $i : X \hookrightarrow \mathbb{R}^N$, $j : X \hookrightarrow \mathbb{R}^M$ be two closed embeddings. We can find a smooth map $\varphi : \mathbb{R}^N \to \mathbb{R}^M$, such that $\varphi \circ i = j$. Let us define an automorphism of $\mathbb{R}^N \times \mathbb{R}^M$ by

$$\theta : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N \times \mathbb{R}^M, \qquad (v, w) \mapsto (v, w - \varphi(v)).$$

Note that θ sends $X_{(i,0)} := i(X) \times \{0\}$ to $X_{(i,j)} := (i \times j)(X)$. Using this and Thom isomorphism, we get

$$\begin{aligned} H^{N-i}(\mathbb{R}^N, \mathbb{R}^N \setminus X) &= H^{N+M-i}(\mathbb{R}^N \times \mathbb{R}^M, (\mathbb{R}^N \times \mathbb{R}^M) \setminus X_{(i,0)}) \\ &= H^{N+M-i}(\mathbb{R}^{N+M}, (\mathbb{R}^{N+M}) \setminus X_{(j,i)}). \end{aligned}$$

Similarly, we have $H^{M-i}(\mathbb{R}^M, \mathbb{R}^M \setminus X) = H^{N+M-i}(\mathbb{R}^{M+N}, (\mathbb{R}^{M+N}) \setminus X_{(j,i)})$, and so it remains to check that

$$H^{N+M-i}(\mathbb{R}^{N+M}, (\mathbb{R}^{N+M}) \setminus X_{(i,j)}) = H^{N+M-i}(\mathbb{R}^{M+N}, (\mathbb{R}^{M+N}) \setminus X_{(j,i)}).$$

However, this isomorphism is realized by the map $\mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^M \times \mathbb{R}^N$ which swaps the factors. □

Note that we can replace a closed embedding into $\mathbb{R}^N$ by an embedding into any smooth manifold M. Indeed, let $X \subset M \subset \mathbb{R}^N$ be closed embeddings, and $m = \dim_{\mathbb{R}} M$. Then

$$H^{m-i}(M, M \setminus X) = H^{N-i}(U, U \setminus A) = H^{N-i}(\mathbb{R}^N, \mathbb{R}^N \setminus A) = H^{N-i}(\mathbb{R}^N, \mathbb{R}^N \setminus X),$$

where U is a tubular neighborhood of M in $\mathbb{R}^N$, and A is its restriction to X.

Remark 6.4. There exist other, equivalent definitions of Borel-Moore homology.

(1) One can prove that for any compactification $X \subset \overline{X}$ such that the boundary $\partial X := \overline{X} \setminus X$ is a retract of its neighborhood (e.g. one-point compactification), we have $H_*^{\mathrm{BM}}(X) = H_*(\overline{X}, \partial X)$;
(2) Consider the complex $C^{\mathrm{lf}(X)}$ of *locally finite* singular chains, that is where we allow infinite sums, which restrict to a finite sum over each compact $K \subset X$. Then $H_*^{\mathrm{BM}}(X) = H_*(C^{\mathrm{lf}}(X))$.

6.2. **Functoriality.** Let $p : X \to Y$ be a proper map. Then by definition p factors as $X \hookrightarrow Y \times \overline{\mathbb{D}} \to Y$, where the first map is a closed embedding, the sedond map is a projection, and $\overline{\mathbb{D}}$ is a closed disk in some $\mathbb{R}^N$. Pick a closed embedding $Y \subset \mathbb{R}^M$, then

$$\begin{aligned} H_i^{\mathrm{BM}}(X) &= H^{M+N-i}(\mathbb{R}^{M+N}, \mathbb{R}^{M+N} \setminus X) \to H^{M+N-i}(\mathbb{R}^{M+N}, \mathbb{R}^{M+N} \setminus (Y \times \overline{\mathbb{D}})) \\ &\simeq H^{M+N-i}(\mathbb{R}^{M+N}, \mathbb{R}^{M+N} \setminus (Y \times \{0\})) \simeq H^{M-i}(\mathbb{R}^M, \mathbb{R}^M \setminus Y) \\ &= H_i^{\mathrm{BM}}(Y) \end{aligned}$$

defines a proper pushforward map $p_* : H_*^{\mathrm{BM}}(X) \to H_*^{\mathrm{BM}}(Y)$. Unraveling this definition, note that for a closed embedding $X \hookrightarrow Y$ this maps is nothing else than the restriction map $H^*(\mathbb{R}^M, \mathbb{R}^M \setminus X) \to H^*(\mathbb{R}^M, \mathbb{R}^M \setminus Y)$.

Now let $j : U \hookrightarrow X$ be an open embedding. Embed X in an m-dimensional manifold M, and let $\mathring{M} = M \setminus (X \setminus U)$. Then

$$H_i^{\mathrm{BM}}(X) = H^{m-i}(M, M \setminus X) \to H^{m-i}(\mathring{M}, \mathring{M} \setminus U) = H_i^{\mathrm{BM}}(U)$$

defines an open pullback $j^* : H_i^{\mathrm{BM}}(X) \to H_i^{\mathrm{BM}}(U)$. Note that if $i : V = X \setminus U \hookrightarrow X$ is the complement, the long exact sequence in relative cohomology induces a long exact sequence in Borel-Moore homology:

$$\cdots \to H_i^{\mathrm{BM}}(V) \xrightarrow{i_*} H_i^{\mathrm{BM}}(X) \xrightarrow{j^*} H_i^{\mathrm{BM}}(U) \to H_{i+1}^{\mathrm{BM}}(V) \to \cdots$$

6.3. **Fundamental classes.** Assume X is smooth of (complex) dimension n. Then we have Poincaré duality:

$$H_i^{\mathrm{BM}}(X) = H^{2n-i}(X, X \setminus X) = H^{2n-i}(X).$$

In particular, in top degree we have $H_{2n}^{\mathrm{BM}}(X) = H^0(X) = \bigoplus_{i \in I} \mathbb{Z}[X_i]$, where X_i, $i \in I$ are connected components of X (each of them has dimension n by smoothness).

Now let X be any algebraic variety. Then it has a non-empty smooth locus $X_{\mathrm{sm}} \subset X$, and by algebraicity the real codimension of the complement $X \setminus X_{\mathrm{sm}}$ is at least 2. The long exact sequence in Borel-Moore homology then implies

$$H_{2\dim X}^{\mathrm{BM}}(X) = H_{2\dim X}^{\mathrm{BM}}(X_{\mathrm{sm}}) = \bigoplus_{i \in I} \mathbb{Z}[X_{\mathrm{sm},i}] = \bigoplus_{i \in I} \mathbb{Z}[X_i],$$

where $i \in I$ runs over connected components of X_{sm} of top dimension, or equivalently over irreducible components of X of top dimension. Note that lower-dimensional components do not contribute, since on each component X_i Poincaré duality establishes an isomorphism between $H^0(X_{\mathrm{sm},i})$ and $H_{2\dim X_i}^{\mathrm{BM}}(X_{\mathrm{sm},i})$. We call the class $[X_i]$ the *fundamental class* of X_i; in particular, for each X we get its fundamental class

$$[X] = \sum_{\dim X_i = \dim X} [X_i] \in H_{2\dim X}^{\mathrm{BM}}(X).$$

Each class $[X_i] \in H_{2\dim X}^{\mathrm{BM}}(X)$ can therefore be understood as the pushforward of $[X_i] \in H_{2\dim X_i}^{\mathrm{BM}}(X_i)$ under the closed inclusion $X_i \subset X$.

Using Poincaré duality, we can also define a pullback along any map $f : X \to Y$ between smooth varieties. Let $n = \dim_{\mathbb{C}} X$, $m = \dim_{\mathbb{C}} Y$, then

$$H_i^{\mathrm{BM}}(Y) = H^{2m-i}(Y) \xrightarrow{f^*} H^{2m-i}(X) = H_{i+2(n-m)}^{\mathrm{BM}}(X).$$

6.4. **Intersection pullback.** Note that $H_*^{\mathrm{BM}}(X)$ is *not* a ring, but only a module over $H^*(X)$. Indeed, one can construct the module structure as follows. Let $X \subset M$ closed embedding, such that M is smooth of dimension m and X is a retract of M. Then cup product in relative cohomology gives us a map

$$\begin{aligned} \cap : H^i(X) \otimes H_j^{\mathrm{BM}}(X) &\simeq H^i(M) \otimes H^{m-j}(M, M \setminus X) \\ &\xrightarrow{\cup} H^{m-j+i}(M, M \setminus X) \simeq H_{j-i}^{\mathrm{BM}}(X), \end{aligned}$$

which is called cap product.

This construction can be generalized. Indeed, let $X, Y \subset M$ be two closed (possibly singular) subvarieties of a smooth variety M. We have a map

$$\begin{aligned}\cap\, :\, H_i^{\mathrm{BM}}(X) \otimes H_j^{\mathrm{BM}}(Y) &\simeq H^{m-i}(M, M \setminus X) \otimes H^{m-j}(M, M \setminus Y)\\ &\xrightarrow{\cup} H^{2m-i-j}(M, (M \setminus X) \cup (M \setminus Y)) = H^{2m-i-j}(M, M \setminus (X \cap Y))\\ &\simeq H_{i+j-m}^{\mathrm{BM}}(X \cap Y).\end{aligned}$$

Note that this map very much depends on the ambient variety M! For instance, even its degree depends on $\dim M$.

Now let $i\, :\, X \hookrightarrow Y$ be a closed embedding of smooth varieties of codimension d, and $Z \subset Y$ any closed subvariety. Denote $Z' = X \cap Z$ and consider the fiber square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d] & Z \arrow[d] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Definition 6.5. The *intersection pullback* is defined by

$$(i')^*\, :\, H_i^{\mathrm{BM}}(Z) \to H_{i-2d}^{\mathrm{BM}}(Z'), \qquad (i')^*(c) = c \cap [X].$$

We will denote this map both $(i')^*$ and i^*, or $(i')^*_i$ if we need to underline its dependence on the map $i\, :\, X \to Y$.

Remark 6.6. Later on, we will drop the requirement for maps $Z \to Y, X \to Y$ to be closed embeddings.

Let us list some useful properties of intersection pullback.

Exercise 6.7 (Functoriality). Consider the diagram with fiber squares

$$\begin{tikzcd} Z_2' \arrow[r, "j'"] \arrow[d] & Z_1' \arrow[r, "i'"] \arrow[d] & Z \arrow[d] \\ X_2 \arrow[r, "j"] & X_1 \arrow[r, "i"] & Y \end{tikzcd}$$

where X_1, X_2, Y are smooth, and all maps are closed embeddings. Then we have an equality $(i \circ j)^* = j^* \circ i^*$ of maps from $H_*^{\mathrm{BM}}(Z)$ to $H_*^{\mathrm{BM}}(Z_2')$.

Exercise 6.8 (Commutativity). Consider the diagram with fiber squares

$$\begin{tikzcd} W'' \arrow[r, "i''"] \arrow[d, "j''"] & W' \arrow[r] \arrow[d, "j'"] & W \arrow[d, "j"] \\ Z'' \arrow[r, "i'"] \arrow[d] & Z' \arrow[r] \arrow[d] & Z \\ X \arrow[r, "i"] & Y & \end{tikzcd}$$

where X, Y, W, Z are smooth, and all maps are closed embeddings. Then we have an equality $j^* \circ i^* = i^* \circ j^*$ of maps from $H_*^{\mathrm{BM}}(Z')$ to $H_*^{\mathrm{BM}}(W'')$.

Proposition 6.9 (Excess intersection). *Consider the diagram with fiber squares*

$$
\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "h'"] & Y'' \arrow[d, "h"] \\
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\tag{6.1}
$$

where X, X', Y, Y' are smooth, and all maps are closed embeddings. Then for any $c \in H_*^{\mathrm{BM}}(Y'')$, we have

$$(f'')^*_f(c) = \mathrm{eu} \cap (f'')^*_{f'}(c).$$

where $\mathrm{eu} := e((g')^*(N_X Y)/N_{X'} Y')$.

Sketch of proof. We deduce from associativity of cup product in relative cohomology that

$$c \cap_Y [X] = (c \cap_{Y'} [Y']) \cap_Y [X] = c \cap_{Y'} ([Y'] \cap_Y [X]).$$

This means that we only need to show that $[Y'] \cap_Y [X] = [X'] \cap \mathrm{eu}$. By excision, we can work in a tubular neighborhood of X'. Moreover, by deformation we can assume that $TX|_{X'} \cap TY'|_{X'} = TX'$. In this case X is given as a zero set of a section of the pullback of normal bundle $N_X(Y)$ to Y, and same for Y, X'. Since the cohomological class of a zero set of a section is exactly the Euler class of the corresponding vector bundle, we have

$$[Y'] \cap_Y [X] = e(N_{Y'}(Y))e(N_X(Y)) = e(N_{X'} Y')\mathrm{eu} = [X'] \cap \mathrm{eu},$$

and so we may conclude. □

Proposition 6.10 (Base change). *Consider the fiber diagram (6.1), and assume that X, Y are smooth, h is proper, and f, g are closed embeddings. Then we have an equality*

$$(f')^* h_* = h'_*(f'')^* : H_*^{\mathrm{BM}}(Y'') \to H_*^{\mathrm{BM}}(X').$$

Proof. Note that since g is not necessarily a closed embedding, the map $(f'')^*$ is not yet defined. We postpone this definition, as well as the proof of the statement until Section 6.8. □

6.5. **Convolution algebras in Borel-Moore homology.** Let us reconsider the setup from the beginning of this section. Let Y be a smooth variety, and $f : Y \to X$ a proper map. Define $Z := Y \times_X Y$, $Z^{(2)} := Y \times_X Y \times_X Y$, and consider the correspondence

$$
\begin{tikzcd}
Z \times Z \arrow[d, hook] & Z^{(2)} \arrow[l, "q"'] \arrow[r, "p"] \arrow[d, hook] & Z \\
Y^2 \times Y^2 & Y^3 \arrow[l, "\Delta"]
\end{tikzcd}
\tag{6.2}
$$

where Δ maps (y_1, y_2, y_3) to (y_1, y_2, y_2, y_3). Taking pullback q^* relatively to the map Δ, we get a product $m = p_* q^*_\Delta$ on $H_*^{\mathrm{BM}}(Z)$

Remark 6.11. When $X = \mathrm{pt}$, we have $Z = Y \times Y$, and so we immediately recover the convolution algebras from Section 5.4.

Proposition 6.12. *The product m is associative.*

Proof. Let us denote $Z^{(3)} := Y \times_X Y \times_X Y \times_X Y$. Consider a diagram as in the proof of Proposition 5.20:

$$\begin{tikzcd}[column sep=small] & & Z^{(3)} \arrow[dl, "q'"'] \arrow[dr] & & \\ & Z^{(2)} \times Z \arrow[dl] \arrow[dr] & & Z^{(2)} \arrow[dl, "q"'] \arrow[dr] & \\ Z \times Z \times Z & & Z \times Z & & Z \end{tikzcd}$$

As before, we want to use Proposition 6.10 to conclude by base change. The only issue is that in the square above, $(q')^*$ is defined with respect to a map $Y^4 \to Y^5$, while q^* is defined with respect to $Y^3 \to Y^4$. However, it is easy to see that the excess intersection class eu from Proposition 6.9 vanishes in this situation, so we can consider q^* relatively to $Y^4 \to Y^5$ instead. $\square$

Exercise 6.13. Let $\Delta := Y \times_Y Y \subset Y \times_X Y$. Show that $[\Delta] \in H^{\mathrm{BM}}_{2\dim Y}(Z)$ is the unit of the convolution product. Furthermore, show that the subspace $[\Delta] \cap f^*(H^*(X))$ belongs to the center of $H^{\mathrm{BM}}_*(Z)$.

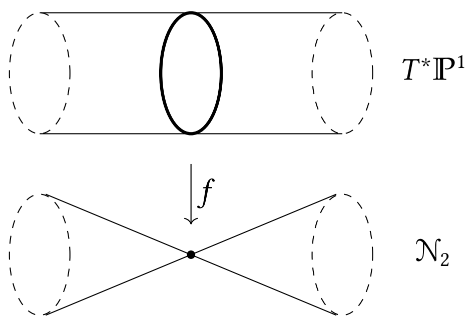
Example 6.14. Let $X = \mathcal{N}_2$ be the set of nilpotent matrices in $\mathfrak{gl}_2$, and

$$Y = \{(L, x) : L \in \mathbb{P}^1, x \in \mathcal{N}_2, x|_L = 0\}.$$

We have a natural map $f : Y \to X$, $f(L, x) = x$. Moreover,

$$\{x \in \mathfrak{gl}_2 : x|_L = 0\} = \mathrm{Hom}(\mathbb{C}^2/L, L) = \mathrm{Hom}(L, \mathbb{C}^2/L)^\vee = T^*_{[L]}\mathbb{P}^1,$$

so that $Y \simeq T^*\mathbb{P}^1$. The map f is an isomorphism over $\mathcal{N}_2 \setminus \{0\}$, and $f^{-1}(0) = \mathbb{P}^1$:



This implies that $Z = T^*\mathbb{P}^1 \times_{\mathcal{N}_2} T^*\mathbb{P}^1$ has two irreducible components, namely $\Delta = T^*\mathbb{P}^1 = T^*\mathbb{P}^1 \times_{T^*\mathbb{P}^1} T^*\mathbb{P}^1$ and $V = \mathbb{P}^1 \times \mathbb{P}^1$. Let us denote $\delta = [\Delta]$, $v = [V]$, and compute the convolution product on

$$H^{\mathrm{BM}}_4(Z) = \mathbb{Z}\delta \oplus \mathbb{Z}v.$$

By Exercise 6.13, it is enough to compute $v \cdot v$. Consider the following diagram:

$$\begin{tikzcd} (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \arrow[d, "i"] & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[l, "q'"'] \arrow[r, "p'"] \arrow[d, "i"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "i"] \\ Z \times Z & Z^{(2)} \arrow[l, "q"'] \arrow[r, "p"] & Z \end{tikzcd}$$

By base change, we have $p_* q^* i_*(v \otimes v) = i_* p'_*(q')^*(v \otimes v)$. Note however that the pullback $(q')^*$ must be taken relatively to the diagonal inclusion $(T^*\mathbb{P}^1)^3 \to (T^*\mathbb{P}^1)^4$. By excess intersection, we have

$$(q')^*_d(v \otimes v) = e(N_{\mathbb{P}^1} T^*\mathbb{P}^1) q^*(v \otimes v) = -2[\mathbb{P}^1 \times \mathrm{pt} \times \mathbb{P}^1].$$

Thus $\nu \cdot \nu = -2p_*i_*([\mathbb{P}^1 \times \mathrm{pt} \times \mathbb{P}^1]) = -2\nu$. In particular, $(\nu + \Delta)^2 = \Delta$, and so $H_4^{\mathrm{BM}}(Z)$ is isomorphic to $\mathbb{Z}[\mathfrak{S}_2]$.

6.6. **Localization of convolution algebras.** Similarly to what we did in the beginning of the course, given a G-space X, one can define G-equivariant Borel-Moore homology $H_*^{\mathrm{BM},G}(X)$, which is a module over $H_G^*(X)$, and so in particular over Λ_G. All constructions from the previous paragraph naturally transport to the equivariant setting.

Let T be an algebraic torus. One of our main tools to study T-equivariant cohomology were the localization theorems. While generalizing GKM-type descriptions is a tall order, we have an analogue of Theorem 3.28.

Theorem 6.15. *Let X be a T-variety, and $i_X : X^T \hookrightarrow X$ the inclusion. Denote by $S \subset \Lambda_T$ the multiplicative set of all non-zero elements. Then*

$$i_{X*} : S^{-1}H_*^{\mathrm{BM},T}(X^T) \to S^{-1}H_*^{\mathrm{BM},T}(X)$$

is an isomorphism. Moreover, assume that $X \subset Y$ is a closed T-equivariant embedding into a smooth variety, and denote $i_Y : Y^T \hookrightarrow Y$. Then the intersection pullback

$$(i_X)_{i_Y}^* : S^{-1}H_*^{\mathrm{BM},T}(X) \to S^{-1}H_*^{\mathrm{BM},T}(X^T)$$

is an isomorphism.

Proof. For i_{X*}, the proof is exactly the same as for Theorem 3.28. For i_X^*, note that we have $i_X^* i_{X*}(-) = e(N_{Y^T}Y) \cap -$ by base change and excess intersection, and we conclude by inverting the Euler class. $\square$

A priori, there is no reason for the map $H_*^{\mathrm{BM},T}(X) \to S^{-1}H_*^{\mathrm{BM},T}(X)$ to be injective. The best general statement is that $H_*^{\mathrm{BM},T}(X)$ is equivariantly formal whenever there exists a filtration of X by T-invariant closed varieties, which satisfies the conclusion of Białynicki-Birula theorem; we will call such filtration *cellular*. Fortunately, all spaces to which we will want to apply localization will satisfy this condition.

Finally, one can show that for G reductive, $H_*^{\mathrm{BM},G}(X) \simeq H_*^{\mathrm{BM},T}(X)^W$, when $G = GL_n$ or over a field where $|W|$ is invertible.

We might ask ourselves a question: is localization compatible with convolution product? The answer is almost.

Definition 6.16. Let $f : Y \to X$ proper, G-equivariant, with Y smooth, and denote $Z = Y \times_X Y$, $Z^{(2)} = Y \times_X Y \times_X Y$. Pick $\gamma \in H_G^*(Y)$. We define the *(γ-)twisted* convolution algebra $\mathcal{A}_\gamma$ to be $H_*^{\mathrm{BM},G}(Y \times_X Y)$ as a vector space, with product given by

$$x \cdot_\gamma y := p_*(\mathrm{pr}_2^*(\gamma) \cap q^*(x \otimes y)),$$

where $q : Z^{(2)} \to Z \times Z$, $p : Z^{(2)} \to Z$ are as in (6.2), and $\mathrm{pr}_2 : Z^{(2)} = Y \times_X Y \times_X Y \to Y$ is the projection on the second factor.

Exercise 6.17. Show that the product above is associative.

Let now $G = T$ be a torus. Note that if $Y \to X$ is proper and Y is smooth, then $Y^T \to X^T$ is proper and Y^T is smooth. This means that $H^{\mathrm{BM},T}(Z^T)$, $Z^T := Y^T \times_{X^T} Y^T$ has a well-defined convolution product.

Theorem 6.18. *Let T be a torus, $f : Y \to X$ proper, T-equivariant, with Y smooth. Denote $Z = Y \times_X Y$, $Z^T = Y^T \times_{X^T} Y^T$, and $\mathcal{A} = H_*^{BM,T}(Z)$, $\mathcal{A}^T = H_*^{BM,T}(Z^T)$. Set $\gamma = e(N_{Y^T}T)^{-1} \in \operatorname{Frac}\Lambda^T$. Assume that $Y^T \to X^T$ is a submersion, so that Z^T is smooth. Then the pullback along $i_Z : Z^T \hookrightarrow Z$ relatively to $g : (Y^T)^4 \hookrightarrow Y^4$ defines a homomorphism of algebras*

$$i_Z^* : \mathcal{A} \to S^{-1}\mathcal{A}_\gamma^T.$$

Proof. Let us denote $\Gamma = \mathrm{pr}_2^*(\gamma)$ for simplicity. Consider the localization diagram

$$\begin{tikzcd} Z \times Z & Z^{(2)} \arrow[l, "q"'] \arrow[r, "p"] & Z \\ Z^T \times Z^T \arrow[u, "i"] \arrow[ur, phantom, "(1)"] & (Z^{(2)})^T \arrow[l, "q'"'] \arrow[r, "p'"] \arrow[u, "i"] \arrow[ur, phantom, "(2)"] & Z^T \arrow[u, "i"] \end{tikzcd}$$

The square (1) commutes by Exercise 6.8, that is $i_g^* q_\Delta^* = (q')_\Delta^* i_g^*$, where $\Delta : Y \to Y \times Y$. Furthermore, by excess intersection we have $(q')_\Delta^* = \Gamma^{-1} \cap (q')^*$, where $(q')^*$ is the non-intersection pullback between two smooth varieties. Thus,

$$i_g^* q_\Delta^* = \Gamma^{-1} \cap (q')^* i_g^*. \tag{6.3}$$

For the square (2), the integration formula implies $i_{g''}^* p_* = p'_*(\Gamma \cap i_{g'}^*)$, where $g' : (Y^T)^3 \hookrightarrow Y^3$, $g'' : (Y^T)^2 \hookrightarrow Y^2$. Yet another application of excess intersection yields $i_{g'}^* = \Gamma \cap i_g^*$. In total,

$$i_{g''}^* p_* = p'_*(\Gamma^2 \cap i_g^*). \tag{6.4}$$

Combining (6.3) and (6.4), we get

$$i_{g''}^* p_* q_\Delta^* = p'_*(\Gamma^2 \cap i_g^* q_\Delta^*) = p'_*(\Gamma^\cap (q')^* i_g^*),$$

which is precisely what we had to prove. □

Remark 6.19. The condition for $Y^T \to X^T$ to be a submersion is not needed; we only use it to simplify the notations in the proof. In all our examples we will have finitely many fixed points, so it is trivially satisfied.

Recall that the convolution algebra $H_*^{\mathrm{BM}}(Z)$ naturally acts on $H_*^{\mathrm{BM}}(Y) \simeq H^*(Y)$. A completely analogous argument shows that this action is also compatible with localization.

Proposition 6.20. *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{A} \otimes H_T^*(Y) \arrow[r] \arrow[d] & H_T^*(Y) \arrow[d] \\ S^{-1}\mathcal{A}_\gamma^T \otimes H_T^*(Y^T) \arrow[r] & H_T^*(Y^T) \end{tikzcd}$$

where vertical arrows are localization maps, the lower horizontal map is given by

$$a \otimes x \mapsto \mathrm{pr}_{1*}(a \cap \mathrm{pr}_2^*(\gamma x)),$$

and the upper horizontal map is the usual action. □

6.7. **Degenerate affine Hecke algebra.** Let us use Theorem 6.18 in order to study an equivariant version of Example 6.14. We will consider the action of $G = GL_2 \times \mathbb{C}^*$ given

by
$$(g,t)(L,x) = (gL, t\cdot \mathrm{Ad}_g(x)); \qquad (g,t)(x) = t\cdot \mathrm{Ad}_g(x).$$
In plain English, GL_2 acts by the natural action and $\mathbb{C}^*$ scales the nilpotent matrix. Let $T = (\mathbb{C}^*)^2 \times \mathbb{C}^* \subset G$ be the maximal torus, and denote its equivariant parameters by t_1, t_2, $\hbar$. Observe that Z has a cellular filtration
$$T_0^*\mathbb{P}^1 \subset T^*\mathbb{P}^1 \subset T^*\mathbb{P}^1 \cup \{(L_0, L) \in \mathbb{P}^1\times\mathbb{P}^1 \,:\, L \neq L_0\} \subset Z,$$
so that the localization map is injective.

We have $X^T = \{0\}$, $Y^T = \{(L_0,0),(L_\infty,0)\}$. Thus we can write elements of the algebra $S^{-1}\mathcal{A}_\gamma^T$ in matrix form. First of all, we have
$$\gamma = e(N_{Y^T}T)^{-1} = \begin{pmatrix} \frac{1}{(t_1-t_2)(t_2-t_1+\hbar)} \\ \frac{1}{(t_2-t_1)(t_1-t_2+\hbar)} \end{pmatrix}.$$
By Proposition 6.20, the action of $S^{-1}\mathcal{A}_\gamma^T$ on $H_T^*(Y^T)$ is given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{af}{(t_1-t_2)(t_2-t_1+\hbar)} & \frac{bg}{(t_2-t_1)(t_1-t_2+\hbar)} \\ \frac{cf}{(t_1-t_2)(t_2-t_1+\hbar)} & \frac{dg}{(t_2-t_1)(t_1-t_2+\hbar)} \end{pmatrix}. \tag{6.5}$$
Since Z decomposes into $\Delta \sqcup (\mathbb{P}^1\times\mathbb{P}^1)$, it is enough to compute the images of these two classes in $H_T^*(Y^T)^{\mathfrak{S}_2}$ (since we only care about G-equivariant cohomology). The image of $[\Delta]$ must be identity by Exercise 6.13. By integration formula, the image of $[\mathbb{P}^1\times\mathbb{P}^1] \in \mathcal{A}$ in $S^{-1}\mathcal{A}_\gamma^T$ is given by $e(TY^2/T(\mathbb{P}^1\times\mathbb{P}^1))$. Computing the fibers at fixed points, we get
$$i_Z^*[\mathbb{P}^1\times\mathbb{P}^1] = \begin{pmatrix} (t_2-t_1+\hbar)(t_2-t_1+\hbar) & (t_2-t_1+\hbar)(t_1-t_2+\hbar) \\ (t_1-t_2+\hbar)(t_2-t_1+\hbar) & (t_1-t_2+\hbar)(t_1-t_2+\hbar) \end{pmatrix}.$$
Plugging this into (6.5), we get for any $f \in \Lambda_T = H_G^*(Y)$
$$\begin{aligned} i_Z^*[\mathbb{P}^1\times\mathbb{P}^1] &= \frac{t_2-t_1+\hbar}{t_1-t_2} f + \frac{t_2-t_1+\hbar}{t_2-t_1} s(f) \\ &= -f + s(f) + \hbar\partial(f). \end{aligned}$$
In particular, $[Z]$ acts on $\Lambda_T = \mathbb{Z}[\hbar][t_1,t_2]$ by $s + \hbar\partial$.

Exercise 6.21. Consider a mild modification of the setup above. Let $\widetilde{X} = \mathfrak{gl}_2$, and
$$\widetilde{Y} = \{(L,x) \in \mathbb{P}^1\times\mathfrak{gl}_2 \,:\, x(L) \subset L\}.$$
Check that $\widetilde{Z} = \widetilde{Y}\times_{\widetilde{X}}\widetilde{Y}$ once again has two irreducible components, and compute the action of each of them on $H_G^*(\widetilde{Y})$. However, verify that the action of $[\widetilde{Z}]$ is different from the action of $[Z]$ above.

In conclusion, we see that the algebra $\mathcal{A}$ is generated by Λ_T and the operator $\sigma = s+\hbar\partial$ in $\mathrm{End}_{\Lambda_G}(\Lambda_T)$. It is easy to check that the defining relations are
$$\sigma^2 = 1, \qquad \sigma t_1 = t_2\sigma + \hbar.$$

One can perform a similar computation for $G = GL_n$ (and any other reductive G, although we will not need this). Namely, define $X = \mathcal{N}_n$ to be the set of nilpotent matrices in $\mathfrak{gl}_n$. Recall the flag variety $\mathcal{F} = G/B$, and write
$$Y = \{(F_\bullet, x) \in \mathcal{F}\times\mathcal{N}_n \,:\, x(F_i) \subset F_{i-1}, 1 \le i \le n\}.$$

A computation similar to the case $n = 2$ shows that $Y \simeq T^*\mathcal{F}$; the map $T^*\mathcal{F} \to \mathcal{N}_n$ is called the *Springer resolution*. One can check that $\mathcal{A} = H_*^{\mathrm{BM},G}(Z)$ is a subalgebra in $\mathrm{End}_{\Lambda_G}(\Lambda_T)$, generated by Λ_T and $\sigma_i = s_i + \hbar\partial_i$ for $1 \le i \le n-1$. The defining relations are $\mathfrak{S}_n$-relations and

$$\sigma_i t_j = \begin{cases} t_j\sigma_i, & |i-j| > 1, \\ t_{i+1}\sigma_i + \hbar, & j = i, \\ t_i\sigma_i - \hbar, & j = i+1. \end{cases}$$

The algebra defined by these generators and relations is known under the name *graded affine Hecke algebra* for GL_n.

Remark 6.22. If $\hbar = 1$, one usually says "degenerate" instead of "graded".

6.8. **Crash course into 6-functor formalism.** This goal of this paragraph is solely to prove and generalize Proposition 6.10, and thus it can be safely skipped.

Let G be a Lie group, and X a variety. To this data, we can associate

$$D^b_{c,G}(X) : \text{ G-equivariant derived category of constructible sheaves on } X.$$

The objects of this category are, roughly speaking, complexes of sheaves E in abelian groups, admitting a G-equivariant stratification $X = \bigsqcup X_\alpha$ such that the cohomology groups of $H^i(E)$ are constant over each α. We will treat this category as a black box, and only use some of its formal properties. We will also write $D(X) = D^b_{c,G}(X)$ for simplicity.

Let $f : X \to Y$ be a map of varieties. It gives rise to functors

$$f_*, f_! : D(X) \to D(Y), \qquad f^*, f^! : D(Y) \to D(X).$$

The stalk of $f_*(E)$ at a point $y \in Y$ is $H^*(E|_{f^{-1}(y)})$, and the stalk of $f_!$ is $H^*_c(E|_{f^{-1}(y)})$. The pullback functors are defined by the property that f^* is left adjoint to f_*, and $f^!$ is right adjoint to $f_!$. Let us list some useful properties:

(1) If f is proper, then $f_* = f_!$. If f is smooth, then $f^! = f^*[2d]$, where $[-]$ is the homological shift, and d is the relative dimension of f;
(2) Verdier duality: there exists a contravariant functor $(-)^\vee : D(X) \to D(X)$, such that $(f_*)^\vee = f_!$, $(f^*)^\vee = f^!$, and vice versa;
(3) Base change: given a fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{6.6}$$

we have $g^! f_* \simeq f'_*(g')^!$.

Let $p : X \to \mathrm{pt}$ be projection to a point. Starting from a constant sheaf $\underline{Z} \in D(\mathrm{pt})$, we define

$$\mathbb{Z}_X = p^*(\underline{\mathbb{Z}}), \quad \mathbb{D}_X = p^!(\underline{\mathbb{Z}}).$$

Remark 6.23. In the equivariant situation, one starts with the regular Λ_G-module.

Note that when X is smooth, we have $\mathbb{D}_X \simeq \mathbb{Z}_X[2\dim X]$. By definition, we have

$$p_*\mathbb{Z}_X = H^*(X), \quad p_!\mathbb{Z}_X = H^*_c(X).$$

Therefore, by Verdier duality $p_!\mathbb{D}_X = H_{-*}(X)$, $p_*\mathbb{D}_X = H^{\mathrm{BM}}_{-*}(X)$.

Let us rewrite the maps in Borel-Moore homology in terms of the sheaf $\mathbb{D}_X$. For $f : X \to Y$ proper, using adjunction for $\mathbb{D}_X \simeq f^!\mathbb{D}_Y$, we get

$$f_*\mathbb{D}_X = f_!\mathbb{D}_X \to \mathbb{D}_Y.$$

Pushing to a point, we recover the proper pushforward $H_*^{\mathrm{BM}}(X) \to H_*^{\mathrm{BM}}(Y)$.

Now let $f : X \to Y$ be a map between smooth varieties, and denote $d = \dim X - \dim Y$. Then

$$f^*\mathbb{D}_Y \simeq f^*\mathbb{Z}_Y[2\dim Y] \simeq \mathbb{Z}_Y[2\dim Y] \simeq \mathbb{D}_X[-2d],$$

and so again the adjunction yields a map $\mathbb{D}_Y \to f_*\mathbb{D}_X[-2d]$. Pushing to a point, we recover the pullback $H_*^{\mathrm{BM}}(Y) \to H_{*+2d}^{\mathrm{BM}}(X)$.

Now suppose we have a fiber square (6.6) with X, Y smooth, but we do not impose any conditions on either f or g. We have

$$\begin{aligned}\mathbb{D}_{Y'} = g^!\mathbb{D}_Y &\xrightarrow{\text{unit}} g^! f_* f^*\mathbb{D}_Y \simeq f'_*(g')^! f^*\mathbb{D}_Y f'_*(g')^! \simeq \mathbb{D}_X[-2d] \\ &\simeq f'_* \simeq \mathbb{D}_{X'}[-2d].\end{aligned}$$

In particular, this yields a generalization $(f')^*_f : H_*^{\mathrm{BM}}(Y') \to H_{*+2d}^{\mathrm{BM}}(X')$ of the intersection pullback map.

With the setup out of the way, we are ready to prove base change in Borel-Moore homology.

Proof of Proposition 6.10. We prove the equality in a more general setup, dropping any conditions on f and g. Let $d = \dim X - \dim Y$, and consider the following diagram:

$$\begin{tikzcd}
h_*\mathbb{D}_{Y''} \arrow[r, equal] \arrow[d] & h_*h^!g^!\mathbb{D}_Y \arrow[r] \arrow[d] & h_*h^!g^!f_*f^*\mathbb{D}_Y \arrow[r, equal] \arrow[d] & h_*h^!f'_*(g')^!f^*\mathbb{D}_Y \arrow[r, "\sim"] \arrow[d] & h_*h^!f'_*\mathbb{D}_{X'}[-2d] \arrow[d] \\
\mathbb{D}_{Y'} \arrow[r, equal] & g^!\mathbb{D}_Y \arrow[r] & g^!f_*f^*\mathbb{D}_Y \arrow[r, equal] & f'_*(g')^!f^*\mathbb{D}_Y \arrow[r, "\sim"] & f'_*\mathbb{D}_{X'}[-2d]
\end{tikzcd}$$

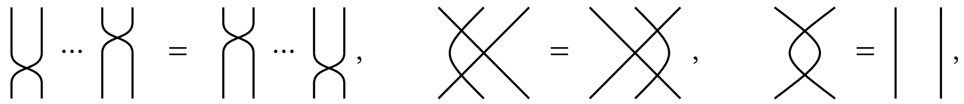
All squares are obtained by applying the counit $h_*h^! \Rightarrow \mathrm{Id}$, so they commute. The composition of leftmost vertical arrow with the bottom row gives the map $(f')^*h_*$ in Borel-Moore homology. For the top row, note that the composition

$$h_*h^!f'_* \Rightarrow h_*f''_*(h')^! = f_*h'_*(h')^! \Rightarrow f'_*$$

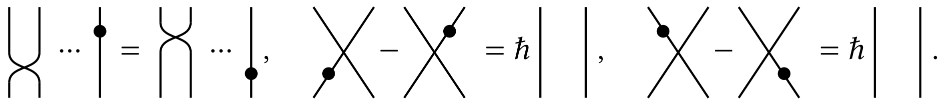
coincides with the counit map $h_*h^!f'_* \Rightarrow f'_*$, since base change commutes with counits. This implies that the composition of the top row with the rightmost vertical arrow gives the map $h'_*(f'')^*$, and so we are done. □

7. Diagrammatics and quiver Hecke algebras

Let us recast the relations in graded affine Hecke algebra of GL_n as diagrams. We will draw the products of σ_i, t_j as diagrams on n strands, where t_i is a dot on i-th strand, and σ_i permutes strands i and $i+1$. Reading the element from left to right will correspond to reading the diagram from top to bottom. Then the $\mathfrak{S}_n$-relations can be drawn as

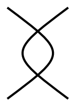
(7.1) 

and the commutation relations between σ's and t's are

(7.2) 

We will often understand the first (commutation) relation in each of the two lines above as implicit.

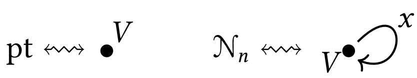
The nilHecke algebra has similar relations, except that we replace $\hbar$ with 1 in (7.2), and the last relation in (7.1) by

 $= 0.$

In this section, we will study a class of algebras that have a similar diagrammatic presentation.

7.1. Quiver representations.

What is common between the maps $\mathcal{F} \to \mathrm{pt}$ and $T^*\mathcal{F} \to \mathcal{N}_n$? We can interpret $\mathcal{F}$ as a vector space V plus a full flag, and $T^*\mathcal{F}$ as a vector space with an endomorphism x, plus a compatible flag. Then the projection maps simply forget the choice of a compatible flag. We can thus draw the elements of pt and $\mathcal{N}_n$ by



This suggests that we could instead begin with any directed graph whatsoever. It is customary to use a different term in this context, to underline that our focus is representation theory, as opposed to graph theory.

Definition 7.1. A *quiver* Q is a finite directed graph. We write $Q = (I, E) = (I, E, s, t)$, where I is the set of vertices, E the set of edges, and $s, t : E \to I$ are the maps sending each edge e to its **s**ource $s(e)$ and **t**arget $t(e)$ vertex respectively.

A *representation* of Q of dimension vector $\mathbf{v} = (v_i) \in \mathbb{Z}^I_{\geq 0}$ is an assignment, for each arrow $e \in E$, of a linear map $\varphi_e \in \mathsf{Hom}(\mathbb{C}^{\mathbf{v}_{s(e)}}, \mathbb{C}^{\mathbf{v}_{t(e)}})$.

Example 7.2. Let $Q = \bullet \to \bullet$. Then a representation of Q of dimension vector (n_1, n_2) is just a linear map $\mathbb{C}^{n_1} \to \mathbb{C}^{n_2}$.

We denote the set of all representations of Q of dimension vector $\mathbf{v}$ by

$$\mathsf{Rep}_{\mathbf{v}}\, Q = \bigoplus_{e \in E} \mathsf{Hom}(\mathbb{C}^{\mathbf{v}_{s(e)}}, \mathbb{C}^{\mathbf{v}_{t(e)}}).$$

There is a natural action of $GL_{\mathbf{v}} := \prod_{i \in I} GL_{v_i}$ on $\mathsf{Rep}_{\mathbf{v}}\, Q$ by changing the bases of vector spaces.

Given a representation $V \in \mathrm{Rep}_{\mathbf{v}} Q$, we say that a collection of subspaces $V' = (V'_i)_{i\in I}$, $V'_i \subset \mathbb{C}^{v_i}$ is a *subrepresentation* of V if for any edge $e \in E$ we have $\varphi_e(V'_{s(e)}) \subset V'_{t(e)}$. In plain English, we require the maps φ_e to restrict to V'. We also say that a representation V of Q is *indecomposable* if it cannot be written as a direct sum of two subrepresentations $V' \oplus V''$.

Exercise 7.3. Let $Q = \bullet \to \bullet \to \cdots \to \bullet$ be the linear quiver of length n. Classify all indecomposable representations of Q.

7.2. **Quiver flag varieties.** Since quiver representations live on several vector spaces, we need to develop some combinatorics before the definition of flag varieties. Let us fix a quiver Q, a dimension vector $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$, and write $|\mathbf{v}| := \sum_i v_i$. For each $i \in I$, denote by ϵ_i the dimension vector whose value at $i' \in I$ is 1 if $i' = i$, and 0 otherwise. Denote

$$\mathsf{Seq}(\mathbf{v}) := \left\{ \eta = (i_1, i_2, \dots, i_{|\mathbf{v}|}) \in I^{|\mathbf{v}|} \ : \ \sum_j \epsilon_{i_j} = \mathbf{v} \right\}.$$

We call $\mathsf{Seq}(\mathbf{v})$ the set of sequences. For any $\eta \in \mathsf{Seq}(\mathbf{v})$, we consider the partial sums

$$\mathbf{v}(\eta, k) = \sum_{j=1}^{k} \epsilon_{i_j}$$

for all $k \in [1, |\mathbf{v}|]$. Let us write $\mathcal{F}_n = GL_n/B$.

Definition 7.4. The *quiver (full) flag variety* of dimension $\mathbf{v}$ and order $\eta \in \mathsf{Seq}(\mathbf{v})$ is defined by

$$\mathsf{Fl}_\eta = \left\{ (V, (F^{(i)}_\bullet)_{i\in I}) \in \mathrm{Rep}_{\mathbf{v}} Q \times \prod_{i\in I} \mathcal{F}_{v_i} \ : \ \forall k, (F^{(i)}_{\mathbf{v}(\eta,k)_i})_{i\in I} \subset V \text{ is a subrepresentation} \right\}.$$

In other words, the points of Fl_η are chains

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{|\mathbf{v}|} = V$$

of representations of Q, such that $\dim(V_{k+1}/V_k) = \varepsilon_{i_k}$ for all k.

Note that we have a natural projection map

$$\mathsf{Fl}_\eta \to \mathrm{Rep}_{\mathbf{v}}, \qquad (V, (F^{(i)}_\bullet)) \mapsto V.$$

Example 7.5. (1) Let $Q = \bullet$. Then dimension vector is just a non-negative number n, and $\mathsf{Seq}(n) = \{(1, \dots, 1)\}$. The corresponding flag variety is $\mathsf{Fl} = \mathsf{Fl}_{(1,\dots,1)} = GL_n/B$.
(2) Let $Q = \bullet \circlearrowleft$. Then $\mathsf{Fl} = \{(F_\bullet, x) \in GL_n/B \times x \in \mathfrak{gl}_n \ : \ x(F_i) \subset F_i\}$. If we additionally require x to be nilpotent, we obtain precisely the Springer resolution.

Example 7.6. Let us consider the quiver $Q = \bullet \to \bullet$ in more detail. We call the left vertex 1, and the right vertex 2.

(1) Let $\mathbf{v} = (1, 1)$. Then $\mathsf{Seq}(\mathbf{v}) = \{12, 21\}$, and $\mathrm{Rep}_{\mathbf{v}} = \mathbb{C}$. For any representation $V = (V_1 \xrightarrow{\varphi} V_2)$ and any subspace $V'_2 \subset V_2$, it's easy to see that $0 \to V'_2$ is a subrepresentation V. Similarly, $V'_1 \to 0$, $V'_1 \subset V$ is a subrepresentation of V if and only if $\varphi|_{V'_1} = 0$. These two facts tell us that

$$\mathsf{Fl}_{12} = \{0\} \subset \mathbb{C}, \qquad \mathsf{Fl}_{21} = \mathbb{C}.$$

(2) Let $\mathbf{v} = (1,2)$. Then $\mathsf{Seq}(\mathbf{v}) = \{122, 212, 221\}$, and $\mathsf{Rep}_{\mathbf{v}} = \mathsf{Hom}(\mathbb{C}, \mathbb{C}^2) = \mathbb{C}^2$. Analogously to the previous case, we see that

$$\mathsf{Fl}_{122} = \{0\} \times \mathbb{P}^1 \subset \mathbb{C}^2 \times \mathbb{P}^1, \qquad \mathsf{Fl}_{221} = \mathbb{C}^2 \times \mathbb{P}^1.$$

Now, a flag of order 212 on $\mathbb{C} \xrightarrow{\varphi} \mathbb{C}^2$ is equivalent to a choice of line $L \subset \mathbb{C}^2$ satisfying $\mathsf{Im}(\varphi) \subset L$. Thus we obtain

$$\mathsf{Fl}_{212} = \{(v, L) \,:\, v \in L\} = \mathsf{Bl}_0\mathbb{C}^2 \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

(3) Let $\mathbf{v} = (2,1)$. Then $\mathsf{Seq}(\mathbf{v}) = \{112, 121, 211\}$, and $\mathsf{Rep}_{\mathbf{v}} = \mathsf{Hom}(\mathbb{C}^2, \mathbb{C}) = (\mathbb{C}^2)^\vee$. Analogously to the previous case, we have

$$\mathsf{Fl}_{112} = \{0\} \times \mathbb{P}^1, \quad \mathsf{Fl}_{121} = \mathsf{Bl}_0(\mathbb{C}^2)^\vee, \quad \mathsf{Fl}_{211} = (\mathbb{C}^2)^\vee \times \mathbb{P}^1.$$

Exercise 7.7. Compute all flag varieties for $Q = \bullet \to \bullet$, $\mathbf{v} = (2,2)$.

7.3. **Quiver Hecke algebras.** In order to consider convolution algebras, we need to verify a couple of properties of quiver flag varieties. From now on, we assume for simplicity that Q doesn't have any edge loops or multiple edges.

Proposition 7.8. *Let $\eta \in \mathsf{Seq}\,\mathbf{v}$. The quiver flag variety Fl_η is smooth, and the projection map $\mathsf{Fl}_\eta \to \mathsf{Rep}_{\mathbf{v}}$ is proper.*

Proof. The properness follows from the fact that Fl_η is cut out by closed conditions in $\mathsf{Rep}_{\mathbf{v}} \times \prod_i \mathcal{F}_{v_i}$, and the projection $\mathsf{Rep}_{\mathbf{v}} \times \prod_i \mathcal{F}_{v_i} \to \mathsf{Rep}_{\mathbf{v}}$ is manifestly proper.

For the smoothness, let us show that the other projection $\mathsf{Fl}_\eta \to \prod_i \mathcal{F}_{v_i}$ is an affine bundle. We think of a point in Fl_η as of a chain of representations $V^1 \subset V^2 \subset \ldots$ of Q, and proceed by induction. Suppose that we proved that the chain $V^1 \subset \ldots \subset V^k$ is an affine bundle, and we want to extend representation V^k to V^{k+1}. Let $i \in I$ be the vertex satisfying $\dim(V^{k+1}/V^k) = \epsilon_i$, and choose a splitting $V_i^{k+1} = V_i^k \oplus \mathbb{C}$. In order to extend the representation to V^{k+1}, we need to extend all maps φ_e, where e either begins or ends at i. In the former case, any map $V_i^k \oplus \mathbb{C} \to V_{t(e)}^k$ contains $V_i^k \to V_{t(e)}^k$ as a subrepresentation, so we acquire a liner space $\mathsf{Hom}(\mathbb{C}, V_{t(e)}^k) = V_{t(e)}^k$. In the latter case, $V_{s(e)}^k \to V_i^k$ is only a subrepresentation of $V_{s(e)}^k \xrightarrow{\varphi} V_i^k \oplus \mathbb{C}$ if $\mathsf{Im}\,\varphi \subset V_i^k$, and so we acquire nothing. Thus restriction from chains of length $k+1$ to chains of length k is an affine bundle, and we conclude by induction. $\square$

Let us denote the disjoint union of all quiver flag varieties of dimension $\mathbf{v}$ by $\mathsf{Fl}_{\mathbf{v}} = \bigsqcup_{\eta \in \mathsf{Seq}(\mathbf{v})} \mathsf{Fl}_\eta$.

Definition 7.9. The *quiver Hecke algebra* (or *KLR algebra*) $R(\mathbf{v})$ is defined to be the convolution algebra

$$R(\mathbf{v}) := H_*^{\mathrm{BM}, G_{\mathbf{v}}}(\mathsf{Fl}_{\mathbf{v}} \times_{\mathsf{Rep}_{\mathbf{v}}} \mathsf{Fl}_{\mathbf{v}}).$$

Let $T_{\mathbf{v}} \subset G_{\mathbf{v}}$ be the maximal torus. As before, the algebra $R(\mathbf{v})$ naturally acts on its polynomial representation

$$\mathsf{Pol}(\mathbf{v}) := H_*^{\mathrm{BM}, G_{\mathbf{v}}}(\mathsf{Fl}_{\mathbf{v}}) \simeq \bigoplus_{\eta \in \mathsf{Seq}(\mathbf{v})} H^*_{G_{\mathbf{v}}}(\mathsf{Fl}_\eta) \simeq \bigoplus_{\eta \in \mathsf{Seq}(\mathbf{v})} \Lambda_{T_{\mathbf{v}}}.$$

We identify this representation with spaces of polynomials in a very specific way. Namely, the quotient V^{k+1}/V^k gives rise to a $G_{\mathbf{v}}$-equivariant line bundle on Fl_η. We denote its first Chern class by x_k. It is straightforward to see that $H^*(\mathsf{Fl}_\eta) \simeq \mathbb{Z}[x_1, \dots, x_{|\mathbf{v}|}]$; indeed, the only thing we did is renaming the standard generators of $H^*(\mathcal{F}_{v_i})$, $i \in I$. We will denote

$$\mathsf{Pol} = \mathbb{Z}[x_1, \dots, x_{|\mathbf{v}|}], \quad \mathsf{Pol}(\mathbf{v}) = \bigoplus_{\eta \in \mathsf{Seq}(\mathbf{v})} \mathsf{Pol}\, e_\eta,$$

where e_η is the fundamental class $[\mathsf{Fl}_\eta] \in H^{\mathrm{BM},G_\mathbf{v}}_{2\dim \mathsf{Fl}_\eta}(\mathsf{Fl}_\eta) = H^0_{G_\mathbf{v}}(\mathsf{Fl}_\eta)$.

Example 7.10. (1) For $Q = \bullet$, we have $R(n) = \mathbf{NH}_n$, $\mathsf{Pol}(v) = \mathbb{Z}[x_1, \dots, x_n]$, and the action of $\mathbf{NH}_n$ on $\mathsf{Pol}(v)$ is the one we studied in Section 5.

(2) For $Q = \bullet\!\circlearrowleft$, we have $R(n) = \mathbb{Z}[x_1, \dots, x_n] \rtimes \mathfrak{S}_n$, $\mathsf{Pol}(v) = \mathbb{Z}[x_1, \dots, x_n]$, and the action is the natural action. If we enlarge the group GL_n to $GL_n \times \mathbb{C}^*$ as in Section 6.7, the algebra $R(n)$ deforms to the graded affine Hecke algebra.

Let us study the case $Q = \bullet \to \bullet$, $\mathbf{v} = (1,1)$ in more detail. We know that $\mathsf{Rep}_\mathbf{v} \simeq \mathbb{C}$, $\mathsf{Fl}_\mathbf{v} \simeq \mathbb{C} \sqcup \{0\}$, and so

$$Z_\mathbf{v} := \mathsf{Fl}_\mathbf{v} \times_{\mathsf{Rep}_\mathbf{v}} \mathsf{Fl}_\mathbf{v} = \begin{Bmatrix} \mathsf{Fl}_{12}, & \mathsf{Fl}_{12} \cap \mathsf{Fl}_{21}, \\ \mathsf{Fl}_{21} \cap \mathsf{Fl}_{12}, & \mathsf{Fl}_{21} \end{Bmatrix} = \begin{Bmatrix} \{0\}, & \{0\}, \\ \{0\}, & \mathbb{C} \end{Bmatrix}.$$

In particular, the algebra $R(1,1)$ is a free module of rank 4 over $\mathsf{Pol} = \mathbb{Z}[x_1, x_2]$, with generators being the fundamental classes of connected components of $Z_\mathbf{v}$. Let us compute their action on $\mathsf{Pol}(\mathbf{v}) = \mathsf{Pol}\, e_{12} \oplus \mathsf{Pol}\, e_{21}$. It is clear that $[\mathsf{Fl}_{12}]$ acts by identity on $\mathsf{Pol}\, e_{12}$, and $[\mathsf{Fl}_{21}]$ acts by identity on $\mathsf{Pol}\, e_{21}$. For $\mathsf{Fl}_{21} \cap \mathsf{Fl}_{12}$, the correspondence that gives rise to action is

$$\begin{tikzcd} & \{0\} \arrow[dl, equal] \arrow[dr, hook] & \\ \{0\} & & \mathbb{C} \end{tikzcd}$$

We have $e(N_0\mathbb{C}) = (x_1 - x_2)$. Moreover, our naming convention means that this operator is not linear in Pol, but rather exchanges x_1 with x_2. Thus

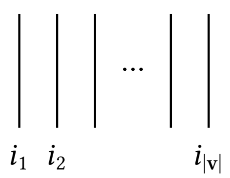
$$[\mathsf{Fl}_{21} \cap \mathsf{Fl}_{12}] : \mathsf{Pol}\, e_{12} \to \mathsf{Pol}\, e_{21}, \qquad f e_{12} \mapsto (x_1 - x_2) s(f) e_{21}.$$

For $\mathsf{Fl}_{12} \cap \mathsf{Fl}_{21}$, we have the same correspondence going the other way. Pushforward will not contribute an Euler class anymore, so that

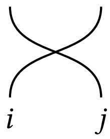
$$[\mathsf{Fl}_{12} \cap \mathsf{Fl}_{21}] : \mathsf{Pol}\, e_{21} \to \mathsf{Pol}\, e_{12}, \qquad f e_{21} \mapsto s(f) e_{12}.$$

If we perform a similar computation for $Q = \bullet \sqcup \bullet$, $\mathbb{C}$ gets replaced by a point, and so both $[\mathsf{Fl}_{12} \cap \mathsf{Fl}_{21}]$ and $[\mathsf{Fl}_{21} \cap \mathsf{Fl}_{12}]$ will act by swapping x_1 with x_2.

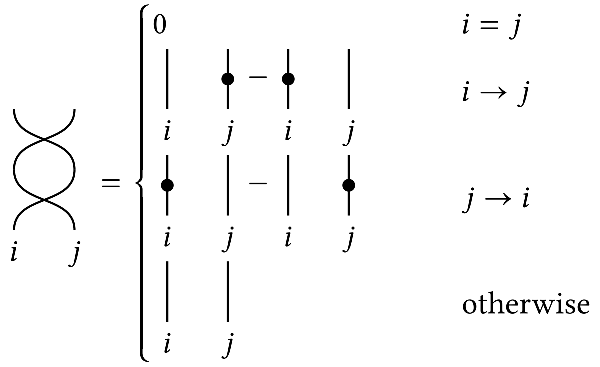
7.4. **KLR diagrammatics.** Let us introduce some diagrammatic notations. First, for any $\eta, \eta' \in \mathsf{Seq}(\mathbf{v})$ let us denote ${}_\eta R(\mathbf{v})_{\eta'} := H^{\mathrm{BM},G_\mathbf{v}}(\mathsf{Fl}_\eta \times_{\mathsf{Rep}_\mathbf{v}} \mathsf{Fl}_{\eta'})$. We will consider diagrams on $|\mathbf{v}|$ strands, each strand colored by an element of I, such that all the colors sum up to $\mathbf{v}$. For $\eta = (i_1, i_2, \dots)$, we draw the fundamental class $[\mathsf{Fl}_\eta] \in H^{\mathrm{BM},G_\mathbf{v}}_*(\mathsf{Fl}_\eta)$ as the arrangement of strands with consecutive colors $i_1, i_2, \dots$:

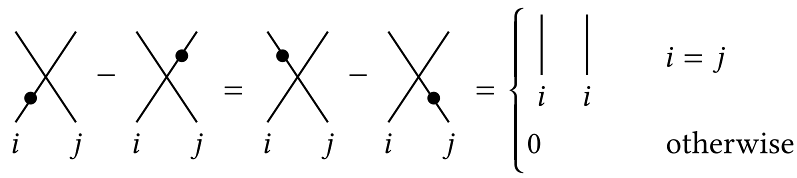


The variable $x_k \in \operatorname{Pol} e_\eta$ will be denoted by a dot on k-th strand. Furthermore, for a pair of consecutive colors i, j in η, let η' be the sequence obtained by swapping them. We have a corresponding operator $[\mathsf{Fl}_{\eta'} \times_{\mathsf{Rep}_\mathbf{v}} \mathsf{Fl}_\eta] \in {}_{\eta'}R(\mathbf{v})_\eta$, computed by base change from $[\mathsf{Fl}_{ji} \times_{\mathsf{Rep}} \mathsf{Fl}_{ij}]$ as in Section 5.6. We draw it as a crossing:

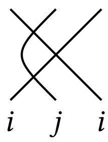
$\tau =$ 

Given two diagrammatic operators $D_1 \in {}_{\nu'}R(\mathbf{v})_\nu$, $D_2 \in {}_{\eta'}R(\mathbf{v})_\eta$, their product D_1D_2 is zero unless $\nu = \eta'$, and is given by stacking D_1 on top of D_2 otherwise. Since we know how all diagrammatic operators act on the polynomial representation, we can easily verify the following relations:

(7.3) 

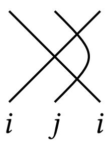
(7.4) 

Let us now consider the cubic relations. Let $i \to j$ be two vertices in Q connected by an arrow, and let us consider the case $|\mathbf{v}| = 3$ for simplicity. Then on one hand, we have



$: f \mapsto (x_1 - x_2)s_1(f) \mapsto (x_1 - x_3)\partial_2 s_1(f) - s_1(f) \mapsto (x_2 - x_3)s_1\partial_2 s_1(f) - f,$

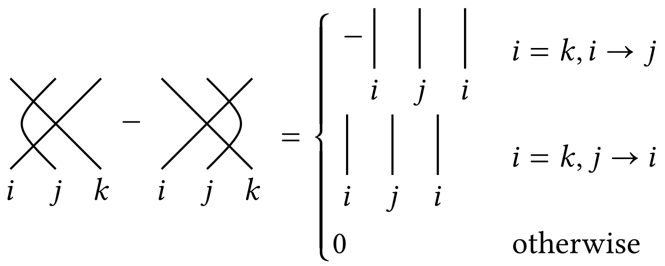
and on the other hand



$: \quad f \mapsto s_2(f) \mapsto \partial_1 s_2(f) \mapsto (x_2 - x_3)s_2\partial_1 s_2(f).$

One can check that this is the only situation when the two expressions do not coincide, so that we have

(7.5)

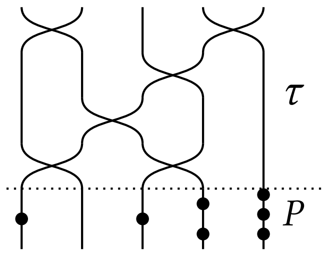


Theorem 7.11 (Khovanov-Lauda, Rouquier, Varagnolo-Vasserot)**.** *The quiver Hecke algebra $R(\mathbf{v})$ is generated by dots and crossings, modulo local relations* (7.3)*,* (7.4)*,* (7.5)*.*

Sketch of proof. Similarly to the proof of Proposition 5.29, one can find a cellular filtration of ., and show that the class of each cell is expressed in terms of dots and crossings by constructing an explicit resolution of every cell. This proves surjectivity. The injectivity follows by comparing the size of the two algebras, see Proposition 7.12. □

It is clear from relations that any symmetric polynomial in dots commutes with any element of $R(\mathbf{v})$. In fact, one can show that the center of $R(\mathbf{v})$ is precisely $\mathsf{Sym} := \mathsf{Pol}^{\mathfrak{S}_{|\mathbf{v}|}}$.

Proposition 7.12. *Let us fix a presentation of each $\tau \in \mathfrak{S}_{|\mathbf{v}|}$ as a diagram in crossings. The set of diagrams as below, where $P \in \mathsf{Pol}$, forms a basis of $R(\mathbf{v})$:*



In particular, $R(\mathbf{v})$ is a free module of rank $(|\mathbf{v}|)^2$ over Sym*.*

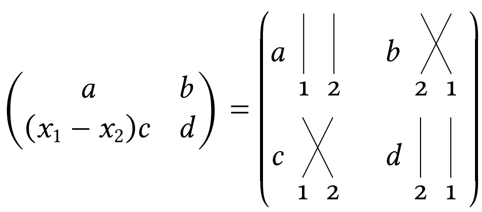
Proof. Using the relations (7.3–7.5), we can push all dots in a given diagram to the bottom; note that we needed to fix a presentation of each permutation because of the relation (7.5). This shows that the set above is a spanning set. In order to check it has the correct size, one uses equivariant localization to compute the size of $R(\mathbf{v})$. □

Note that despite its size, $R(\mathbf{v})$ is typically not isomorphic to $\mathsf{Mat}_{|\mathbf{v}|\times|\mathbf{v}|}(\mathsf{Sym})$.

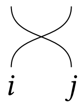
Example 7.13. Let $Q = \bullet \to \bullet$, $\mathbf{v} = (1,1)$. Then $R(1,1)$ can be realized as a subalgebra of $\mathsf{Mat}_{2\times 2}(\mathsf{Sym})$:

$$R(1,1) = \left\{ \begin{pmatrix} a & b \\ (x_1 - x_2)c & d \end{pmatrix} \ : \ a, b, c, d \in \mathsf{Sym} \right\}.$$

The bijection is given by



7.5. **Simple and projective representations.** Before studying $R(\mathbf{v})$-modules, note that $R(\mathbf{v})$ has a natural grading by setting

deg() $= 2,$ deg() $= \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } i \to j \text{ or } j \to i, \\ 0 & \text{otherwise.} \end{cases}$

From now on, all modules we consider will be understood to be *graded.*

Let $R(\mathbf{v})$–mod be the category of finitely generated (left) $R(\mathbf{v})$-modules. It contains two subcategories of interest:

- The category $R(\mathbf{v})$–fmod of finite-dimensional modules;
- The category $R(\mathbf{v})$–pmod of projective modules.

The first question one needs to answer in order to understand $R(\mathbf{v})$–fmod is what are the simple modules.

Proposition 7.14. *The category $R(\mathbf{v})$–mod has finitely many simple modules.*

Proof. Recall that the center of $R(\mathbf{v})$ is the polynomial algebra $\mathrm{Sym} = \mathrm{Pol}^{\mathfrak{S}_{|\mathbf{v}|}}$. Note that Sym is also a polynomial ring in finitely many variables, which are all positively graded.

Let $\mathrm{Sym}^+ \subset \mathrm{Sym}$ be the augmentation ideal of all polynomials without constant term. Then for any $M \in R(\mathbf{v})$–fmod, $\mathrm{Sym}^+ M$ is a proper submodule of M. If M is simple, this means that $\mathrm{Sym}^+ M = 0$, so that the action of $R(\mathbf{v})$ on M factors through the quotient $R(\mathbf{v})/\mathrm{Sym}^+$. Since $R(\mathbf{v})/\mathrm{Sym}^+$ is a finite-dimensional algebra (more precisely, of dimension $|\mathbf{v}|^2$), it has finitely many simple modules. $\square$

In particular, there finitely many simple modules in $R(\mathbf{v})$–fmod.

Example 7.15. Let $Q = \bullet$, and $\mathbf{v} = 2$. Then we know that $R(2) \simeq \mathrm{Mat}_{2\times 2}(\mathrm{Sym})$. By the proof of Proposition 7.14, all simple $R(2)$-modules come from $R(2)/\mathrm{Sym}^+ \simeq \mathrm{Mat}_{2\times 2}(\Bbbk)$. Let E_{ij}, $i, j = 1, 2$ be the matrix elements. Clearly, E_{11} and E_{22} are idempotents, therefore a representation of $\mathrm{Mat}_{2\times 2}(\Bbbk)$ is the same as a quiver representation

$$V_1 \underset{E_{21}}{\overset{E_{12}}{\rightleftarrows}} V_2$$

with an extra condition that E_{12} is the inverse of E_{21} Up to isomorphism, it is exactly the same as specifying a vector space $V_1 \simeq V_2$. Since a vector space is indecomposable if and only if it has dimension one, $R(2)$ has the unique simple module, namely the two-dimensional "vector" module.

Example 7.16. Let $Q = \bullet \to \bullet$, and $\mathbf{v} = (1, 1)$. Analogously to the previous example, every simple module over $R(1, 1)$ comes from the quotient $R(1, 1)/\mathrm{Sym}^+$. It follows from Example 7.13 representations of this quotient are the same as quiver representations

$$V_1 \underset{y}{\overset{x}{\rightleftarrows}} V_2$$

with an extra condition $xy = yx = 0$. Note that for any such representation both $0 \rightleftarrows \mathrm{Im}\, x$ and $\mathrm{Im}\, y \rightleftarrows 0$ are non-trivial submodules. This implies that for any simple module, we have $x = y = 0$. Therefore $R(1, 1)$ has two simple modules, one-dimensional modules, corresponding to $\mathbb{C} \rightleftarrows 0$ and $0 \rightleftarrows \mathbb{C}$.

Let us summarize some properties of graded algebras.

Definition 7.17. Let A be an algebra, and M an A-module. A *projective cover* of M is a surjective map $\psi : P \to M$ from a projective A-module M, such that for any $\alpha \in \mathrm{Hom}_A(P, P)$ equality $\psi \circ \alpha = \psi$ implies that α is invertible. When a projective cover exists, it is always unique.

Proposition 7.18. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded algebra, such that $\dim A_i < \infty$ for all $i \in \mathbb{Z}$, and $A_i = \{0\}$ for $i \ll 0$. Then up to a shift of grading, every projective A-module can be written uniquely as a sum of indecomposable projectives. Every indecomposable projective is a direct summand of A, so has a form Ae, where $e \in A$ is an idempotent. Every simple A-module S admits a projective cover P_S, and the map $S \mapsto P_S$ establishes a bijection between simple modules in $A-\mathrm{fmod}$ and indecomposable modules in $A-\mathrm{pmod}$.*

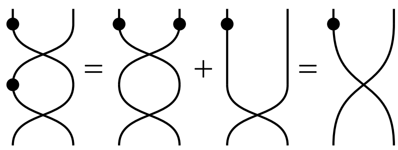
Proof. This follows from the fact that $A-\mathrm{mod}$ is a Krull-Schmidt category. □

Thus classifying simples in $R(\mathbf{v})-\mathrm{fmod}$ amounts to classifying indecomposables in $R(\mathbf{v})-\mathrm{pmod}$, which in turn amounts to classifying *primitive* idempotents in $R(\mathbf{v})$, i.e. $e \in R(\mathbf{v})$ with $e^2 = e$, such that $e \neq e_1 + e_2$, where e_i's satisfy the same property.

It is clear from diagrammatics developed in Section 7.4 that the elements

$$e_\eta := [\mathsf{Fl}_\eta] \in {}_\eta R(\mathbf{v})_\eta, \qquad \eta \in \mathsf{Seq}(\mathbf{v})$$

are idempotents in $R(\mathbf{v})$, summing up to 1. However, these idempotents are neither pairwise distinct nor indecomposable. Indeed, already in the case of $Q = \bullet$, we have $e_{11} = x_1\tau + (-\tau x_2)$, and

$(x_1\tau)^2 =$  $= x_1\tau.$

Let us denote this idempotent by $e_{1^{(2)}} = x_1\tau$. One can check that it is primitive; moreover it is the unique primitive idempotent by Example 7.15, and the module $R(\mathbf{v})e_{1^{(2)}}$ is precisely the polynomial representation Pol_2.

For the same quiver $Q = \bullet$ and arbitrary dimension $n > 0$, one can similarly check that the element

$$e_{1^{(n)}} = x_1^{n-1}x_2^{n-2}\dots x_{n-1}\partial_{w_0}, \qquad w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$$

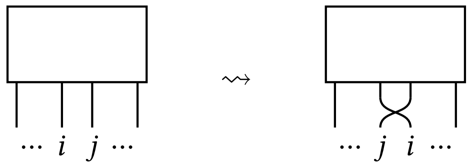
is the unique primitive idempotent in $R(n)$, and $R(n)e_{1^{(n)}} \simeq \mathsf{Pol}_n$. We will call it the *divided power* idempotent. By comparing the dimensions, one sees that $R(n) \simeq (\mathsf{Pol}_n)^{\oplus n!}$, up to grading shifts (we will return to these later).

Let us slightly mofify the indexing set $\mathsf{Seq}(\mathbf{v})$. Namely, for every sequence $\eta \in \mathsf{Seq}(\mathbf{v})$ and every color $i \in I$, we replace each occurrence of $\dots ji^k j' \dots$ in η with $\dots ji^{(k)} j' \dots$. We will denote the resulting set of sequences by $\mathsf{Seq}'(\mathbf{v})$; it is clear that any element $\eta \in \mathsf{Seq}'(\mathbf{v})$ gives rise to an idempotent $e_\eta \in R(\mathbf{v})$ by concatenation of divided power idempotents. We will write $P_\eta = R(\mathbf{v})e_\eta$ for the corresponding projectives.

Proposition 7.19. *Let $i \neq j \in I$. There exist isomorphisms of projective modules*

$$P_{\dots ij \dots} \simeq P_{\dots ji \dots}, \qquad \text{if there is no arrow between } i \text{ and } j,$$
$$P_{\dots iji \dots} \simeq P_{\dots i^{(2)} j \dots} \oplus P_{\dots j i^{(2)} \dots}, \qquad \text{if } i \to j \text{ or } j \to i.$$

Proof. Let us consider a map $P_{\dots ij \dots} \to P_{\dots ji \dots}$, given by appending a crossing to the bottom of each diagram:

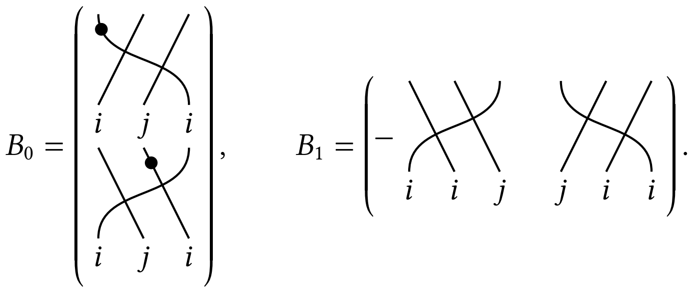


We have an analogous map $P_{\dots ji \dots} \to P_{\dots ij \dots}$. If i and j are not connected by an edge, these two maps are inverses of each other by relation (7.3), hence the first claim.

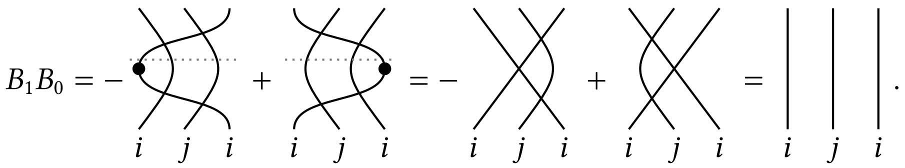
For the second claim, let us assume that $i \to j$, the other case being analogous. We construct explicit maps

$$B_0 : P_{\dots iji \dots} \to P_{\dots i^{(2)} j \dots} \oplus P_{\dots j i^{(2)} \dots}, \quad B_1 : P_{\dots i^{(2)} j \dots} \oplus P_{\dots j i^{(2)} \dots} \to P_{\dots iji \dots}$$

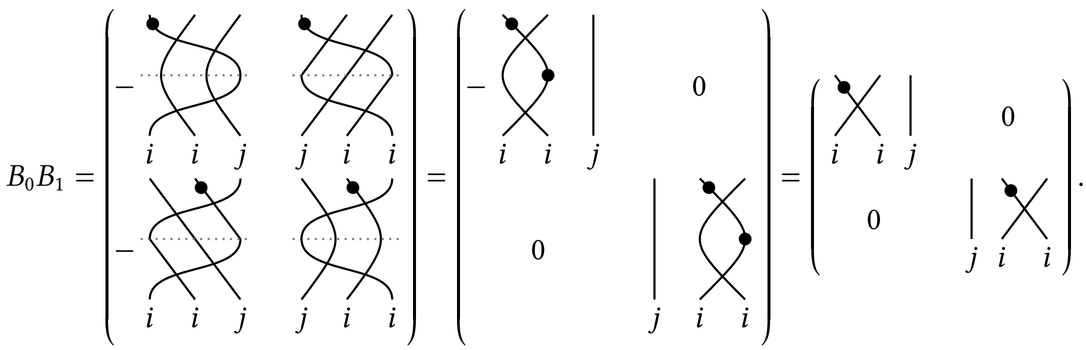
by appending the following diagrams:



Note that $B_1 B_0 = 1$:



Similarly $B_0 B_1 = 1$:



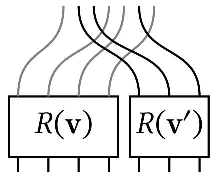
Thus we may conclude. □

7.6. **Grothendieck groups.** Let $B \subset A$ be an inclusion of unital algebras which does not preserve a unit, and denote the image of 1_B by e. This is an idempotent in A. In this situation, we have adjoint functors of induction and restriction:

$$\mathsf{Ind} \,:\, B\text{–mod} \underset{\longleftarrow}{\overset{\longrightarrow}{}} A\text{–mod} \,:\, \mathsf{Res}$$

where $\mathsf{Ind}(M) = B \otimes_A M$, and $\mathsf{Res}(N) = eN$.

For quiver Hecke algebras, we have an obvious inclusion of algebras $R(\mathbf{v}) \otimes R(\mathbf{v}') \subset R(\mathbf{v} + \mathbf{v}')$, obtained by putting the diagrams horizontally next to each other. Note that by definition, we have $\mathsf{Ind}(P_\eta, P_{\eta'}) = P_{\eta\eta'}$. On the other hand, it is easy to see that $\mathsf{Res}(R(\mathbf{v}+\mathbf{v}'))$ is free over $R(\mathbf{v}) \otimes R(\mathbf{v}')$, with basis given by "shuffle" permutation, which leave the order of first $|\mathbf{v}|$ and last $|\mathbf{v}'|$ strands intact:



In particular, both Ind and Res restrict to functors between the category of projective modules.

Definition 7.20. Let $\mathcal{C}$ be an exact category. The *Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is defined by

$$K_0(\mathcal{C}) := \bigoplus_{C \in \mathcal{C}/\sim} \mathbb{Z}[C] / \langle [C] = [A] + [B] \,:\, 0 \to A \to C \to B \to 0 \rangle$$

where the sum runs over all isomorphism classes of objects in $\mathcal{C}$, and the relations run over all short exact sequences in $\mathcal{C}$.

Let $\mathcal{C} = R(\mathbf{v})\text{–pmod}$, and denote $K_0(R(\mathbf{v})) := K_0(R(\mathbf{v})\text{–pmod})$. We know that all exact sequences in this category split; therefore,

$$K_0(R(\mathbf{v})) = \bigoplus_P \mathbb{Z}[P],$$

where the sum runs over all indecomposable projective $R(\mathbf{v})$-modules. In particular, the number of simple $R(\mathbf{v})$-modules is the same as the rank of $K_0(R(\mathbf{v}))$ over $\mathbb{Z}$.

Since all algebras and modules we consider are graded, we have an additional piece of structure, namely a grading shift. We will denote the class of $P[1]$ in $K_0(R(\mathbf{v}))$ by $q[P]$; this makes $K_0(R(\mathbf{v}))$ into a free module over $\mathbb{Z}_q := \mathbb{Z}[q, q^{-1}]$.

Example 7.21. Let us consider the nilHecke algebra $R(2)$ again. We saw that $R(2) \simeq P_{1^{(2)}}$ as a left $R(2)$-module; however, this does not take into account the grading shift. One can check that $R(2)(\partial x_1)$ starts in degree 0, while $R(2)(x_2\partial)$ starts in degree (-2). This means that in the Grothendieck group, we have

$$P_{1^2} = (1 + q^{-2}) P_{1^{(2)}}. \tag{7.6}$$

The functors Ind, Res induce $\mathbb{Z}_q$-linear maps $K_0(R(\mathbf{v})) \otimes K_0(R(\mathbf{v}')) \rightleftarrows K_0(R(\mathbf{v} + \mathbf{v}'))$. This suggests to consider $K_0(R(\mathbf{v}))$ for all $\mathbf{v}$ at the same time.

Definition 7.22. Define $K_0(R) := \bigoplus_{\mathbf{v} \in \mathbb{Z}^I_{\geq 0}} K_0(R(\mathbf{v}))$.

Note that the algebra has a $b\,bZ^I$-grading, where $K_0(R(\mathbf{v}))$ lives in degree $\mathbf{v}$.

Proposition 7.23. *The functor* Ind *induces a (graded) associative product on* $K_0(R)$*. The functor* Res *induces a (graded) coassociative coproduct on* $K_0(R)$*.*

Proof. This follows from our computation of Ind and Res for projective modules. □

The product and coproduct on $K_0(R)$ are dual to each other in a certain sense. Let us denote the graded dimension of the center Sym of $R(\mathbf{v})$ by $(\mathbf{v})_q$:

$$(\mathbf{v})_q = \mathrm{grdim}(\mathsf{Sym}) = \prod_{i\in I}\prod_{a=1}^{v_i} \frac{1}{1-q^{2a}}.$$

Definition 7.24. Define a $\mathbb{Z}_q$-bilinear form on $K_0(R(\mathbf{v}))$ by

$$\begin{aligned}(-,-) \,:\, K_0(R(\mathbf{v})) \times K_0(R(\mathbf{v})) &\to \mathbb{Z}_q[(\mathbf{v})_q],\\ (R(\mathbf{v})e_1, R(\mathbf{v})e_2) &:= \mathrm{grdim}(e_1 R(\mathbf{v}) e_2),\end{aligned}$$

where e_1, e_2 are idempotents in $R(\mathbf{v})$.

Proposition 7.25. *The bilinear form satisfies the following properties:*

(1) $(1,1) = 1$ *for* $\mathbf{v} = 0$*;*
(2) $(P_i, P_j) = \delta_{ij}(1-q^2)^{-1}$ *for* $|\mathbf{v}| = 1$*;*
(3) $(x, \mathsf{Ind}(y, y')) = (\mathsf{Res}(x), y \otimes y')$*.*

Proof. The first two properties are obvious. For the third one, we have

$$\begin{aligned}(P_\eta, \mathsf{Ind}(P_\nu, P_{\nu'})) &= \mathrm{grdim}(e_\eta R(\mathbf{v}+\mathbf{v}') \otimes_{R(\mathbf{v}+\mathbf{v}')} R(\mathbf{v}+\mathbf{v}')(e_\nu \otimes e_{\nu'}))\\ &= \mathrm{grdim}(e_\eta R(\mathbf{v}+\mathbf{v}') e_{\nu\nu'}) = (\mathsf{Res}(P_\eta), P_\nu \otimes P_{\nu'}).\end{aligned}$$

and so we're done. □

The proposition implies that the bilinear form can be defined inductively.

Remark 7.26. One can ask whether product and coproduct on $K_0(R)$ are algebraically compatible. They almost form a bialgebra; namely, one needs to twist the factorwise product on $K_0(R) \otimes K_0(R)$ by some powers of q. We will not need this statement.

7.7. Universal enveloping algebras.

Let us recall some facts about universal enveloping algebras.

Definition 7.27. Let $\mathfrak{g}$ be a Lie algebra. The *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$ is defined by

$$U(\mathfrak{g}) := T(\mathfrak{g})/\langle x\otimes y - y \otimes x = [x,y] \,:\, x, y \in \mathfrak{g}\rangle,$$

where $T(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathfrak{g}^{\otimes k}$ is the tensor algebra.

The algebra $U(\mathfrak{g})$ is universal in the sense that every map of Lie algebras $\mathfrak{g} \to A$, where A is an associative algebra equipped with the commutator Lie bracket $[a,b] := ab - ba$, factors through $U(\mathfrak{g})$. Therefore the representation of $\mathfrak{g}$ is contained in the (richer) representation theory of $U(\mathfrak{g})$.

The size of $U(\mathfrak{g})$ is well known:

Theorem 7.28 (Poincaré-Birkhoff-Witt)**.** *We have an isomorphism of vector spaces*

$$U(\mathfrak{g}) \simeq S(\mathfrak{g}),$$

where $S(\mathfrak{g}) = \bigoplus_{k\geq 0} S^k\mathfrak{g} := \bigoplus_{k\geq 0}(\mathfrak{g}^{\otimes k})^{\mathfrak{S}_k}$. □

Now we restrict to a particular example. Let $\mathfrak{n} \subset \mathfrak{gl}_n$ be the Lie algebra of strictly upper-triangular matrices. The adjoint action of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ of diagonal matrices gives rise to a $\mathbb{Z}^n$-grading on $\mathfrak{n}$, and so on $U(\mathfrak{n})$. Let $\{E_{ij} : i < j\}$ be the basis of $\mathfrak{n}$ consisting of matrix elements, and denote $e_i = E_{i,i+1}$ for $1 \leq i \leq n-1$. Since $E_{ij}E_{kl} = \delta_{j,k}E_{il}$ we have the following relations in $U(\mathfrak{n})$:

$$e_ie_j - e_je_i = 0 \text{ for } |i-j| > 1, \qquad e_i^2e_{i+1} - 2e_ie_{i+1}e_i + e_{i+1}e_i^2 = 0. \tag{7.7}$$

It is easy to check that these are the defining relations of $U(\mathfrak{n})$.

Definition 7.29. We define $\mathbb{Z}(q)$-algebra $U_q(\mathfrak{n})$ to be generated by elements e_i, $1 \leq i \leq n-1$, modulo the following deformation of relations (7.7):

$$e_ie_j - e_je_i = 0 \text{ for } |i-j| > 1, \qquad e_i^2e_{i+1} - (q+q^{-1})e_ie_{i+1}e_i + e_{i+1}e_i^2 = 0. \tag{7.8}$$

Let $[n] = \frac{q^n - q^{-n}}{q-q^{-1}}$, $[n]! := [n][n-1]\dots[1] \in \mathbb{Z}_q$, and $e_i^{(k)} := \frac{e_i^k}{[k]!} \in U_q(\mathfrak{n})$. We define $U_q^{\mathcal{A}}(\mathfrak{n}) \subset U_q(\mathfrak{n})$ be the $\mathbb{Z}_q$-subalgebra generated by $e_i^{(k)}$, $i \in I$, $k \geq 1$.

The algebra $U_q(\mathfrak{n})$ is known as the positive half of quantum $\mathfrak{gl}_n$, and $U_q^{\mathcal{A}}(\mathfrak{n})$ is its Lusztig's integral form.

7.8. **A categorification result.** It turns out that the algebra $K_0(R)$ often has an explicit description. In order to simplify exposition, we restrict our attention to the case of a linear quiver, as in Exercise 7.3.

Theorem 7.30. *Let Q be a linear quiver, with any orientation of edges. Then there exists an isomorphism of $\mathbb{Z}_q$-algebras*

$$U_q^{\mathcal{A}}(\mathfrak{n}) \xrightarrow{\sim} K_0(R),$$

which sends $e_i^{(k)}$ to $[P_{i^{(k)}}]$. It preserves the $\mathbb{Z}^n$-grading.

Sketch of proof. First, we define a homomorphism $U_q(\mathfrak{n}) \to K_0(R) \otimes_{\mathbb{Z}_q} \mathbb{Z}(q)$ of $\mathbb{Z}(q)$-algebras. This amounts to checking the defining relations (7.8) of $U_q(\mathfrak{n})$, which follow from Proposition 7.19. Indeed, for $|i-j| > 1$ we have

$$e_ie_j = [P_{ij}] = [P_{ji}] = e_je_i,$$

and for the cubic relation

$$e_ie_{i+1}e_i = [P_{iji}] = [P_{i+1,i^{(2)}}] + [P_{i+1,i^{(2)}}] = e_{i+1}e_i^{(2)} + e_i^{(2)}e_{i+1} = (e_{i+1}e_i^2 + e_i^2e_{i+1})/(q+q^{-1})$$
$$\Rightarrow e_{i+1}e_i^2 + e_i^2e_{i+1} = (q+q^{-1})e_ie_{i+1}e_i.$$

This morphism restricts to a map $\Phi : U_q^{\mathcal{A}}(\mathfrak{n}) \to K_0(R)$. In order to check its injectivity, we recall that there exists a non-degenerate bilinear form on $U_q^{\mathcal{A}}(\mathfrak{n})$, defined by the properties in Proposition 7.25. Therefore Φ is compatible with bilinear forms, and as such must be injective. The surjectivity of Φ is slightly more technical, and so we omit it. ☐

What is the point of this theorem? On one hand, by PBW theorem it computes the rank of each individual $K_0(R(\mathbf{v}))$, which is what we started out with. On the other hand, it tells us that the universal enveloping algebra $U(\mathfrak{n})$ is a shadow of a "categorified" picture. Namely, we can replace the algebra $U(\mathfrak{n})$, by the collection of categories $R(\mathbf{v})$–pmod. A slightly better way to view this is as a *2-category*, where objects are points in $\mathbb{Z}^n$, and morphism categories are $\mathrm{Hom}(\mathbf{v}', \mathbf{v} + \mathbf{v}') = R(\mathbf{v})$–pmod. One eventually extends this to $U_q(\mathfrak{gl}_n)$. Then instead of representation theory of an algebra, one can study categorical representations.

While all of this might sounds abstract, here is one application. For any finite dimensional representation V of $\mathfrak{sl}_2 = \langle E, F, H \rangle$, we have an isomorphism of weight spaces $V_{-k} \simeq V_k$, given by E^k. Similarly, one can show that a categorical action of $U(\mathfrak{sl}_2)$ induces similar equivalences of weight categories. This observation was leveraged by Chuang and Rouquier to a great effect in order to understand the blocks of the category of representations of $\mathfrak{S}_n$ over a field of positive characteristic.

A note on references

Sections 2 to 4 roughly follow selected chapters from the excellent lecture notes by Anderson-Fulton [AF23]. Section 5.1 is an extreme butchering of a beautiful and important topic. One would need a separate series of lectures to do it justice; if you want to dive into that, my preferred textbooks are [Hum12] and [OV12]. The rest of Section 5 is based on the seminal paper of Demazure [Dem73], and its later extension by Holm-Sjamaar [HS08]. My exposition of convolution algebras in Section 6 started out by following Chriss-Ginzburg [CG97], but ended up being more inspired by the techniques found in Fulton [Ful13]. For a less rushed look at the equivariant categories of constructible sheaves, look at lecture notes by Yun [Yun06]. Finally, I am not aware of a good textbook treating quiver Hecke algebras; however, I suggest looking at the expository article by Brundan [Bru13], and Przezdziecki's master thesis [Prz15]. I based my exposition on the original paper of Khovanov-Lauda [KL09].

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